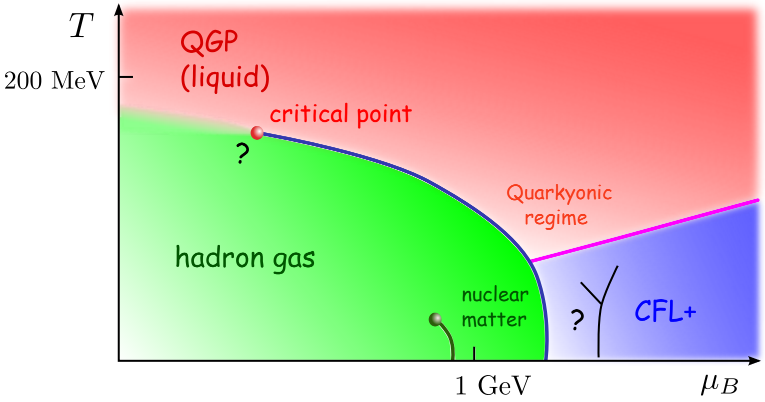
Hydrodynamics and critical slowing down

M. Stephanov and Y. Yin

UIC, MIT

QCD Phase Diagram *(a theorist's view)*



Outline

- Equilibrium
- Non-equilibrium

Why fluctuations are large at a critical point?

- The key equation:

$$P(\sigma) \sim e^{S(\sigma)} \qquad \textit{(Einstein 1910)}$$

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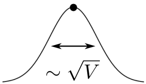

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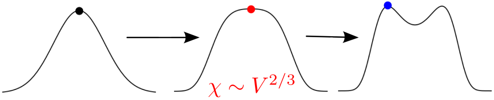

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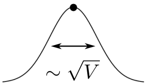

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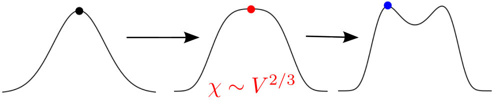

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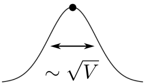

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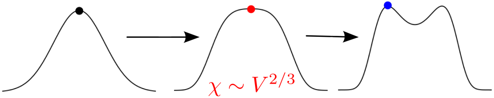
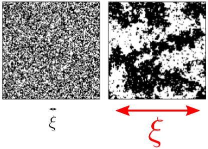
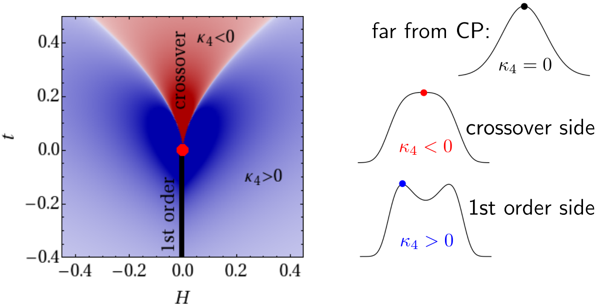




CLT? σ is not a sum of ∞ many *uncorrelated* contributions: $\xi \to \infty$

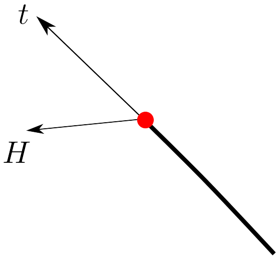
Higher order cumulants

- Higher cumulants (shape of $P(\sigma)$) depend stronger on ξ.
  E.g., $\langle\sigma^2\rangle \sim V\xi^2$ while $\langle\sigma^4\rangle_c \sim V\xi^7$
- Higher moments also depend on which side of the CP we are.
  This dependence is also universal.
- Using Ising model variables:



Mapping Ising to QCD phase diagram

T vs μ_B:



- In QCD $(t, H) \to (\mu - \mu_{\rm CP}, T - T_{\rm CP})$

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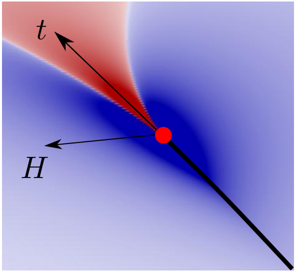

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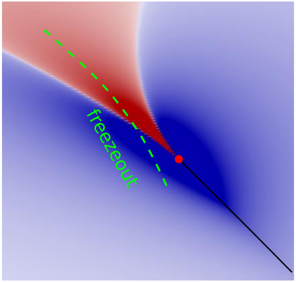
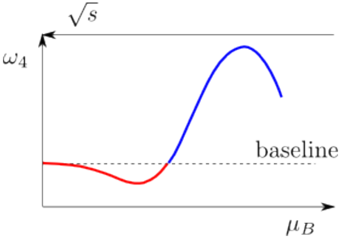
Beam Energy Scan





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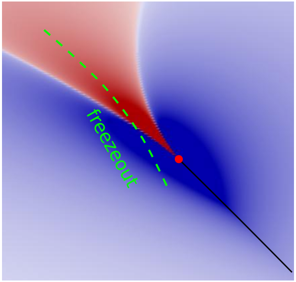

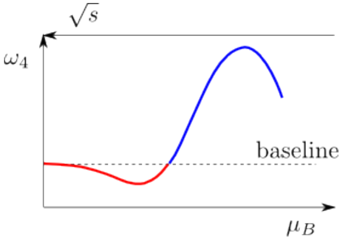

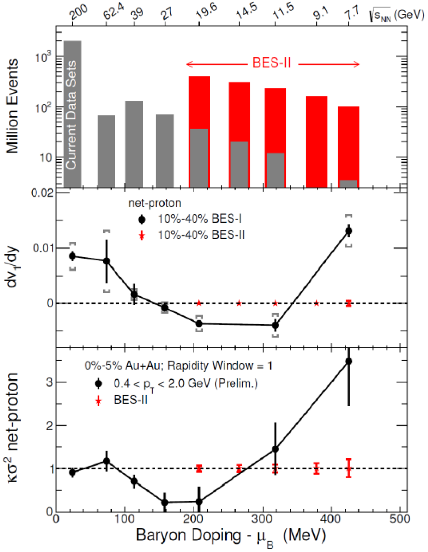

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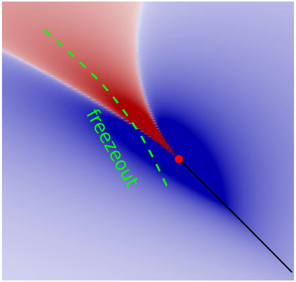

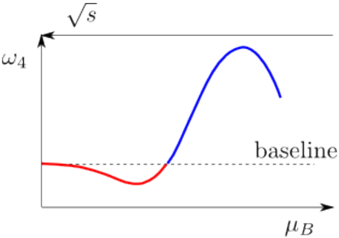

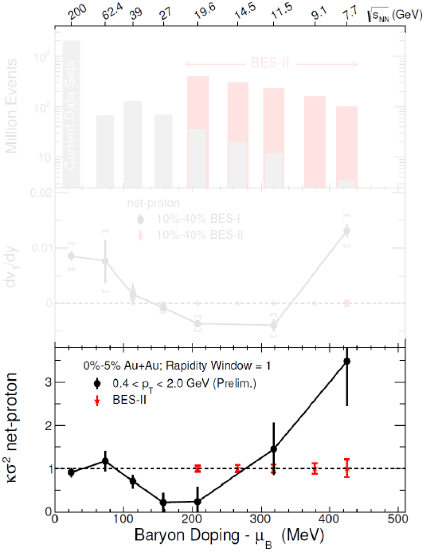

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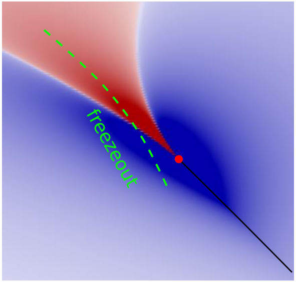

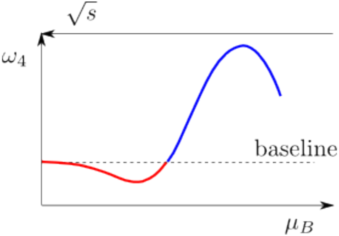

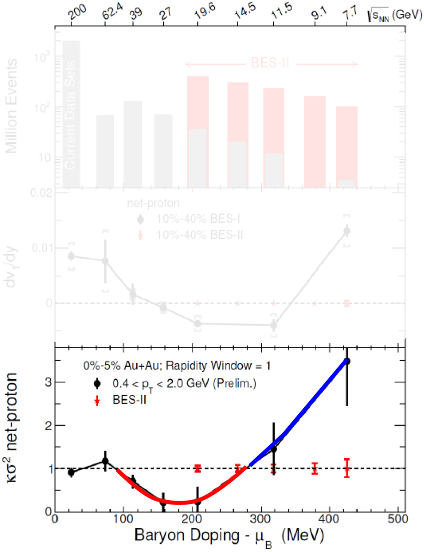

“intriguing hint” (2015 LRPNS)

Non-equilibrium physics is essential near the critical point.

The goal for BEST COLLABORATION

Why ξ is finite

System expands and is *out of equilibrium*

Kibble-Zurek mechanism.

Critical slowing down means $\tau_{\mathrm{relax}} \sim \xi^z$.
Given $\tau_{\mathrm{relax}} \lesssim \tau$ (expansion time scale):

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$z \approx 3$ (universal).

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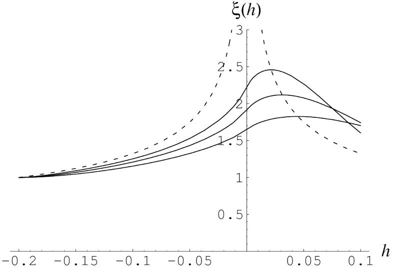
Estimates: $\xi \sim 2 - 3$ fm
(Berdnikov-Rajagopal)

KZ scaling for $\xi(t)$
and cumulants
(Mukherjee-Venugopalan-Yin)



Lessons

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  $\longrightarrow$ Observable critical fluctuations
- Can we get *critical* fluctuations from hydrodynamics *directly*?

Hydrodynamics breaks down at CP

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When $k\xi^3 \sim 1$ hydrodynamics breaks down, while $k\xi \ll 1$ still.

Why does it happen before $k \sim \xi^{-1}$?

Critical slowing down and bulk viscosity

Bulk viscosity is the effect of system taking time to adjust to local equilibrium (Khalatnikov-Landau).

$$p_{\rm hydro} = p_{\rm equilibrium} - \zeta\, \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$\boldsymbol{\nabla} \cdot \boldsymbol{v}$ – expansion rate

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Hydrodynamics breaks down because of large relaxation time.

Similar to breakdown of an effective theory due to a low-energy mode which cannot be integrated out.

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- Regime I: $\tau_\phi \ll \tau_{\rm hydro}$ – ordinary hydro (with $\zeta \sim \xi^3$ near CP).

  Crossover occurs when $\tau_{\rm hydro} \sim \tau_\phi$, or $k \sim \xi^{-3}$.
- Regime II: $\xi^{-3} < k < \xi^{-1}$ – "Hydro+" regime.

Advantages/motivation of Hydro+

- Extends the range of validity of “vanilla” hydro near CP.
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- No large kinetic coefficients.

  Since noise $\sim \zeta$, also the noise is not large.

Ingredients of "Hydro+"

- Nonequilibrium entropy, or quasistatic EOS:

$$s^*(\varepsilon, n, \phi)$$

Equilibrium entropy is the maximum of s^*:

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- The 6th equation (constrained by 2nd law):

$$(u \cdot \partial)\phi = -\gamma_\phi \pi - G_\phi(\partial \cdot u), \qquad \text{where } \pi = \frac{\partial s^*}{\partial \phi}$$

Noise can be added in accordance with FD.

Linearized Hydro+

Linearized Hydro+ has 4 longitudinal modes (sound$\times 2$ + density + ϕ).

In addition to the usual c_s, D, etc. Hydro+ has two more parameters $\Delta c^2 = c_*^2 - c_s^2$ and $\Gamma = 1/\tau_\phi$.

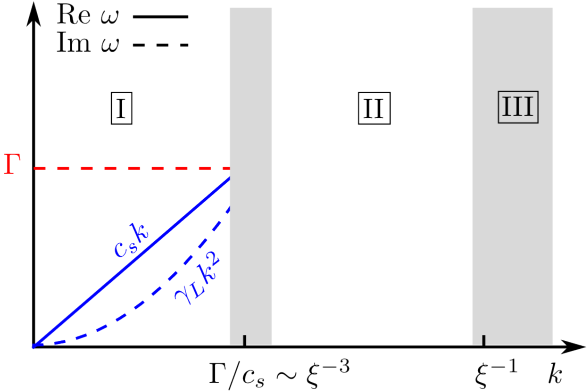
The sound velocities are different in Regime I ($c_s k \ll \Gamma$) and II:

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_{s/n,\pi=0} \quad \text{and} \quad c_*^2 = \left(\frac{\partial p^*}{\partial \varepsilon}\right)_{s/n,\phi}$$

The bulk viscosity receives large contribution from the slow mode given by Landau-Khalatnikov formula

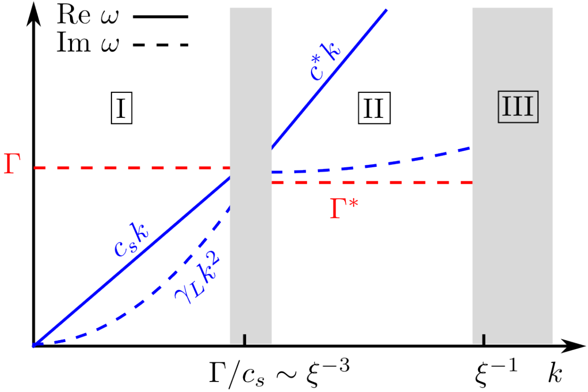
$$\Delta\zeta = w\Delta c^2/\Gamma$$

Modes



Re ω
Im ω
I
II
III
Γ
$c_s k$
$\gamma_L k^2$
$\Gamma/c_s \sim \xi^{-3}$
ξ^{-1}
k

Modes



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$c^* k$
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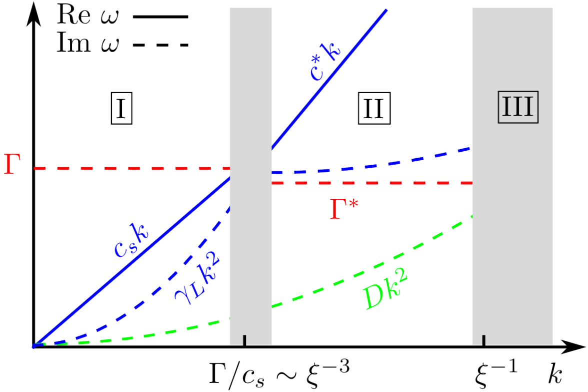
Modes



Re ω
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c^*k
$c_s k$
$\gamma_L k^2$
Dk^2
$\Gamma/c_s \sim \xi^{-3}$
ξ^{-1}
k

Microscopic origins of Hydro+

Understanding the microscopic origin of the slow mode:

The fluctuations around equilibrium are controlled by the entropy functional $P \sim e^S$.

Near the critical point convenient to "rotate" the basis of variables to "Ising"-like critical variables $\mathcal{E}$ and $\mathcal{M}$.

$$\delta\mathcal{S}[\delta\mathcal{E}, \delta\mathcal{M}] = \left[\frac{1}{2}\, a_{\mathcal{M}}\, (\delta\mathcal{M})^2 + \frac{1}{2}\, a_{\mathcal{E}}\, (\delta\mathcal{E})^2 + b\,\delta\mathcal{E}\,\delta\mathcal{M}^2 + \ldots\right].$$

Since $a_{\mathcal{M}} \ll a_{\mathcal{E}}$ fluctuations of $\mathcal{M}$ are large and are slow to equilibrate.

Their magnitude is related to the slow relaxation mode ϕ.

Hydro + mode distribution

Separate “hard” $k > \xi^{-1}$ and “soft” $k \ll \xi^{-1}$ modes.

The new variable, “mode distribution function”:

$$n_{\mathcal{M}}(t, \boldsymbol{x}, \boldsymbol{Q}) = \int_{\boldsymbol{y}} \langle\, \delta\mathcal{M}(t, \boldsymbol{x} + \boldsymbol{y}/2)\, \delta\mathcal{M}(t, \boldsymbol{x} - \boldsymbol{y}/2)\,\rangle\; e^{-i\boldsymbol{Q}\cdot\boldsymbol{y}}$$

The additional mode distribution function relaxation equation:

$$(u \cdot \partial) n_{\mathcal{M}}(t, \boldsymbol{x}, \boldsymbol{Q}) = 2\Gamma_{\mathcal{M}}(\boldsymbol{Q}) \left[a_{\mathcal{M}}^{-1} - n_{\mathcal{M}}(t, \boldsymbol{x}, \boldsymbol{Q}) \right]$$

where $\Gamma_{\mathcal{M}}(\boldsymbol{Q})$ is known from model H (Kawasaki, Onuki).

Single mode from a mode distribution

In terms of mode distribution function $n_{\mathcal{M}}$, the slow mode is

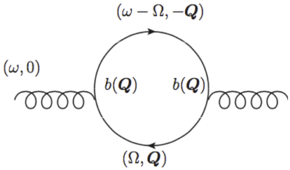
$$\phi(t, \boldsymbol{x}) \sim \int_{\boldsymbol{Q}} n_{\mathcal{M}}(t, \boldsymbol{x}, \boldsymbol{Q})$$

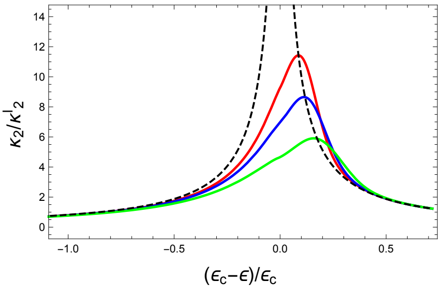
In Hydro+ we replace $\Gamma(\boldsymbol{Q})$ by a representative rate Γ. Since the integrands are peaked around $Q^* \sim \xi^{-1}$, $\Gamma \approx 2\Gamma_{\mathcal{M}}(\xi^{-1})$.

More precisely, e.g.,

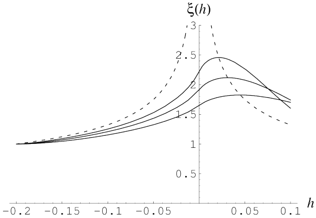
$$\Delta\zeta = TG_{\mathcal{E}} \int_{\boldsymbol{Q}} \frac{b^2(\boldsymbol{Q})}{2a^2_{\mathcal{M}}(\boldsymbol{Q})\Gamma_{\mathcal{M}}(\boldsymbol{Q})}$$

$$\Delta c_s^2 = TG_{\mathcal{E}} \int_{\boldsymbol{Q}} \frac{b^2(\boldsymbol{Q})}{a^2_{\mathcal{M}}(\boldsymbol{Q})}$$

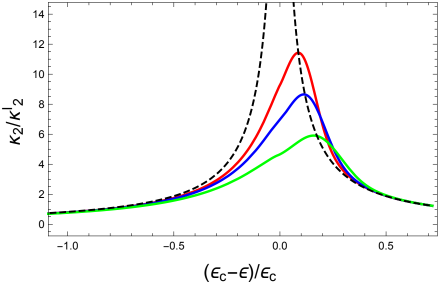


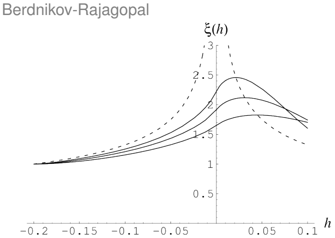


Berdnikov-Rajagopal

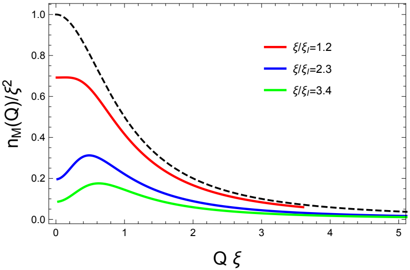


Simple model. Bjorken expansion.





mode distribution function falls out of equilibrium



Summary

- A fundamental question for Heavy-Ion collision experiments:

  Is there a critical point on the boundary between QGP and hadron gas phases?

  Theoretical framework is needed – the goal for BEST COLLABORATION.

- Large (non-gaussian) fluctuations – universal signature of a critical point.

- In H.I.C., the magnitude of the signatures is controlled by dynamical non-equilibrium effects. The physics of the interplay of critical and dynamical phenomena can be captured by hydrodynamics with a critically slow mode – Hydro+.