Fluctuations of conserved charges at zero and finite density

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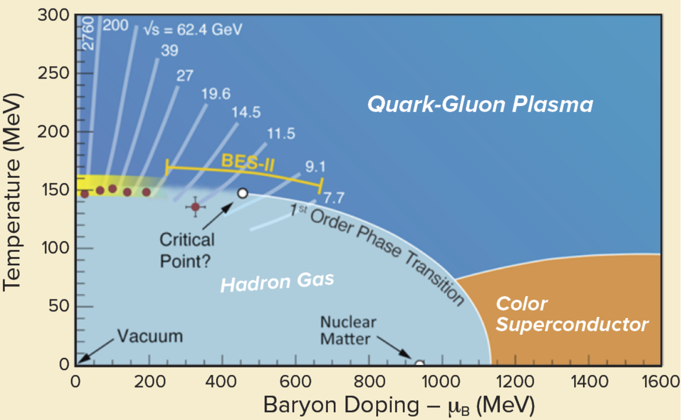
Challenge: extend $\mu_B = 0$ lattice thermodynamics results to finite densities.



T_c: [hep-lat/0611014,hep-lat/0609068,0903.4155,1005.3508], [1111.1710]

Equation of state: [1005.3508,1309.5258,1606.07494], [1407.6387]

Fluctuations: [1112.4416,1507.04627], [1203.0784,1701.04325]

Fluctuations in a grand canonical ensemble

The expectation value of a conserved charge is a derivative with respect to the chemical potential.

$$\langle N_q \rangle = T \frac{\partial \log Z(T, V, \{\mu_q\})}{\partial \mu_q}$$

The response of the system to the thermodynamic force μ_q is proportional to the fluctuation of the conserved charge:

$$\frac{\partial \langle N_i \rangle}{\partial \mu_j} = T \frac{\partial^2 \log Z(T, V, \{\mu_q\})}{\partial \mu_j \partial \mu_i} = \frac{1}{T}(\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle)$$

The higher derivatives are the generalized quark number susceptibilities:

$$\chi^{u,d,s,c}_{i,j,k,l} = \frac{\partial^{i+j+k+l}(p/T^4)}{(\partial \hat{\mu}_u)^i (\partial \hat{\mu}_d)^j (\partial \hat{\mu}_s)^k (\partial \hat{\mu}_c)^l}$$

with $\hat{\mu}_q = \mu_q / T$.

We have to calculate $\mu_Q(\mu_B)$ and $\mu_S(\mu_B)$ that are defined by the constraints of the experimental setup:

$$\langle S \rangle = 0 \qquad \langle Q \rangle / \langle B \rangle = Z^{Au}/A^{Au} \approx 0.4$$

$$\begin{aligned}
M_Q &= V\chi_1^Q(\mu_B) = V\mu_B\left[\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}\right] + \mathcal{O}(V\mu_B^3)\\
\sigma_Q^2 &= V\chi_2^Q(\mu_B) = V\chi_B^Q + \mathcal{O}(V\mu_B^2)\\
S_Q\sigma_Q^3 &= V\chi_3^Q(\mu_B) = V\mu_B\left[\chi_{13}^{BQ} + \chi_{31}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_4^Q\frac{d\mu_Q}{d\mu_B}\right] + \mathcal{O}(V\mu_B^3)\\
\kappa\sigma_Q^4 &= V\chi_4^Q(\mu_B) = V\chi_B^Q + \mathcal{O}(V\mu_B^2)
\end{aligned}$$

To leadig order, at infinitesimal μ_B:

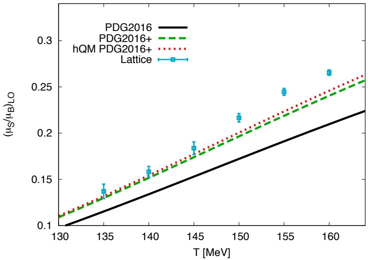
$$\frac{M_Q}{\mu_B\sigma_Q^2} = \frac{\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}}{\chi_2^Q}; \qquad \frac{S_Q\sigma_Q^3}{M_Q} = \frac{\chi_{13}^{BQ} + \chi_{31}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_4^Q\frac{d\mu_Q}{d\mu_B}}{\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}}$$

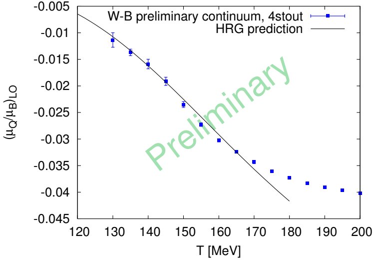
Strangeness and electric charge chemical potential

We have to calculate $\mu_Q(\mu_B)$ and $\mu_S(\mu_B)$ that are defined by the constraints of the experimental setup:

$$\langle S \rangle = 0 \qquad \langle Q \rangle / \langle B \rangle = Z^{Au} / A^{Au} \approx 0.4$$

Our latest continuum extrapolations with up to $N_\tau = 24$:

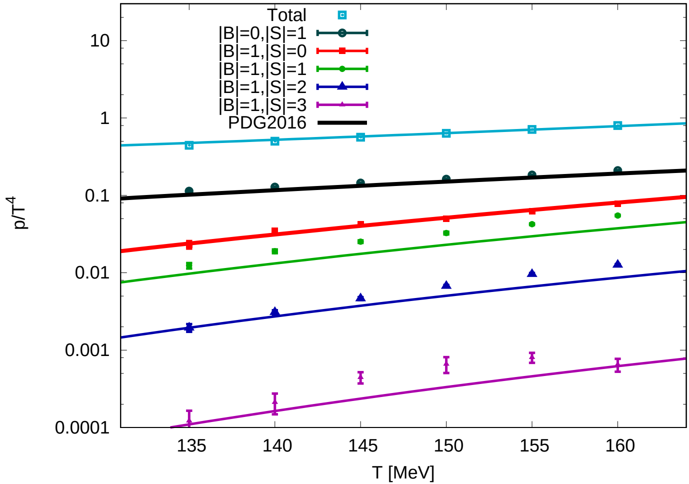




See also:
HotQCD: [1208.1220,1404.6511]
Wuppertal-Budapest: [1305.5161]

Strangeness sectors from the lattice

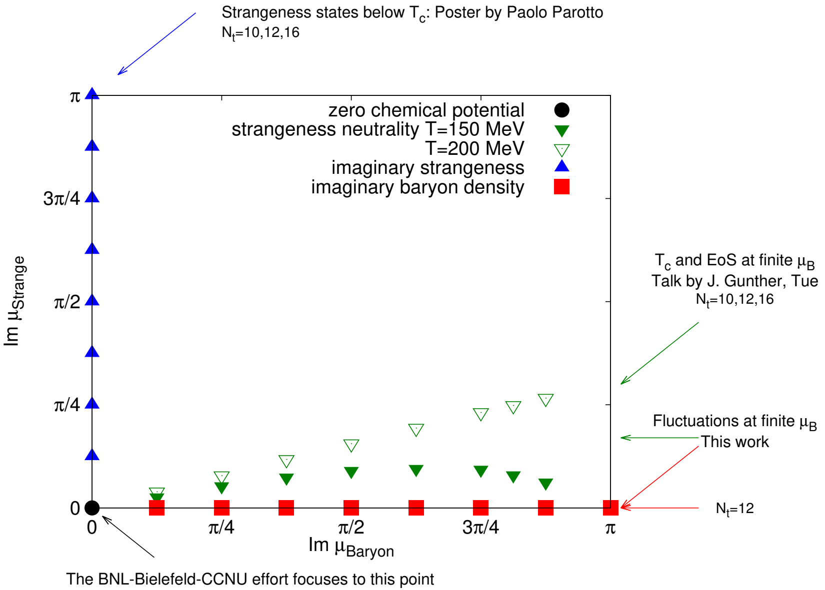
Lattice can calculate the partial pressure sector by sector.



Discrepancy to HRG: *indication for undiscovered resonances?*

poster by Paolo Parotto **G19** [1702.01113]

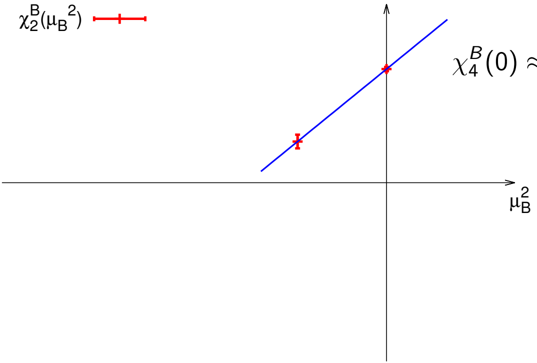
Simulation landcape with imaginary μ



The standard method to calculate higher order fluctuations is to calculate the non-gaussianity of the baryon number distribution at $\mu_B = 0$.

$$\chi_4^B(0) = \frac{1}{VT^3}\left(\langle N_B^4\rangle - 3\langle N_B^2\rangle^2\right)$$

Alternatively one measures the variance $\chi_2^B(\mu_B^2)$ at non-zero chemical potentials.



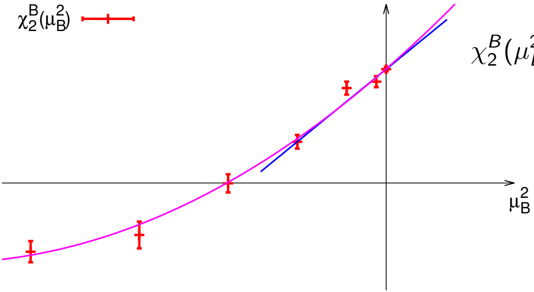
$$\chi_4^B(0) \approx 2\frac{\chi_2^B(0) - \chi_2^B(-\Delta\mu_B^2)}{\Delta\mu_B^2}$$

The imaginary μ_B approach

The standard method to calculate higher order fluctuations is to calculate the non-gaussianity of the baryon number distribution at $\mu_B = 0$.

$$\chi_4^B(0) = \frac{1}{VT^3}\left(\langle N_B^4\rangle - 3\langle N_B^2\rangle^2\right)$$

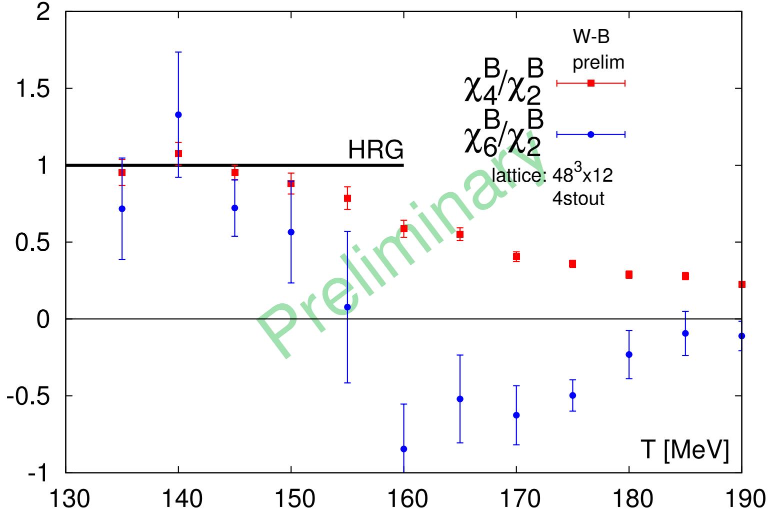
Alternatively one measures the variance $\chi_2^B(\mu_B^2)$ at several non-zero chemical potentials.



$$\chi_2^B(\mu_B^2) \approx \chi_2^B(0) + \frac{1}{2}\mu_B^2\chi_4^B(0) + \frac{1}{24}\mu_B^4\chi_6^B(0)$$

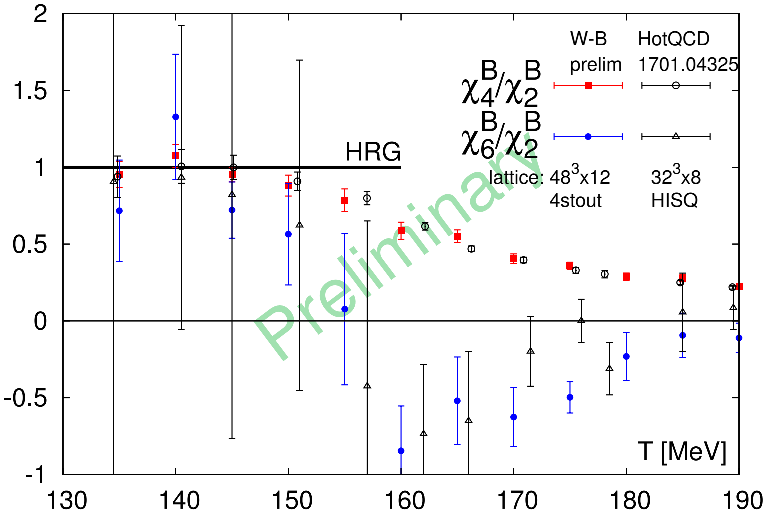
See also: [Wuppertal-Budapest 1607.02493, D'Elia et al 1611.08285]

Baryon kurtosis and beyond



See also D'Elia et al 1611.08285; Datta et al 1612.06673 ;
BNL-Bielefeld-CCNU: 1701.04325

Baryon kurtosis and beyond

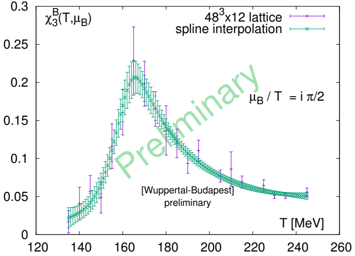


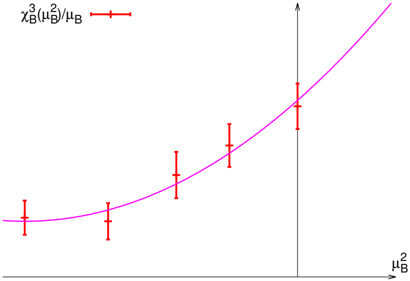
See also D'Elia et al 1611.08285; Datta et al 1612.06673 ;
BNL-Bielefeld-CCNU: 1701.04325

The imaginary μ_B approach

A spline interpolation of independent lattice results reduces statistical errors, *but might introduce systematic errros*.

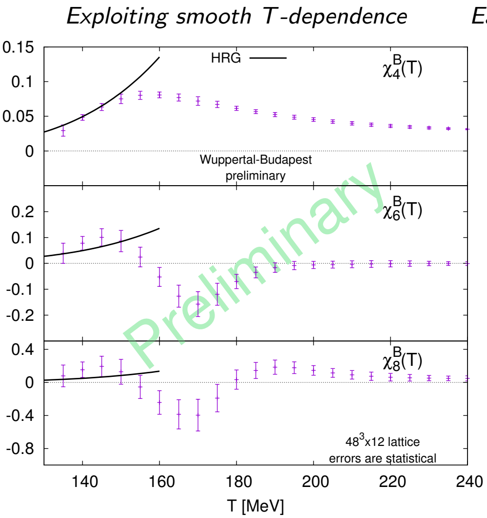
We assume that there is no phase transition at imaginary the studied part of the imaginary-μ_B phase diagram. [Bonati et al 1602.01426]

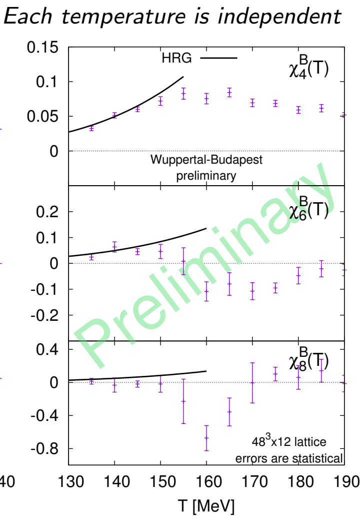




$$\frac{\chi_3^B(\mu_B^2)}{\mu_B} \approx \chi_4^B(0) + \frac{1}{6}\mu_B^2\chi_6^B(0) + \frac{1}{120}\mu_B^4\chi_8^B(0)$$

Baryon kurotosis and beyond

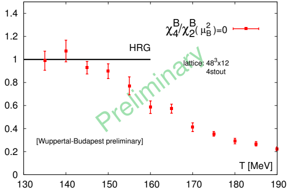


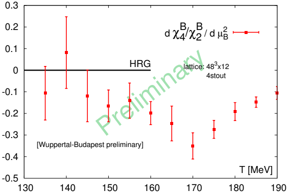


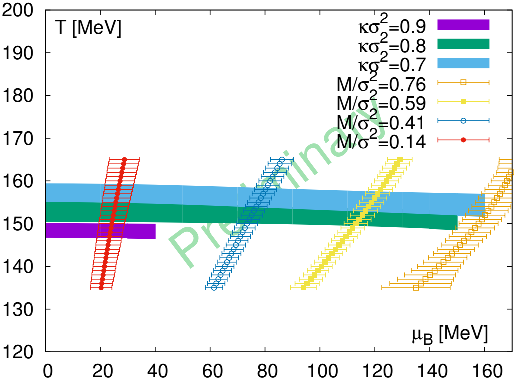
Systematic errors are not shown

Baryon kurotosis extrapolated

Using the $48^3 \times 12$ data set we can draw contours of constant kurtosis on the phase diagram. The mean/variance contours are continuum extrapolated.

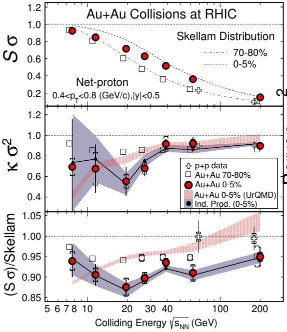


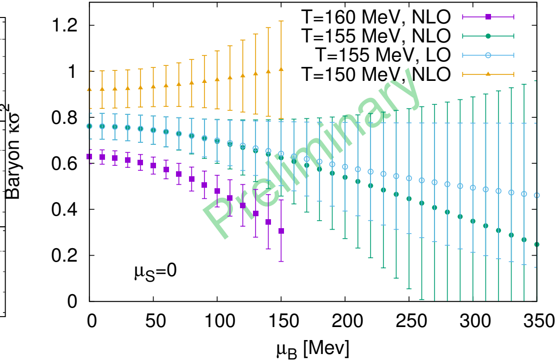




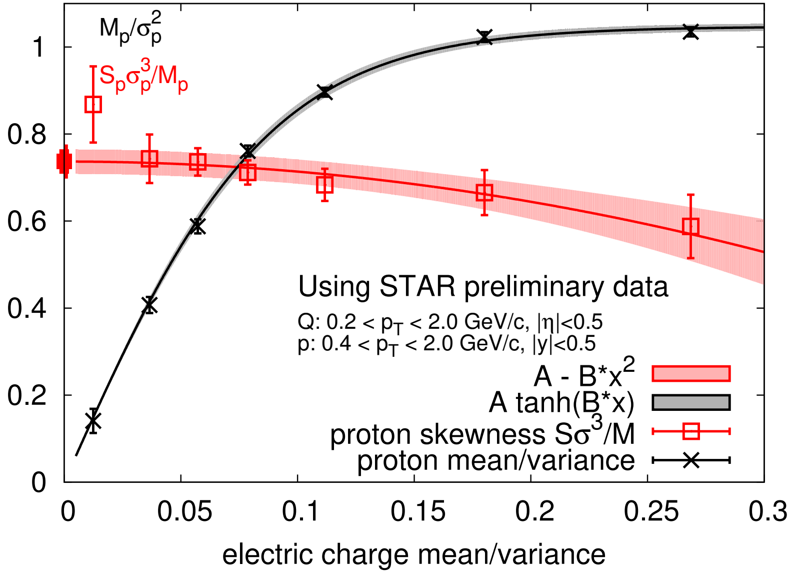
Using the $48^3 \times 12$ data set with spline interpolation we can make use of the $\chi_2^B \ldots \chi_8^B$ fit to extrapolate the kurtosis to finite μ_B.

Here we simply used the parameters $\mu_S = \mu_Q = 0$.





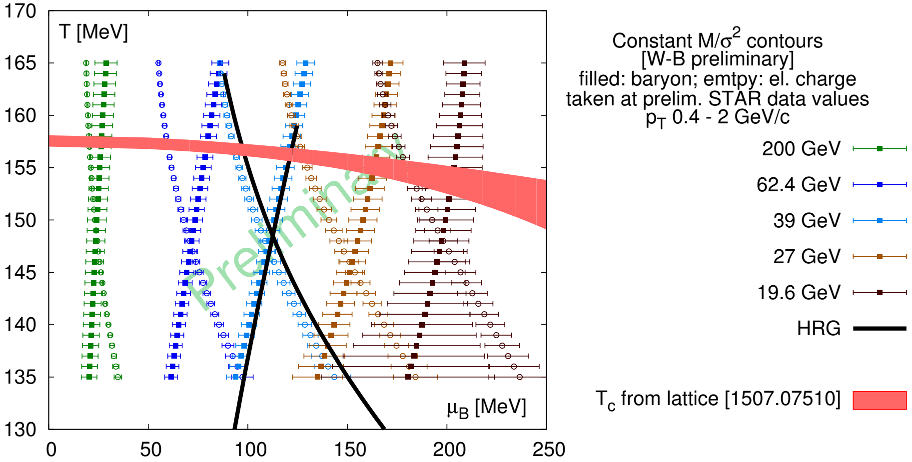
Fluctuations by the STAR experiment



Contours of constant fluctuations on the phase diagram

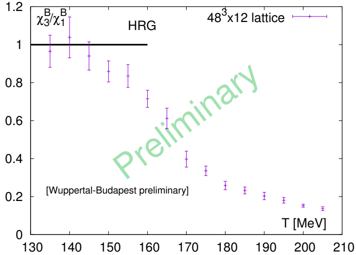
As a realistic example we take the preliminary STAR proton and electric charge fluctuation data [1309.5681,1402.1558].

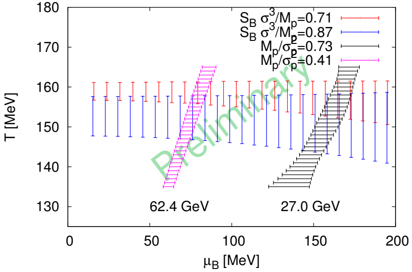
$\sqrt{s_{NN}}$ [GeV]	200	62.4	39	27	19.6
σ_p^2/M_p	7.10(1)(138)	2.46(0)(11)	1.70(0)(5)	1.31(0)(2)	1.12(0)(1
σ_Q^2/M_Q	80.2(2)(1)	27.32(7)(3)	17.46(3)(1)	12.71(3)(1)	8.95(3)(1



Baryon skewness extrapolated

Using the $48^3 \times 12$ data set we can draw contours of constant baryon skewness ($S\sigma^3/M$) on the phase diagram. The mean/variance contours are continuum extrapolated.





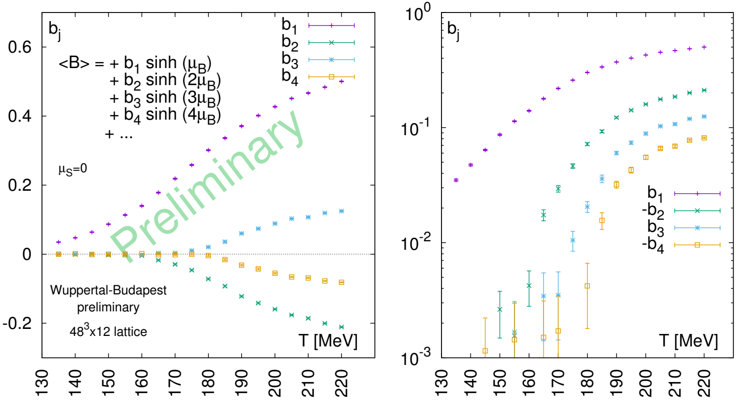
In a direct comparison STAR's new fluctuation data would correspond to temperatures where the HRG result is no longer trustworthy.

A Fourier analysis of the imaginary density

HRG prediction *(Boltzmann approximation)*:

$$\langle B \rangle \sim VT^3 \chi_2^B(T) \sinh(\mu_B/T)$$

Using imaginary μ_B we represent $\langle B \rangle = i \sum_{j=1} b_j \sin(\mathrm{Im}\mu_B/T)$

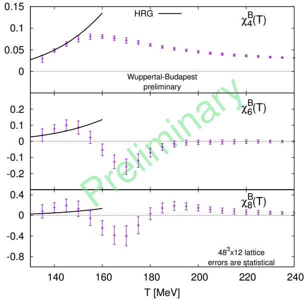


The emergence of a negative $\sinh(2\mu_B/t)$ *contribution signals deconfinement.*

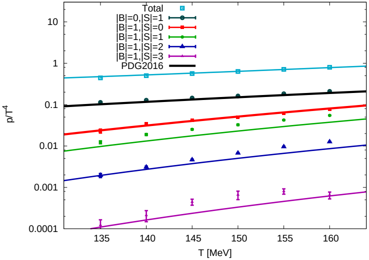
Summary

- Lattice QCD is making good progress in exploring thermodynamics.
  - Low order fluctuations are well established in the continuum imit, even at small μ_B, higher order fluctuations are subject of current research.
  - Higher order fluctuations can be explored with large statistics $\mu_B = 0$ simulations (BNL-Bielefeld way) or with studying the response to imaginary chemical potential (Wuppertal-Budapest strategy).
- Two applications:

High net-baryon moments:



The pressure contribution of $|S|$=0,1,2,3 mesons and baryons.



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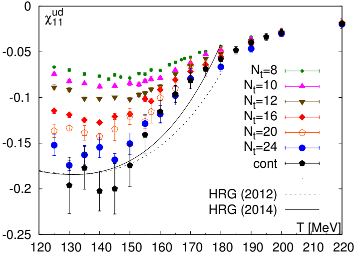
The partition function of the lattice gauge theory with staggered fermions is

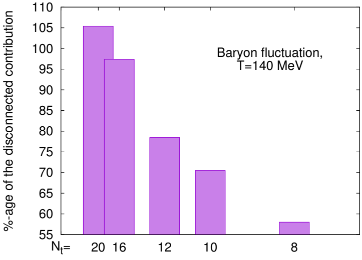
$$Z = \int \mathcal{D}U\, e^{-S_g} (\det M_u(\mu_u))^{1/4} (\det M_d(\mu_d))^{1/4} (\det M_s(\mu_s))^{1/4} = \int \mathcal{D}U\, e^{-S_{\rm eff}}$$

where S_g is the gauge action. For $\mu > 0$ the determinant becomes complex. The fermion determinant $\det M = |\det M| e^{i\theta}$ has a fluctuating phase at small $\mu > 0$: [Allton hep-lat/02040130, Wuppertal-Budapest 1507.04627]

$$\left\langle \theta^2 \right\rangle = -\frac{1}{9} \mu_B^2 L^3 T N_f^2 \chi_{11}^{ud}$$

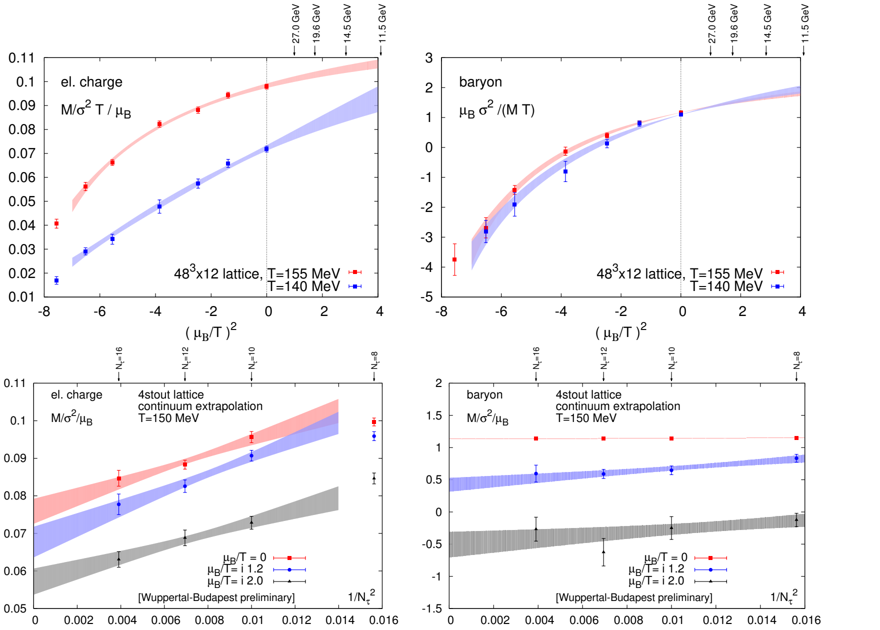
(~ 1 at $\mu_B \approx 100\ \mathrm{MeV}$, with $T = T_c$ and $LT = 3$.)





[Wuppertal-Budapest 1507.04627]

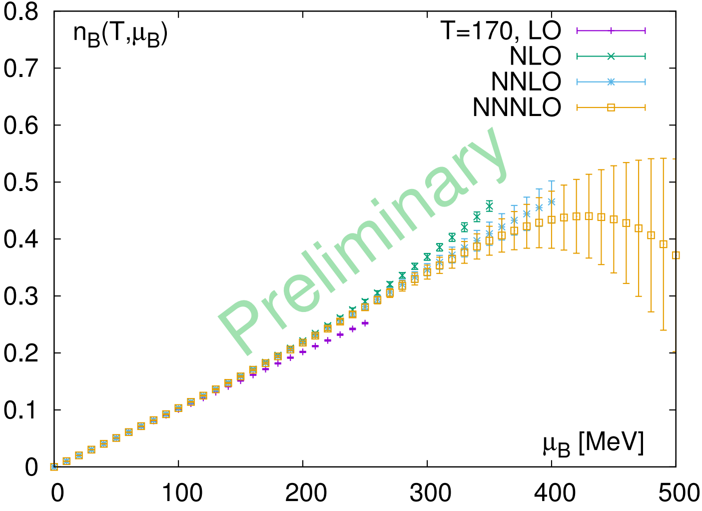
Extending to imaginary μ_B



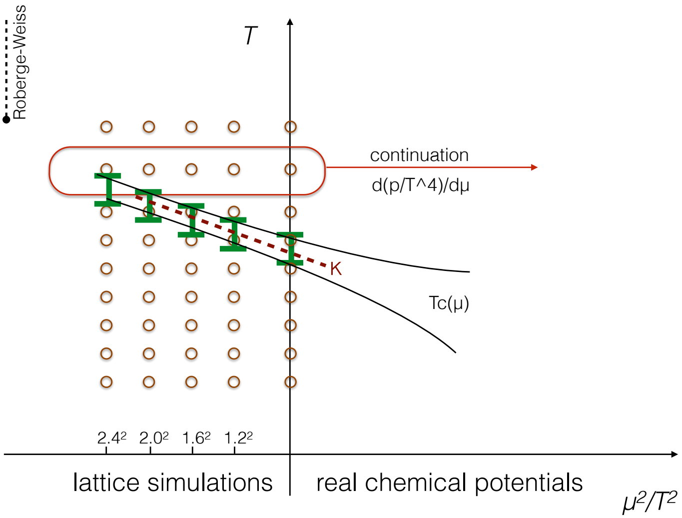
Baryon density at various orders

Slightly above T_c $\chi_6^B(T)$ and $\chi_8^B(T)$ show negative peaks.

How does this affect the extrapolation of the baryon density?



Analytic continuation



Many exploratory studies: [de Forcrand & Philipsen hep-lat/0205016]

[Philipsen 0708.1293] [Philipsen 1402.0838] [Cea et al hep-lat/0612018,0905.1292,1202.5700]

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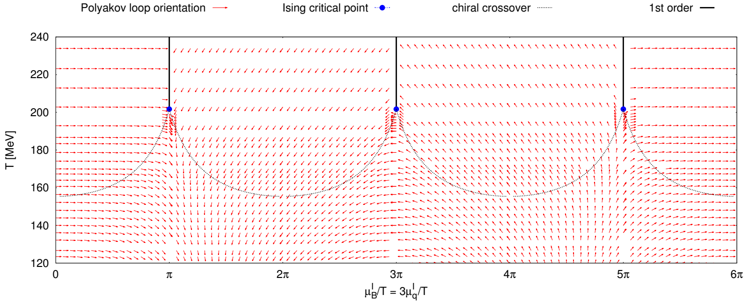

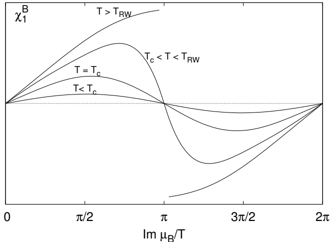

At low T only B states contribute:

$$\chi_1^B \sim \sin(\mu_B/T)$$

Then $\chi_4^B/\chi_2^B = 1$ as we find in simulations.

At high T fractional charges are required to make a first order transition at $\mu_B = \pi T$:

$$\chi_1^B \sim \sin(\mu_B/T/3)$$