Lattice QCD up to the perturbative regime

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Introduction

When is a weak coupling description of the QGP accurate?
Use recent high $T$ lattice results to check. In particular:

1. The equation of state
2. Fluctuations of conserved charges
3. Topological Susceptibility

Why these observables?

- All of these are important for phenomenology (heavy ion/cosmology/both)
- Can be calculated both on the lattice and with weak coupling techniques

Continuum extrapolated lattice results will be compared with:

- Resummed perturbation theory (HTL and DR) for 1-2
- Dilute instanton gas (DIGA) for 3
1. Equation of state with dynamical charm

- $N_f = 2 + 1 + 1$ with physical quark masses
- Lattice data up to $T = 1\text{GeV}$
- Dynamical charm is included since it is needed for cosmology
- Based on hep-lat/1606.07494, Nature 539 (2016) no.7627
- For the equation of state of the pure Yang-Mills theory up to high temperatures, see e.g.: WB, hep-lat/1204.6184, JHEP 1208 (2012) 0 53 and Giusti, Pepe, hep-lat/1612.00265
A good parametrization of the EoS is:

\[
\frac{I(T)}{T^4} = \exp\left(-\frac{h_1}{t} - \frac{h_2}{t^2}\right) \cdot \left(h_0 + f_0 \frac{\tanh(f_1 \cdot t + f_2) + 1}{1 + g_1 \cdot t + g_2 \cdot t^2}\right)
\]

<table>
<thead>
<tr>
<th></th>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$f_0$</th>
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<th>$f_2$</th>
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<td>0.534</td>
<td>1.75</td>
<td>6.80</td>
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<td>0.160</td>
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<td>0.293</td>
<td>6.10</td>
<td>-4.90</td>
<td>-0.787</td>
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</tr>
</tbody>
</table>

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**Notes:**
- Attila Pásztor (Wuppertal)
- IQCD up to the perturbative regime
Lattice vs HTL


Both perturbative treatments assume massless quarks, but that is clearly not a good approximation for the charm in this regime.
EoS: Heavy quark thresholds

- In pQCD with massless quarks, a good approximation is:
  \[ \frac{p^{(4)}(T)}{p^{(3)}(T)} \approx \frac{SB(4)}{SB(3)} \]

- Tree level correction for the charm is:
  \[ \frac{p^{(3+1)}(T)}{p^{(4)}(T)} = \frac{SB(3) + F_Q(m_c, T)}{SB(4)} \]

  where the dimensionless free energy density of a free massive quark is:

  \[ F_Q = \frac{6}{\pi^2} \left( \frac{m}{T^2} \right)^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} K_2 \left( \frac{m_i k}{T} \right) \]

- There is also a \( O(g^2) \) result by Laine and Schroder: hep-ph/0603048
  The pressure itself does not match our lattice results but the ratio of \( 2 + 1 \)
  and \( 2 + 1 + 1 \) flavour pressures does.

- We now also have lattice simulation with dynamical charm, and as we will
  see, already the tree level estimate of the ratio is quite reasonable and agrees
  with the lattice
EoS: Charm quark threshold

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\[ p = \# + \# g^2 + \# g^3 + \# g^4 + \# g^4 \log(g) + \# g^5 + \# g^6 \log(g) + \# g^6 \]

The \( g^6 \) term has a non-perturbative coefficient. After we introduce the tree-level correction coming from the charm quark mass, we fit it to the lattice results, and obtain \(-3200 < q_c < -2700\).

Next, keeping \( q_c \) fixed, we can introduce the bottom quark threshold at tree level to obtain:

\[ p^{(2+1+1+1)}(T) = p^{(2+1+1)}(T) \frac{SB(4) + F_Q(m_b/T)}{SB(4)} \]
Extrapolating the EoS to high temperature

Effect of the bottom threshold

EoS estimate for $N_f = 2 + 1 + 1 + 1$
The full cosmological equation of state


Energy dens. \( \rho = g_\rho \frac{\pi^2}{30} T^4 \) entropy dens. \( s = g_s \frac{2\pi^2}{45} T^3 \) heat cap. \( c = g_c \frac{2\pi^2}{15} T^3 \)

Cooling rate in the early universe:

\[
\frac{dT}{dt} = - \frac{T^3}{M_{Pl}} \frac{2\pi^{3/2}}{3\sqrt{5}} \frac{\sqrt{g_\rho g_s}}{g_c}
\]
2. Fluctuations of conserved charges at high temperature for $N_f = 2 + 1$

- Various uses in heavy ion phenomenology: freezeout, chemical composition, etc.
- Based on hep-lat/1507.04627, PRD92 (2015) no.11, 114505
- See also Ding et al, hep-lat/1507.06637, PRD 92 (2015) no.7, 074043
- Lots of activity on these at this conference: Jana Günther Tue 14:40, Sayantan Sharma Tue 15:20, Frithjof Karsch We 14:40, Szabolcs Borsanyi We 16:30
Higher order susceptibilities - definition

The expectation value of a conserved charge:

$$\langle N_q \rangle = T \frac{\partial \log Z}{\partial \mu_q}$$

The response to $\mu_q$ is given by the fluctuations of the conserved charge:

$$\frac{\partial \langle N_i \rangle}{\partial \mu_j} = T \frac{\partial^2 \log Z}{\partial \mu_i \partial \mu_j} = \frac{1}{T} (\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle)$$

The higher order susceptibilities:

$$\chi_{u,d,s,c}^{i,j,k,l} = \frac{\partial^{i+j+k+l} \left( p/T^4 \right)}{(\partial \hat{\mu}_u)^i (\partial \hat{\mu}_d)^j (\partial \hat{\mu}_s)^k (\partial \hat{\mu}_c)^l}$$

$$\chi_{B,S,Q}^{i,j,k} = \frac{\partial^{i+j+k} \left( p/T^4 \right)}{(\partial \hat{\mu}_B)^i (\partial \hat{\mu}_S)^j (\partial \hat{\mu}_Q)^k}$$

where $\hat{\mu} = \mu/T$. The relationship between the chemical potentials:

$$\mu_u = \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q \quad \mu_d = \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q \quad \mu_s = \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_S$$
Testing perturbation theory

Simulations at high temperatures

For temperatures \( T \geq 300 \text{MeV} \) we do not keep the lattice geometry constant in our temperature scan, but keep the physical volume constant with \( LT_c > 2 \). E.g. for \( N_t = 16 \) we have the lattices:

<table>
<thead>
<tr>
<th>Volume</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 64^3 \times 16 )</td>
<td>300MeV</td>
</tr>
<tr>
<td>( 80^3 \times 16 )</td>
<td>360MeV</td>
</tr>
<tr>
<td>( 96^3 \times 16 )</td>
<td>440MeV</td>
</tr>
<tr>
<td>( 112^3 \times 16 )</td>
<td>520MeV</td>
</tr>
<tr>
<td>( 128^3 \times 16 )</td>
<td>600MeV</td>
</tr>
</tbody>
</table>

Keeping the volume large is important for testing perturbation theory. These volumes are the cause of the error getting bigger at higher temperatures on some of our plots. It is more customary to keep the aspect ratio \( LT \) fixed. That approach was used in Ding et al, Phys.Rev. D92 7 074043, hep-lat/1507.06637.

Perturbative results from

Fluctuations at high temperature: HTL vs lattice
3. Topological Susceptibility of QCD

- Based on hep-lat/1606.07494, Nature 539 (2016) no.7627
- See also: M. Lombardo’s Talk, We 12:00
Topological Susceptibility

Definition

\[ \chi_t = \frac{\langle Q^2 \rangle}{V_4}, \quad \text{with} \quad Q = \frac{1}{32\pi^2} \int d^4x F_{\mu\nu} \tilde{F}^{\mu\nu} \]

Axion cosmology

Temperature dependent axion mass is given by: \( f_A^2 m_A^2 = \chi_t \)

Instanton density

Probability for \( Q=0 \rightarrow p_0 \) Probability for \( Q=1,-1 \rightarrow p_1 \) At fixed volume, and high enough temperature \( 1 \approx p_0 + 2p_1 \)

\[ (V/T)\chi_t = \langle Q^2 \rangle \approx 0^2 p_0 + 1^2 p_1 + (-1)^2 p_1 = 2p_1 \]

So the topological susceptibility basically gives the instanton density. This suggests that a necessary condition for a perturbative treatment to be feasible is a small topological susceptibility.
Results for the topological susceptibility

\[ \chi \text{[fm}^{-4}] \]

\[ T \text{[MeV]} \]

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IQCD up to the perturbative regime
At finite $T$ there are finite action configurations with non-integer $Q$. These configurations have, $\Omega \neq 1$ as $|\vec{x}| \to \infty$, where $\Omega(\vec{x}) = \mathcal{P} \exp(\int_0^{1/T} A_0 d\tau)$ is the Polyakov loop.

But at $T > T_c$ the medium is deconfined, and there will be a screening mass. This induces a term in the high temperature effective action $\int d^3x \frac{1}{2} m_{el}^2 \text{Tr} \left( (\Omega - 1)^2 \right)$. As the volume increases, this term suppresses configurations with non-integer $Q$.

Large instantons are also suppressed by the screening, therefore the semiclassical approximation becomes more reliable.

The high temperature behavior will be dominated by the $|Q| = 0, 1$ sectors.

The action for a single instanton is $2\pi/\alpha$ this implies $\chi_t/T^4 \sim e^{-2\pi/\alpha}$

The running coupling is given by the $\beta$ function: $\alpha \sim \frac{2\pi}{(11-2N_f/3) \log(\mu/\Lambda)}$

With fermions: extra factors of $m_f/T$ from the fermion det.

Putting all of this together we get: $\chi_t \sim T^{4-11+2N_f/3-N_f} \sim T^{-7-N_f/3}$
Results for the exponent

The exponent is in approximate agreement with DIGA for $T \sim 1\text{GeV}$, but the prefactor is off by an order of magnitude.
Summary

- Continuum extrapolated lattice simulations at the physical point in the high temperatures well in the plasma phase are now available.
- We gave a perturbative parametrization that matched both the pressure and the trace anomaly in the range $T = 0.5\text{GeV} \ldots 1.0\text{GeV}$ and included the charm threshold effects.
- We then gave a parametrization that also included the bottom in a similar manner, giving an estimate of the EoS up to cosmologically relevant scales.
- For second and forth order susceptibilities in general HTL seems to work from say $T \sim 300\text{MeV}$ and up.
- The topological susceptibility at $T \sim \text{GeV}$ is overestimated by DIGA by an order of magnitude, but still, it is small enough that instantons are quite rare.
- The decay rate of the susceptibility is approximately captured by DIGA.
Backup Slides
The 4stout staggered action

Action

- 2+1+1 dynamical flavors
- 4 levels of stout smearing in the fermion action, with the smearing parameter $\rho = 0.125$
- The masses are set by bracketing both the pion and the kaon masses within a few percent, keeping $m_c/m_s = 11.85$
- The scale is set 2 ways: $f_\pi$ and $w_0$ (with Wilson flow). The scale setting procedure is one of the sources of the systematic error in all of the plots.

Ensembles

- For the equation of state, continuum limit from $N_t = 6, 8, 10, 12$ lattices
- For the generalized susceptibilities, continuum limit from $N_t = 10, 12, 16, 20, 24$ lattices
EoS - renormalization

- Needs the gauge action, the chiral condensate, running couplings and masses

\[
\frac{I(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = N_t^4 \left( - \frac{d\beta}{d\log a} \right) \left[ \frac{\partial}{\partial \beta} + \sum_f \left( \frac{d \log m_f}{d\beta} \right) m_f \frac{\partial}{\partial m_f} \right] \frac{\log Z}{N_t N_s^3}
\]

Note: The systematics of the LCP choice are particularly important here.

- There is an additional additive UV divergence, that is commonly renormalized by subtracting the zero temperature result. In this scheme, \(p(T = 0) = 0\).

Technical problem: renormalization with \(T = 0\) lattices becomes prohibitively expensive

- So we calculate the continuum limit of \([I(T) - I(T/2)]/T^4\) instead. We can do this since the UV divergence is temperature independent.

- The trace anomaly is given by:

\[
\frac{I(T)}{T^4} = \sum_{k=0}^{n-1} \frac{I(T/2^k) - I(T/2^{k+1})}{(T/2^k)^4} + 2^{-4n} \frac{I(T/2^n)}{(T/2^n)^4}
\]

- The pressure is \(p(T_1)/T_1^4 - p(T_2)/T_2^4 = \int_{T_2}^{T_1} \frac{\epsilon - 3p}{T^5}\)
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\]

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Continuum limit for EoS

\[ \frac{I(T)}{T^4} \]

\[ T \ [\text{MeV}] \]

WB 2013, 2+1f

4stout 2+1+1f, continuum

4stout (tree-level imp.) \( N_t=6 \)

\( N_t=8 \)

\( N_t=10 \)

\( N_t=12 \)

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IQCD up to the perturbative regime
Continuum limit for EoS

\[ \frac{I(T) - I(T/2)}{T^4} \]

- WB 2013, 2+1f
- 4stout 2+1+1f, continuum
- 4stout (tree-level imp.) \( N_t = 6 \)
- \( N_t = 8 \)
- \( N_t = 10 \)
- \( N_t = 12 \)

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Comparisons with $N_f = 4$ perturbation theory

2+1+1 flavor EoS from lattice

2+1+1 flavor EoS from lattice
Parametrization for the speed of sound: $2+1$ vs $2+1+1$ flavor

![Graph showing the parametrization for the speed of sound: $2+1$ vs $2+1+1$ flavor. The graph compares the speed of sound $v_s$ at different temperatures $T$ in units of MeV. The green and blue lines represent the parametrizations for $N_f=2+1$ and $N_f=2+1+1$ flavors, respectively. The graph indicates a clear distinction between the two parametrizations, with the $N_f=2+1+1$ parametrization generally showing a higher speed of sound at higher temperatures.](image-url)
Model estimates at low and high temperatures

Low temperatures: Hadron Resonance Gas

The interaction of the hadrons are introduced by adding all their resonances to the heat bath, as free particles.

\[
\frac{p_{\text{HRG}}}{T^4} = \frac{1}{VT^3} \left( \sum_{i \in \text{meson}} \log Z^M (T, V, m_i, \{\mu\}) + \sum_{i \in \text{baryon}} \log Z^B (T, V, m_i, \{\mu\}) \right)
\]

High temperature: weakly interacting quarks

For an ideal gas we have:

\[
\frac{p}{T^4} = \frac{8\pi^2}{45} + \frac{7\pi^2}{60} N_f + \frac{1}{2} \sum_f \left( \frac{\mu_f^2}{T^2} + \frac{\mu_f^4}{2\pi^2 T^4} \right)
\]

This means e.g. that \( \chi_4^u = 0.608 \) or \( \chi_{11}^{ud} = 0 \) etc. This estimate can be improved with resummed PT: Hard Thermal Loop, Dimensional Reduction
Continuum extrapolation of the fluctuations

The coarsest $N_t = 8$ results are usually included only in non-linear continuum extrapolations, like $A + B/N_t^2 + C/N_t^4$. For some observables, like $\chi_2^B$, there is a long range of safe linear extrapolation, but observables that are related to pion physics, like $\chi_2^Q$, show a strong, non-linear $1/N_t^2$ dependence, and only for very fine lattices, $N_t \geq 16$ we see a linear regime. Calculating these quantities is one of the achievements of our paper.
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Fluctuations: Hierarchy

Perturbatively:

\[ \chi_4 - \frac{6}{\pi^2} \sim \alpha \]

\[ \chi_{22} \sim \alpha^{3/2} \]

\[ \chi_{11} \sim \alpha^3 \log \alpha \]

\[ \chi_{31} \sim \alpha^5 \log \alpha \]

Vuorinen hep-ph/0212283
The Strong CP problem

In general, gauge symmetry allows for two CP breaking terms in the QCD Lagrangian:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{a\mu\nu} + i \bar{\psi} (i \gamma^\mu D_\mu - M e^{i\theta'} \gamma^5) \psi - \frac{n_f g^2 \theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}_{a\mu\nu} \]

The phase of the quark mass matrix and the \( \theta \) parameter are not really independent, in the sense that physics can only depend on the combination:

\[ e^{-2i\theta} \prod_f m_f, \]

since this is the combination left invariant under the change of variables \( \Psi_f \rightarrow e^{i\alpha_f \gamma^5} \Psi_f \) in the path integral, therefore we can work with only one total effective angle we can simply call \( \theta \).

- The CP violating term would induce an electric dipole moment of the neutron, and gives a strong constraint \( |\theta| < 10^{-9} \).
- If any of the quark masses were zero, that would imply CP. But this is probably not the case, from:
  - Lattice
  - \( \chi PT \) + experimental data
- Explaining the smallness of the CP violating term is the Strong CP problem.
The Peccei-Quinn mechanism

Assume a global $U(1)_{PQ}$ symmetry that breaks spontaneously at $T f_A$. The emerging would be Goldstone boson is the axion($\phi$). There are various variants, but the generic feature is that the only non-derivative coupling is to the topological charge density:

$$
\mathcal{L}_{\text{non-der.}} = i \frac{\phi}{f_A} \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}
$$

The PQ mechanism works because the effective potential for the (homogeneous) axion field has a CP conserving minimum:

$$
e^{-V_{eff}(\phi)} = \left| \int DA \det(\gamma_\mu D_\mu + M) e^{-S+i(\theta+\phi/f_A)Q} \right|
\geq \int DA \det(\gamma_\mu D_\mu + M) e^{-S} \left| e^{i(\bar{\theta} + \phi/f_A)Q} \right| = e^{-V_{eff}(\phi=-f_A\theta)}
$$

So $\langle \theta + \phi/f_A \rangle = 0$ as we wanted. Shifting the axion field, we define $\theta + \phi/f_A \rightarrow \theta_a$. Thus, the PQ mechanism effectively trades $\theta$, a free parameter, with a dynamical field evolving around a CP conderving minimum.
The axion potential

The effective Lagrangian:

\[ \mathcal{L} = (\partial \phi)^2 + i \frac{\phi}{f_A} q + \text{derivative terms} + \mathcal{L}_{\text{QCD}} \]

\[ \frac{\partial V_{\text{eff}}}{\partial \phi} = - \frac{1}{f_A} \frac{g^2}{64\pi^2} \left\langle F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \right\rangle m^2_{\phi} = \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} = \chi_t / f_A^2 \]

So \( f_A^2 m_A^2 = \chi_t \) is a purely QCD quantity, that can be calculated on the lattice, or estimated from semiclassical approximations.

On the other hand, if only \( Q = 0, -1, 1 \) contribute:

\[ \exp(-V_0 V_{\text{eff}}) = \left\langle e^{iQ\theta} \right\rangle = p_0 + p_1 e^{i\theta} + p_1 e^{-i\theta} = 1 - 2p_1 (1 - \cos \theta) \]

The topological susceptibility at a fixed volume \( V_0 \) is

\[ (V_0/T) \chi_t = \left\langle Q^2 \right\rangle \approx 0^2 p_0 + 1^2 p_1 + (-1)^2 p_1 = 2p_1 \]

In the thermodynamic limit \( V = V_0 N \) with \( N \to \infty \). We get:

\[ -\frac{V}{T} V_{\text{eff}} = N \log \left( 1. \frac{\chi_t V}{NT} (1 - \cos \theta) \right) = -\frac{\chi_t V}{T} (1 - \cos \theta) + O \left( \frac{1}{N} \right) \]
Axion production mechanisms

Three production mechanisms known:
- **Thermal**: negligible
- **Misalignment**: As the temperature drops, the topological susceptibility is increased, the axion develops a non-negligible mass, and relaxes to its minimum (oscillator equation damped by Hubble expansion).
- **Strings**: Such a broken symmetry supports the formation of topological strings, whose radiation produces axions.

When can the PQ symmetry breaking occur?
- **Pre-inflation**: Only misalignment plays a rule, the strings get inflated away
- **Post-inflation**: Both non-thermal mechanisms play a role
Axion production from misalignment

See book by Kolb & Turner, see also Wantz, Shellard, astro-ph.CO/0910.1066
Assume a homogeneous field.

\[ \mathcal{L} = \frac{1}{2} \dot{\theta}_a^2 - \frac{1}{2} m_A^2 (1 - \cos \theta_a) = \frac{1}{2} \dot{\theta}_a^2 - \frac{1}{2} m_A^2 \theta_a^2 + \ldots \]

Equation of motion can be obtained by varying \( S = \int d^4 x R^3 \mathcal{L} \), to get:

\[ \frac{d^2 \theta_a}{dt^2} + 3H(T) \frac{d\theta_a}{dt} + m_A(T)^2 \sin \theta_a 
= \frac{d^2 \theta_a}{dt^2} + 3H(T) \frac{d\theta_a}{dt} + m_A(T)^2 \theta_a + \mathcal{O} \left( \theta_a^2 \right) = 0 \]

The \( \theta_a \) field cannot roll into the potential minimum until \( 3H > m_a \), since the oscillator equation is overdamped.
Remarking: There is additional damping due to sphalerons, but it appears to be negligible.
Axion mass predictions

Remember $f_A^2 m_A^2 = \chi_t$ is what we can calculate. The axion scale $f_A$ can be estimated from a guess of how much of the DM comes from misalignment.

- All DM from misalignment: $m_A = 28(2)\mu eV$
- 50% from misalignment: $m_A = 40(4)\mu eV$
- 1% from misalignment: $m_A = 1500\mu eV$
Integral method for the topological susceptibility

\[ Z = \int \mathcal{D}U e^{-S} = \sum_Q \int \mathcal{D}U |_Q e^{-S} = \sum_Q Z_Q \]

At high temperature: \( Z_0 \gg Z_1 \gg Z_2 \gg \ldots \) By \( \mathcal{P} \) symmetry: \( Z_Q = Z_{-Q} \)

Keeping \( R = LT \) const.

\[ \frac{V}{T} \chi_t = R^3 \frac{\chi_t}{T^4} = \langle Q^2 \rangle = \frac{0 \cdot Z_0 + 1 \cdot Z_1 + 1 \cdot Z_{-1} + \ldots}{Z_0 + \ldots} \approx \frac{2Z_1}{Z_0} \]

Finally:

\[ \chi_t \sim T^{-b}, \text{ with } b = -4 - \frac{d\beta}{dT} \langle S_g \rangle_{1-0} - \sum_f \frac{dm_f}{dT} m_f \langle \bar{\phi} \psi_f \rangle_{1-0} \]

If we know \( T \) at some temperature, we can calculate it at higher temperatures.

An important point here is that:

- Very large cutoff effects on the fermionic part (in particular, any non-chiral action fails miserably)
- Gauge action part is much noiser, than the fermion part
Evaluate susceptibility and decay exponent at a quark mass, where the simulation is less expensive than at phys. pt.: Three flavor symmetric point. Carry out an integration in light quark mass from $m_s$ down to $m_{ud}$. 

$n_f=3+1$

$n_f=2+1+1\;\text{phys}$

Staggered
Staggered fix Q
Overlap fix Q
Integral method in the mass direction

With chiral fermions Q never changes in the Monte Carlo history,

\[ \frac{\partial \log \chi_t}{\partial \log m} = \frac{\partial \log Z_1}{\partial \log m} - \frac{\partial \log Z_0}{\partial \log m} \]

\[ (\frac{\chi_t}{T^4})_{m_A} = \int_{m_A}^{m_B} d(\log m) m \langle \bar{\Psi} \Psi \rangle_{1-0} (\frac{\chi_t}{T^4})_{m_B} \]

Thus, to integrate down in the mass direction:

- Calculate \( \chi_t \) at the flavour physical point with high statistics.
- Calculate \( \langle \bar{\Psi} \Psi \rangle \) at different quark masses using chiral fermions.
On cut-off effects

The strong cut-off effects are related to the lack of exact zero modes. (In the continuum, these zero-modes are a consequence of the Atiyah-Singer index theorem.)

Possible solutions:
1) Eigenvalue reweighting
2) Using chiral fermions

We use 1) for the 3 flavor theory and 2) for going down to the physical point.
Reweighting method

Strong cut-off effects are related to the lack of exact zero modes of the Dirac operator.

- In the continuum non-trivial sectors are suppressed by the contribution of zero-modes to the fermion determinant.
- On the lattice the suppression is altered: $m \to m + i\lambda_0$, where $\lambda_0$ is a would-be zero-mode. Weaker suppression $\to \chi_t$ overestimated.
- To improve:
  1. identify would-be zero-modes
  2. restore the continuum weight $\to$ reweight

$$w[U] \sim m/(m + i\lambda_0)$$
Reweighting works

For large $T$ goes to 0 instead of approaching the SB limit, this is a lattice artefact. Non-chiral fermions (without reweighting) fail spectacularly for large temperatures!
As we approach the continuum limit, the would be zero modes become, smaller, so we expect $W_Q \to 1$ as $a \to 0$. 
\[ \chi_t = 2 \int_0^\infty d\rho D(\rho) G(\rho \pi T) \]

\( D(\rho) \) Density of instantons of size \( \rho \)

\( G(x) \) is the cut-off function that includes the effects of Debye screening in the medium.

\[ D(\rho) = \frac{d_{MS}}{\rho^5} \left( \frac{2\pi}{\alpha_s(\mu)} \right)^6 e^{-\frac{2\pi}{\alpha_s(\mu)} (\rho \mu) (\beta_0 + (\beta_1 - 12\beta_0 + 8N_f) \frac{\alpha_s}{4\pi})} \prod_{i=1}^{N_f} (\rho m_i) \]

\[ G(x) = \exp \left( -\frac{1}{3} (6 + N_f)x^2 - 12A(x)(1 + \frac{1}{6}(3 - N_f)) \right) \]

\[ A(x) = -\frac{1}{12} \log (1 + x^2/3) + c_1 \frac{1}{(1 + c_2 x^{-3/2})^8} \]

\[ c_1 = 0.01289764 \quad c_2 = 0.15858 \]
Quenched case

\[ \chi / Tc \]

\[ T / Tc \]

[1508.06917]

DIGA

integral 8x32

b(T) 8x32

8x64

10^{-8}

10^{-6}

10^{-4}

10^{-2}

10^{-1}

10^{0}

10^{1}

10^{2}

10^{3}

10^{4}

10^{5}

10^{6}

10^{7}

10^{8}

10^{9}

10^{10}

T / Tc

1  2  3  4  5  6  7  8  9  10

Attila Pásztor (Wuppertal)
n$_f$=3+1 flavor ("three flavor symmetric point")