Lattice QCD up to the perturbative regime

Attila Pásztor [1]
apasztor@bodri.elte.hu

for the Wuppertal-Budapest collaboration.

[1] University of Wuppertal

Quark Matter 2017, Chicago

Introduction

When is a weak coupling description of the QGP accurate?

Use recent high T lattice results to check. In particular:

1. The equation of state
2. Fluctuations of conserved charges
3. Topological Susceptibility

Why these observables?

- All of these are important for phenomenology (heavy ion/cosmology/both)
- Can be calculated both on the lattice and with weak coupling techniques

Continuum extrapolated lattice results will be compared with:

- Resummed perturbation theory (HTL and DR) for 1-2
- Dilute instanton gas (DIGA) for 3

1. Equation of state with dynamical charm

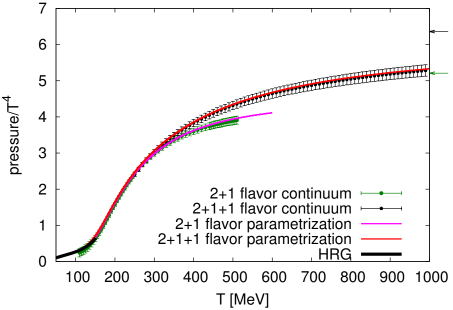
- $N_f = 2 + 1 + 1$ with physical quark masses
- Lattice data up to $T = 1$GeV
- Dynamical charm is included since it is needed for cosmology
- Based on hep-lat/1606.07494, Nature 539 (2016) no.7627
- In collaboration with: Sz. Borsanyi, Z. Fodor, J. Gunther, K.-H. Kampert, S. D. Katz, T. Kawanai, T. G. Kovacs, S. W. Mages, F. Pittler, J. Redondo, A. Ringwald, K. K. Szabo
- For the equation of state of the pure Yang-Mills theory up to high temperatures, see e.g.: WB, hep-lat/1204.6184, JHEP 1208 (2012) 0 53 and Giusti, Pepe, hep-lat/1612.00265

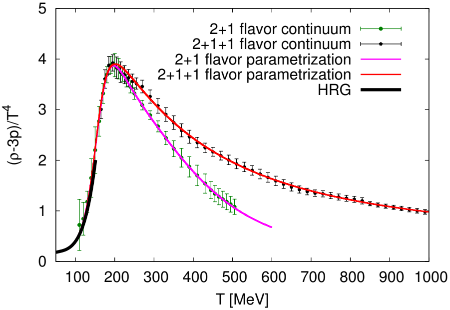
EoS: 2+1 vs 2+1+1 flavour

A good parametrization of the EoS is:

$$\frac{I(T)}{T^4} = \exp(-h_1/t - h_2/t^2) \cdot \left(h_0 + f_0 \frac{\tanh(f_1 \cdot t + f_2) + 1}{1 + g_1 \cdot t + g_2 \cdot t^2} \right)$$

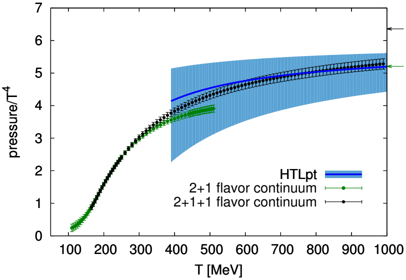
	h_0	h_1	h_2	f_0	f_1	f_2	g_1	g_2
2+1+1	0.353	-1.04	0.534	1.75	6.80	-5.18	0.525	0.160
2+1	-0.00433	-1.00	-0.288	0.293	6.10	-4.90	-0.787	0.289

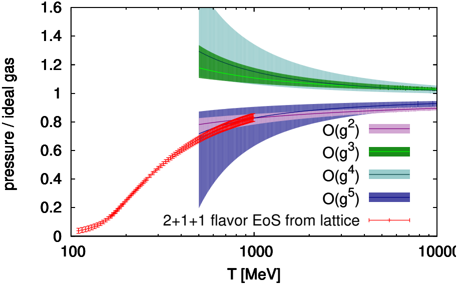




Lattice vs HTL

Left: comparison with 4 flavour NNLO HTL results from Andersen et al, Phys.Lett. B696 (2011) 468-472, hep-ph/1009.4644
Right: comparison with strict expansion in g, Arnold, Zhai: hep-ph/9408276v1, Zhai, Kastening: hep-ph/9507380, Kajantie et al: hep-ph/0211321





Both perturbative treatements assume massless quarks, but that is clearly not a good approximation for the charm in this regime.

EoS: Heavy quark thresholds

- In pQCD with massless quarks, a good approximation is: $\frac{p^{(4)}(T)}{p^{(3)}(T)} \approx \frac{SB(4)}{SB(3)}$
- Tree level correction for the charm is:

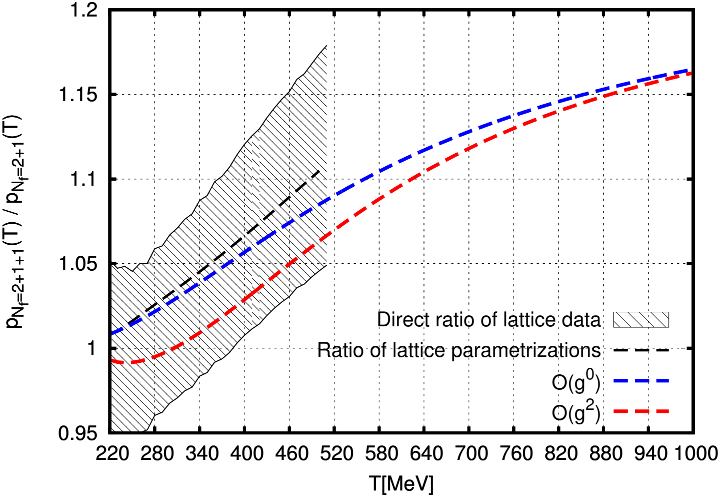
$$\frac{p^{(3+1)}(T)}{p^{(4)}(T)} = \frac{SB(3) + F_Q(m_c, T)}{SB(4)}$$

  where the dimensionless free energy density of a free massive quark is:

$$F_Q = \frac{6}{\pi^2} \left(\frac{m}{T^2}\right)^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} K_2 \left(\frac{m_i k}{T}\right)$$

- There is also a $\mathcal{O}\left(g^2\right)$ result by Laine and Schroder: hep-ph/0603048
  The pressure itself does not match our lattice results but the ratio of $2+1$ and $2+1+1$ flavour pressures does.
- We now also have lattice simulation with dynamical charm, and as we will see, already the tree level estimate of the ratio is quite reasonable and agrees with the lattice

EoS: Charm quark threshold



EoS: Perturbative parametrization at high T

- Kajantie et al: hep-ph/0211321, PRD67 (2003) 105008 gives the result for massless quarks:

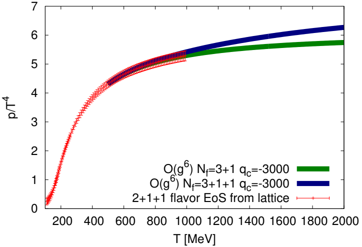
$$p = \# + \#g^2 + \#g^3 + \#g^4 + \#g^4 \log(g) + \#g^5 + \#g^6 \log(g) + \#g^6$$

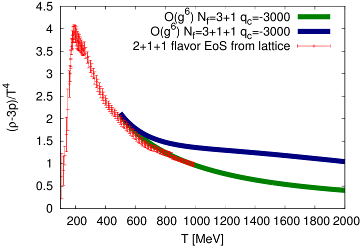
- The g^6 term has a non-perturbative coefficient. After we introduce the tree-level correction coming from the charm quark mass, we fit it to the lattice results, and obtain $-3200 < q_c < -2700$.
- Next, keeping q_c fixed, we can introduce the bottom quark treshold at tree level to obtain:

$$p^{(2+1+1+1)(T)} = p^{(2+1+1)}(T)\frac{SB(4) + F_Q(m_b/T)}{SB(4)}$$

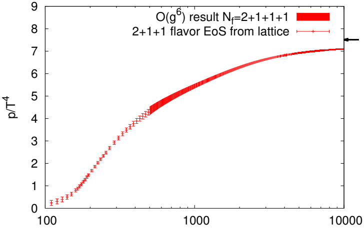
Extrapolating the EoS to high temperature

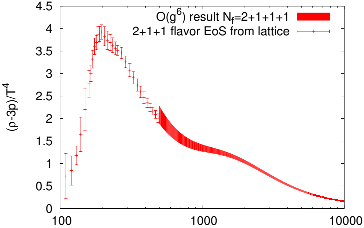
Effect of the bottom threshold



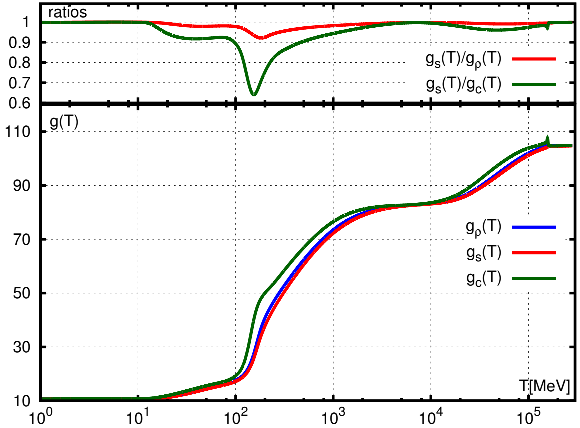


EoS estimate for $N_f = 2+1+1+1$





The full cosmological equation of state



Electroweak contribution: Laine, Meyer 1503.04935, JCAP 1507 (2015)

Energy dens. $\rho = g_\rho \frac{\pi^2}{30} T^4$ entropy dens. $s = g_s \frac{2\pi^2}{45} T^3$ heat cap. $c = g_c \frac{2\pi^2}{15} T^3$

Cooling rate in the early universe: $\frac{dT}{dt} = -\frac{T^3}{M_{\rm Pl}} \frac{2\pi^{3/2}}{3\sqrt{5}} \frac{\sqrt{g_\rho} g_s}{g_c}$

2. Fluctuations of conserved charges at high temperature for $N_f = 2 + 1$

- Various uses in heavy ion phenomenology: freezeout, chemical composition, etc.
- Based on hep-let/1507.04627, PRD92 (2015) no.11, 114505
- In collaboration with: R. Bellwied, Sz. Borsanyi, Z. Fodor, S.D. Katz, C. Ratti, K.K. Szabo
- See also Ding et al, hep-lat/1507.06637, PRD 92 (2015) no.7, 074043
- Lots of activity on these at this conference: Jana Günther Tue 14:40, Sayantan Sharma Tue 15:20, Frithjof Karsch We 14:40, Szabolcs Borsanyi We 16:30

Higher order susceptibilities - definition

The expectation value of a conserved charge:

$$\langle N_q \rangle = T \frac{\partial \log \mathcal{Z}}{\partial \mu_q}$$

The response to μ_q is given by the fluctuations of the conserved charge:

$$\frac{\partial \langle N_i \rangle}{\partial \mu_j} = T \frac{\partial^2 \log \mathcal{Z}}{\partial \mu_i \partial \mu_j} = \frac{1}{T} \left(\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle \right)$$

The higher order susceptibilities:

$$\chi^{u,d,s,c}_{i,j,k,l} = \frac{\partial^{i+j+k+l} \left(p/T^4 \right)}{(\partial \hat{\mu}_u)^i (\partial \hat{\mu}_d)^j (\partial \hat{\mu}_s)^k (\partial \hat{\mu}_c)^l} \quad \chi^{B,S,Q}_{i,j,k} = \frac{\partial^{i+j+k} \left(p/T^4 \right)}{(\partial \hat{\mu}_B)^i (\partial \hat{\mu}_S)^j (\partial \hat{\mu}_Q)^k}$$

where $\hat{\mu} = \mu/T$. The relationship between the chemical potentials:

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \qquad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \qquad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$$

Testing pertubation theory

Simulations at high temperatures

For temperatures $T \geq 300\text{MeV}$ we do not keep the lattice geometry constant in our temperature scan, but keep the physical volume constant with $LT_c > 2$. E.g. for $N_t = 16$ we have the lattices:

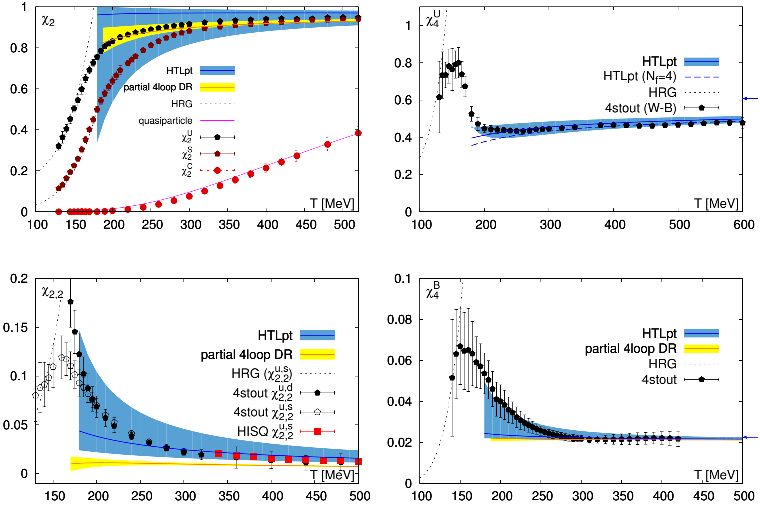
$64^3 \times 16$	$80^3 \times 16$	$96^3 \times 16$	$112^3 \times 16$	$128^3 \times 16$
300MeV	360MeV	440MeV	520MeV	600MeV

Keeping the volume large is important for testing perturation theory. These volumes are the cause of the error getting bigger at higher temperatures on some of our plots. It is more customary to keep the aspect ratio LT fixed. That approach was used in Ding et al, Phys.Rev. D92 7 074043, hep-lat/1507.06637.

Perturbative results from

- N. Haque, A. Bandyopadhyay, J. O. Andersen, M. G. Mustafa, M. Strickland and N. Su, JHEP **1405** (2014) 027, hep-ph/1402.6907
- S. Mogliacci, J. O. Andersen, M. Strickland, N. Su and A. Vuorinen, JHEP **1312** (2013) 055, hep-ph/1307.8098

Fluctuations at high temperature: HTL vs lattice



3. Topological Susceptibility of QCD

- Based on hep-lat/1606.07494, Nature 539 (2016) no.7627
- In collaboration with: Sz. Borsanyi, Z. Fodor, J. Gunther, K.-H. Kampert, S. D. Katz, T. Kawanai, T. G. Kovacs, S. W. Mages, F. Pittler, J. Redondo, A. Ringwald, K. K. Szabo
- See also: Petreczky, Schadler, Sharma Phys.Lett. B762 (2016) 498-505, hep-lat/1606.03145
- See also: M. Lombardo's Talk, We 12:00

Topological Susceptibility

Definition

$$\chi_t = \frac{\langle Q^2 \rangle}{V_4}, \text{ with } \mathrm{Q} = \frac{1}{32\pi^2} \int \mathrm{d}^4 \mathrm{x} \mathrm{F}_{\mu\nu} \tilde{\mathrm{F}}^{\mu\nu}$$

Axion cosmology

Temperature dependent axion mass is given by: $f_A^2 m_A^2 = \chi_t$

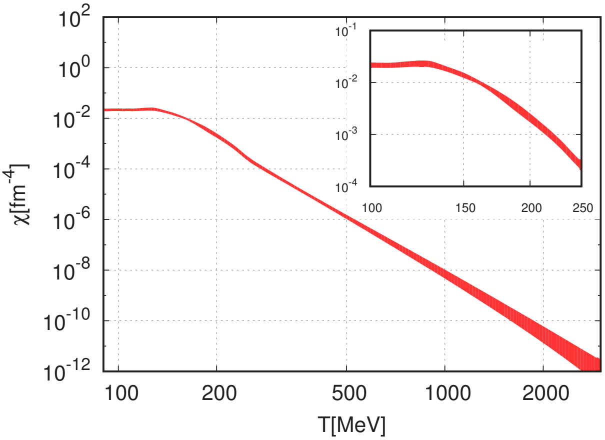
Instanton density

Probability for Q=0 $\rightarrow\ p_0$ Probability for Q=1,-1 $\rightarrow\ p_1$ At fixed volume, and high enough temperature $1 \approx p_0 + 2p_1$

$$(V/T)\chi_t = \langle Q^2 \rangle \approx 0^2 p_0 + 1^2 p_1 + (-1)^2 p_1 = 2p_1$$

So the topological susceptibility basically gives the instanton density. This suggests that a necessary condition for a perturbative treatement to be feasible is a small topological susceptibility.

Results for the topological susceptibility

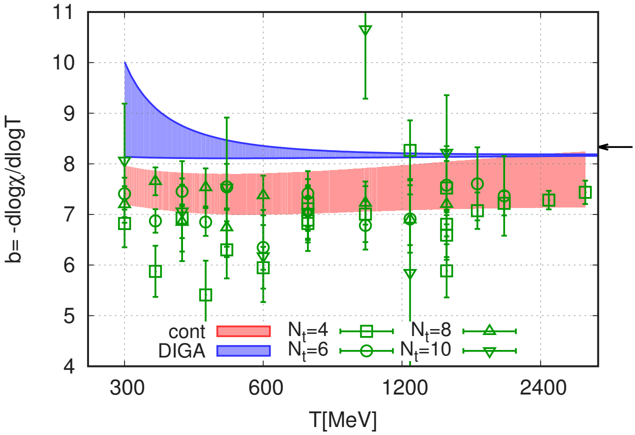


The dilute instanton gas approximation (DIGA)

Gross, Pisarski Rev.Mod.Phys. 53 (1981) 43
Ringwald, Schrempp (DESY), Phys.Lett. B459 (1999) 249-258, hep-lat/9903039

- At finite T there are finite action configurations with non-integer Q. These configurations have, $\Omega \neq \mathbf{1}$ as $|\vec{x}| \to \infty$, where $\Omega(\vec{x}) = \mathcal{P}\exp(\int_0^{1/T} A_0 d\tau)$ is the Polyakov loop.
- But at $T > T_c$ the medium is deconfined, and there will be a screening mass. This induces a term in the high temperature effective action $\int d^3x \frac{1}{2} m_{el}^2 \,\mathrm{Tr}\left((\Omega - 1)^2\right)$. As the volume increseas, this term suppresses configurations with non-integer Q.
- Large instantons are also suppressed by the screening, therefore the semiclassical approximation becomes more reliable.
- The high temperature behavior will be dominated by the $|Q| = 0, 1$ sectors.
- The action for a single instanton is $2\pi/\alpha$ this implies $\chi_t/T^4 \sim e^{-2\pi/\alpha}$
- The running coupling is given by the β function: $\alpha \sim \frac{2\pi}{(11-2N_f/3)\log(\mu/\Lambda)}$
- With fermions: extra factors of m_f/T from the fermion det.
- Putting all of this together we get: $\chi_t \sim T^{+4-11+2N_f/3-N_f} \sim T^{-7-N_f/3}$

Results for the exponent



The exponent is in approximate agreement with DIGA for $T \sim 1$GeV, but the prefactor is off by an order of magnitude.

Summary

- Continuum extrapolated lattice simulations at the physical point in the high temperatures well in the plasma phase are now available
- We gave a perturbative parametrization that matched both the pressure and the trace anomaly in the range $T = 0.5\mathrm{GeV} \ldots 1.0\mathrm{GeV}$ and included the charm threshold effects
- We then gave a parametrization that also included the bottom in a similar manner, giving an estimate of the EoS up to cosmologically relevant scales
- For second and forth order susceptibilities in general HTL seems to work from say $T \sim 300$MeV and up.
- The topological susceptibility at $T \sim$GeV is overestimated by DIGA by an order of magnitude, but still, it is small enough that instantons are quite rare.
- The decay rate of the susceptibility is approximately captured by DIGA.

Backup Slides

The 4stout staggered action

Action

- 2+1+1 dynamical flavors
- 4 levels of stout smearing in the fermion action, with the smearing parameter $\rho = 0.125$
- The masses are set by bracketing both the pion and the kaon masses within a few percent, keeping $m_c/m_s = 11.85$
- The scale is set 2 ways: f_π and w_0 (with Wilson flow). The scale setting procedure is one of the sources of the systematic error in all of the plots.

Ensembles

- For the equation of state, continuum limit from $N_t = 6, 8, 10, 12$ lattices
- For the generalized susceptibilities, continuum limit from $N_t = 10, 12, 16, 20, 24$ lattices

EoS - renormalization

- Needs the gauge action, the chiral condensate, running couplings and masses

$$\frac{I(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = N_t^4 \left(-\frac{d\beta}{d\log a}\right) \left[\frac{\partial}{\partial\beta} + \sum_f \left(\frac{d\log m_f}{d\beta}\right) m_f \frac{\partial}{\partial m_f}\right] \frac{\log Z}{N_t N_s^3}$$

  Note: The systematics of the LCP choice are particularly important here.
- There is an additional additive UV divergence, that is commonly renormalized by subtracting the zero temperature result. In this scheme, $p(T=0)=0$. Technical problem: renormalization with $T=0$ lattices becomes prohibitively expensive
- So we calculate the continuum limit of $\left[I\left(T\right) - I\left(T/2\right)\right]/T^4$ instead. We can do this since the UV divergence is temperature independent.
- The trace anomaly is given by:

$$\frac{I(T)}{T^4} = \sum_{k=0}^{n-1} \frac{I(T/2^k) - I(T/2^{k+1})}{(T/2^k)^4} + 2^{-4n} \frac{I(T/2^n)}{(T/2^n)^4}$$

- The pressure is $p(T_1)/T_1^4 - p(T_2)/T_2^4 = \int_{T_2}^{T_1} \frac{\epsilon - 3p}{T^5}$

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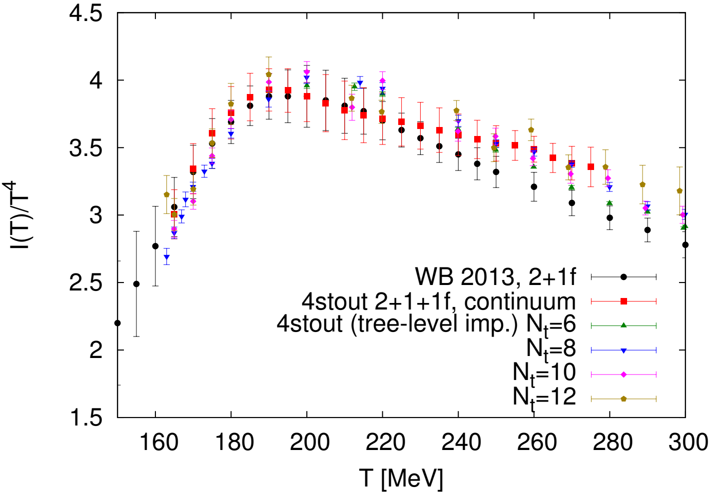
Continuum limit for EoS



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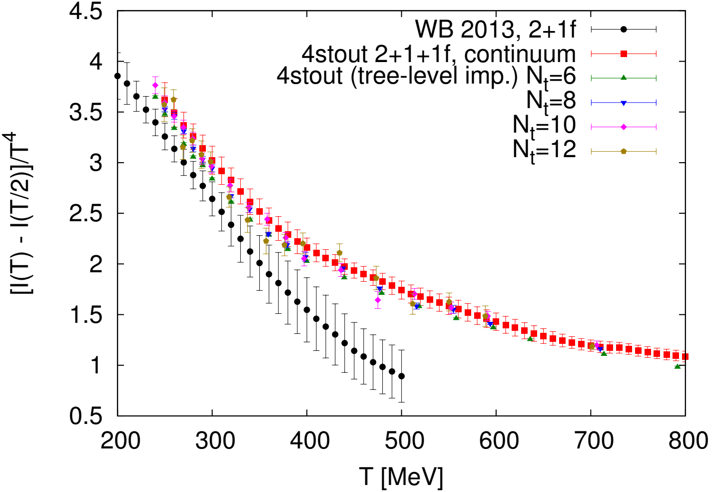
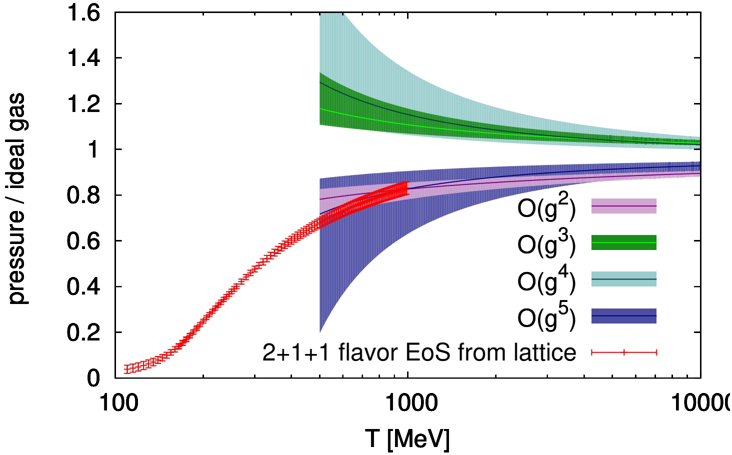
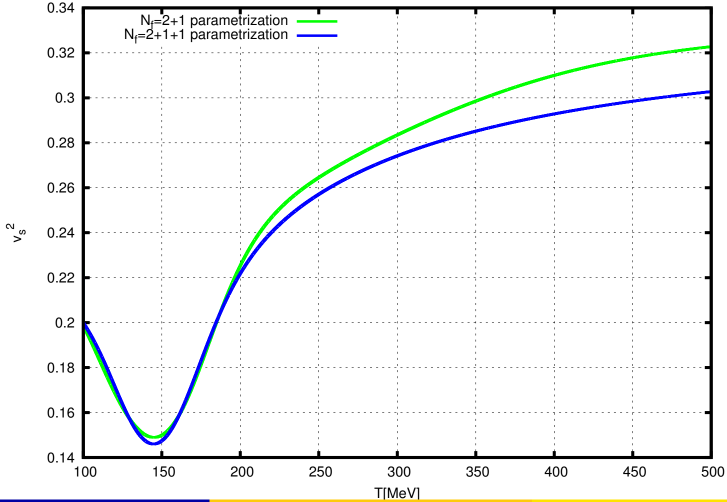


Comparisons with $N_f = 4$ perturbation theory



Parametrization for the speed of sound: 2+1 vs 2+1+1 flavor



Model estimates at low and high temperatures

Low temperatures: Hadron Resonance Gas

The interaction of the hadrons are introduced by adding all their resonances to the heat bath, as free particles.

$$\frac{p^{\mathrm{HRG}}}{T^4} = \frac{1}{VT^3}\left(\sum_{i\in\mathrm{meson}} \log \mathcal{Z}^M\left(T,V,m_i,\{\mu\}\right) + \sum_{i\in\mathrm{baryon}} \log \mathcal{Z}^B\left(T,V,m_i,\{\mu\}\right)\right)$$

High temperature: weakly interacting quarks

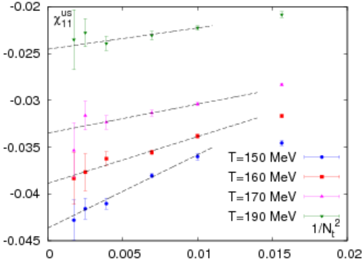
For an ideal gas we have:

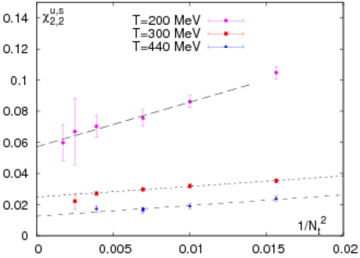
$$\frac{p}{T^4} = \frac{8\pi^2}{45} + \frac{7\pi^2}{60}N_f + \frac{1}{2}\sum_f\left(\frac{\mu_f^2}{T^2} + \frac{\mu_f^4}{2\pi^2T^4}\right)$$

This means e.g. that $\chi_4^u = 0.608$ or $\chi_{11}^{ud} = 0$ etc. This estimate can be improved with resummed PT: Hard Thermal Loop, Dimensional Reduction

Continuum extrapolation of the fluctuations

The coarsest $N_t = 8$ results are usually included only in non-linear continuum extrapolations, like $A + B/N_t^2 + C/N_t^4$. For some observables, like χ_2^B, there is a long range of safe linear extrapolation, but observables that are related to pion physics, like χ_2^Q, show a strong, non-linear $1/N_t^2$ dependence, and only for very fine lattices, $N_t \geq 16$ we see a linear regime. Calculating these quanitities is one of the achievements of our paper.





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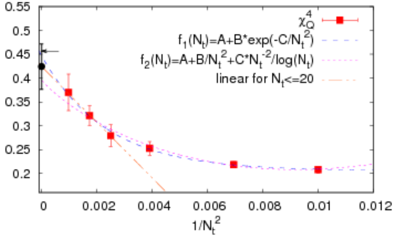
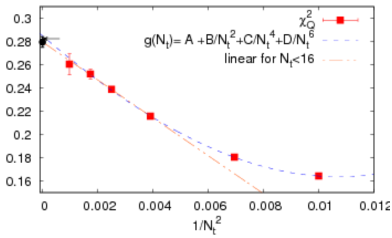


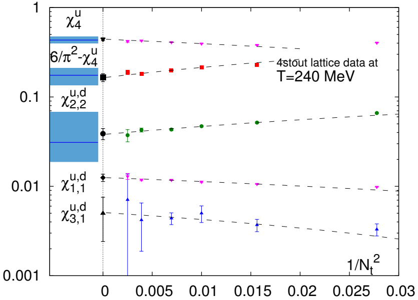


Fluctuations: Hierarchy

Perturbatively:

$$\chi_4 - \frac{6}{\pi^2} \sim \alpha$$
$$\chi_{22} \sim \alpha^{3/2}$$
$$\chi_{11} \sim \alpha^3 \log \alpha$$
$$\chi_{31} \sim \alpha^5 \log \alpha$$

Toimela PLB (1983) B124,407;
Blaizot hep-ph/0110360
Vuorinen hep-ph/0212283



The Strong CP problem

In general, gauge symmetry allows for two CP breaking terms in the QCD Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\psi}(i\gamma^\mu D_\mu - Me^{i\theta'\gamma_5})\psi - \frac{n_f g^2\theta}{32\pi^2}F^a_\mu\nu\tilde{F}^{a\mu\nu}$$

The phase of the quark mass matrix and the θ parameter are not really independent, in the sense that physics can only depend on the combination: $e^{-2i\theta}\prod_f m_f$, since this is the combination left invariant under the change of variables $\Psi_f \to e^{i\alpha_f\gamma_5}\Psi_f$ in the path integral, therefore we can work with only one total effective angle we can simply call θ

- The CP violating term would induce an electric dipole moment of the neutron, and gives a strong contraint $|\theta| < 10^{-9}$
- If any of the quark masses were zero, that would imply CP. But this is probably not the case, from:
  - Lattice
  - χPT + experimental data
- Explaining the smallness of the CP violating term is the Strong CP problem

The Peccei-Quinn mechanism

Assume a global $U(1)_{PQ}$ symmetry that breaks spontaneasly at $T\ f_A$. The emerging would be Goldstone boson is the axion(ϕ). There are various variants, but the generic feature is that the only non-derivative coupling is to the topological charge density:

$$\mathcal{L}_a^{non-der.} = i\frac{\phi}{f_A}\frac{g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$$

The PQ mechanism works because the effective potential for the (homogeneous) axion field has a CP conversing minimum:

$$e^{-VV_{eff}(\phi)} = \left|\int \mathcal{D}A \det(\gamma_\mu D_\mu + M)e^{-S+i(\theta+\phi/f_A)Q}\right|$$
$$\geq \int \mathcal{D}A \det(\gamma_\mu D_\mu + M)e^{-S}\left|e^i i(\bar{\theta}+\phi/f_A)Q\right| = e^{-VV_{eff}(\phi=-f_A\theta)}$$

So $\langle\theta+\phi/f_A\rangle = 0$ as we wanted. Shifting the axion field, we define $\theta+\phi/f_A \to \theta_a$. Thus, the PQ mechanism effectively trades θ, a free parameter, with a dynamical field evolving around a CP conderving minimum.

The axion potential

The effective Lagrangian:

$$\mathcal{L} = (\partial\phi)^2 + i\frac{\phi}{f_A}q + \text{derivative terms} + \mathcal{L}_{\text{QCD}}$$

$$\frac{\partial V_{eff}}{\partial \phi} = -\frac{1}{f_A}\frac{g^2}{64\pi^2}\left\langle F^a\mu\nu\tilde{F}^{a\mu\nu}\right\rangle m_\phi^2 = \frac{\partial^2 V_{eff}}{\partial\phi^2} = \chi_t/f_A^2$$

So $f_A^2 m_A^2 = \chi_t$ is a purely QCD quantity, that can be calculated on the lattice, or estimated from semiclassical approximations.
On the other hand, if only $Q = 0, -1, 1$ contribute:

$$exp(-V_0 V_e f f) =< e^{iQ\theta} >= p_0 + p_1 e^{i\theta} + p_1 e^{-i\theta} = 1 - 2p_1(1-\cos\theta)$$

The topological susceptibility at a fixed volume V_0 is

$$(V_0/T)\chi_t = \left\langle Q^2 \right\rangle \approx 0^2 p_0 + 1^2 p_1 + (-1)^2 p_1 = 2p_1$$

In the thermodynamic limit $V = V_0 N$ with $N \to \infty$. We get:

$$-\frac{V}{T}V_{eff} = N\log\left(1.\frac{\chi_t V}{NT}(1-\cos\theta)\right) = -\frac{\chi_t V}{T}(1-\cos\theta) + \mathcal{O}\left(\frac{1}{N}\right)$$

Axion production mechanisms

Three production mechanism known:

- Thermal: negligable
- Misalignment: As the temperature drops, the topological susceptibility is increased, the axion develops a non-negligable mass, and relaxes to its minimum (oscillator equation damped by Hubble expansion).
- Strings: Such a broken symmetry supports the formation of topological strings, whose radiation produces axions.

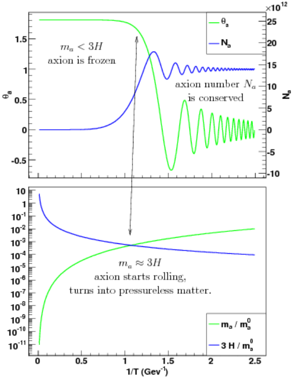
When can the PQ symmetry breaking occur?

- Pre-inflation: Only misalignment plays a rule, the strings get inflated away
- Post-inflation: Both non-thermal mechanisms play a role

Axion production from misalignment

See book by Kolb & Turner, see also Wantz, Shellard, astro-ph.CO/0910.1066
Assume a homogeneous field.

$$\mathcal{L} = \frac{1}{2}\dot{\theta_a}^2 - \frac{1}{2}m_A^2(1-\cos\theta_a) = \frac{1}{2}\dot{\theta_a}^2 - \frac{1}{2}m_A^2\theta_a^2 + \dots$$



Equation of motion can be obtained by varying $S = \int d^4x R^3 \mathcal{L}$, to get:

$$\frac{d^2\theta_a}{dt^2} + 3H(T)\frac{d\theta_a}{dt} + m_A(T)^2 \sin\theta_a$$
$$= \frac{d^2\theta_a}{dt^2} + 3H(T)\frac{d\theta_a}{dt} + m_A(T)^2\theta_a + \mathcal{O}\left(\theta_a^2\right) = 0$$

The θ_a field cannot roll into the potential minimum until $3H > m_a$, since the oscillator equation is over damped.
Remark: There is additional damping due to sphalerons, but it appears to be negligable.

Axion mass predictions

Remember $f_A^2 m_A^2 = \chi_t$ is what we can calculate. The axion scale f_A can be estimated from a guess of how much of the DM comes from misalignment.

All DM from misalignment	$m_A = 28(2)\mu eV$
50% from misalignment	$m_A = 40(4)\mu eV$
1% from misalignment	$m_A = 1500\mu eV$

Integral method for the topological susceptibility

$$Z = \int \mathcal{D}U e^{-S} = \sum_Q \int \mathcal{D}U|_Q e^{-S} = \sum_Q Z_Q$$

At high temperature: $Z_0 \gg Z_1 \gg Z_2 \gg \dots$ By $\mathcal{P}$ symmetry: $Z_Q = Z_{-Q}$
Keeping $R = LT$ const.

$$\frac{V}{T}\chi_t = R^3 \frac{\chi_t}{T^4} = \langle Q^2 \rangle = \frac{0 \cdot Z_0 + 1 \cdot Z_1 + 1 \cdot Z_{-1} + \dots}{Z_0 + \dots} \approx \frac{2Z_1}{Z_0}$$

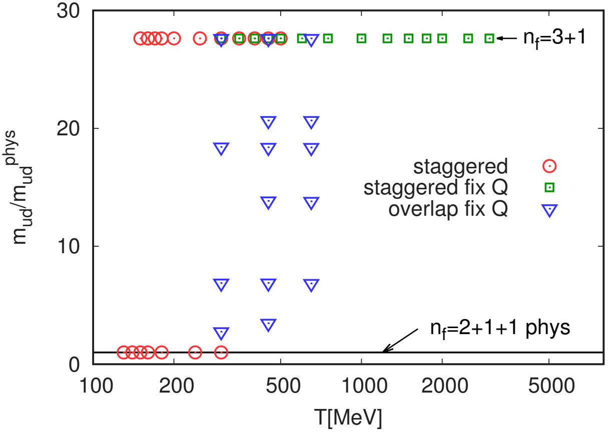
Finally:

$$\chi_t \sim T^{-b}, \text{ with } \mathrm{b} = -4 - \frac{\mathrm{d}\beta}{\mathrm{dT}} \langle \mathrm{S_g} \rangle_{1-0} - \sum_{\mathrm{f}} \frac{\mathrm{dm_f}}{\mathrm{dT}} \mathrm{m_f} \left\langle \bar{\phi}\psi_{\mathrm{f}} \right\rangle_{1-0}$$

If we know T at some temperature, we can calculate it at higher temperatures.
An important point here is that:

- Very large cutoff effects on the fermionic part (in particular, any non-chiral action fails miserably)
- Gauge action part is much noiser, than the fermion part

Simulation strategy for the topological susceptibility



Evaluate susceptibility and decay exponent at a quark mass, where the simulation is less expensive than at phys. pt.: Three flavor symmetric point.
Carry out an integration in light quark mass from m_s down to m_{ud}.

Integral method in the mass direction

With chiral fermions Q never changes in the Monte Carlo history,

$$\frac{\partial \log \chi_t}{\partial \log m} = \frac{\partial \log Z_1}{\partial \log m} - \frac{\partial \log Z_0}{\partial \log m}$$

$$\left(\frac{\chi_t}{T^4}\right)_{m_A} = \int_{m_A}^{m_B} d(\log m) m \left\langle \bar{\Psi}\Psi \right\rangle_{1-0} \left(\frac{\chi_t}{T^4}\right)_{m_B}$$

Thus, to integrate down in the mass direction:

- Calculate χ_t at the flavour physical point with high statistics.
- Calculate $\left\langle \bar{\Psi}\Psi \right\rangle$ at different quark masses using chiral fermions.

On cut-off effects

- The strong cut-off effects are related to the lack of exact zero modes. (In the continuum, these zero-modes are a consequence of the Atiyah-Singer index theorem.)
- Possible solutions:
  1) Eingenvalue reweighting
  2) Using chiral fermions
- We use 1) for the 3 flavor theory and 2) for going down to the physical point

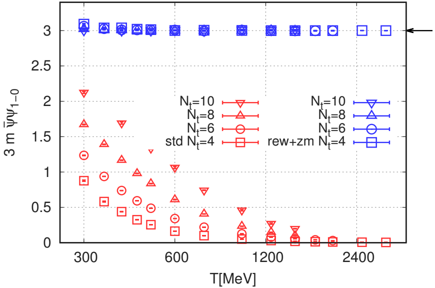
Reweighting method

Strong cut-off effects a re related to the lack of exact zero modes of the Dirac operator.

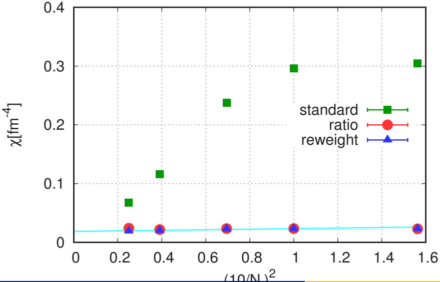
- In the continuum non-trivial sectors are suppressed by the contribution of zero-modes to the fermion determinant
- On the lattice the suppression is altered: $m \to m + i\lambda_0$, where λ_0 is a would be zero-mode. Weaker suppression $\to \chi_t$ overestimated
- To improve:
  1. identify would be zero-modes 2. restore the continuum weight $\to$ reweight

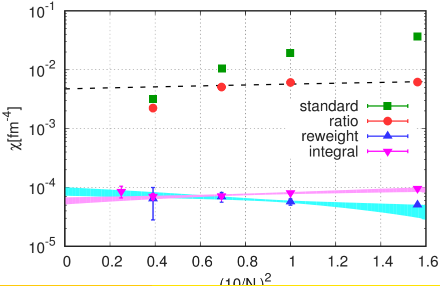
$$w[U] \sim m/(m + i\lambda_0)$$

Reweighting works

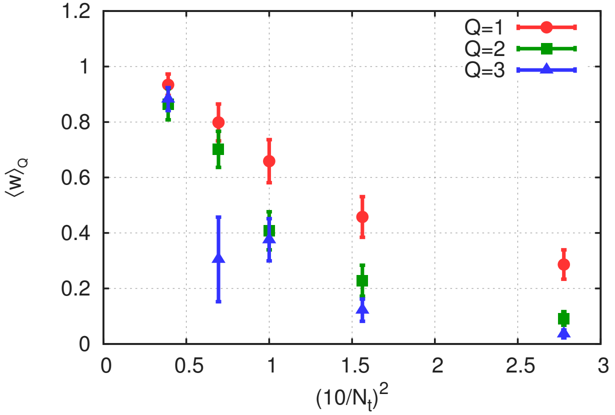


For large T goes to 0 instead of approaching the SB limit, this is a lattice artefact. Non-chiral fermions (without reweighting) fail spectacularly for large temperatures!





Reweighting factors



As we approach the continuum limit, the would be zero modes become, smaller, so we expect $W_Q \to 1$ as $a \to 0$.

DIGA: More details

Gross, Pisarski Rev.Mod.Phys. 53 (1981) 43
Ringwald, Schrempp (DESY), Phys.Lett. B459 (1999) 249-258, hep-lat/9903039

$$\chi_t = 2\int_0^\infty d\rho D(\rho) G(\rho\pi T)$$

$D(\rho)$ Density of instantons of size ρ
$G(x)$ is the cut-off function that includes the effects of Debye screening in the medium.

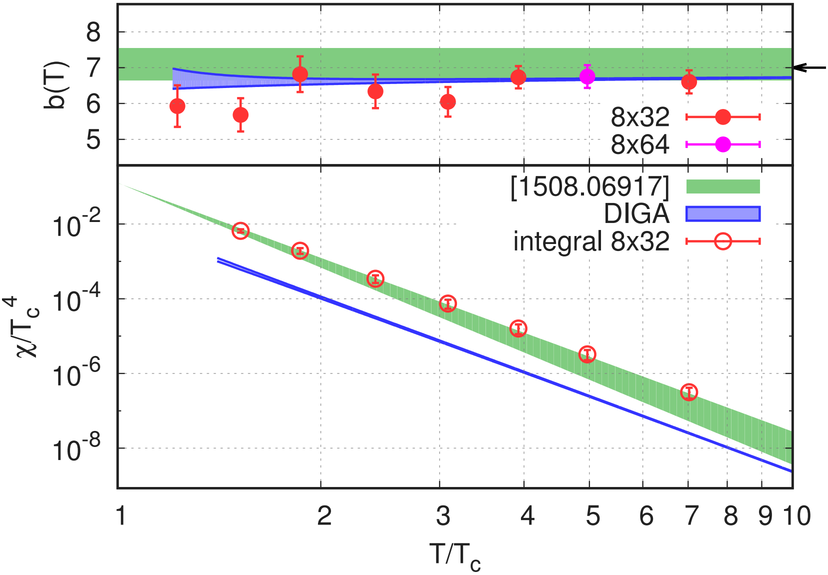
$$D(\rho) = \frac{d_{\overline{MS}}}{\rho^5}\left(\frac{2\pi}{\alpha_s(\mu)}\right)^6 e^{-\frac{2\pi}{\alpha_s(\mu)}}(\rho\mu)^{\beta_0+(\beta_1-12\beta_0+8N_f)\frac{\alpha_s}{4\pi}}\prod_{i=1}^{N_f}(\rho m_i)$$

$$G(x) = \exp\left(-\frac{1}{3}(6+N_f)x^2 - 12A(x)(1+\frac{1}{6}(3-N_f))\right)$$

$$A(x) = -\frac{1}{12}\log\left(1+x^2/3\right) + c_1\frac{1}{(1+c_2x^{-3/2})^8}$$

$$c_1 = 0.01289764 \qquad c_2 = 0.15858$$

Quenched case



Three flavour symmetric point