Effective Model of QCD Monopoles Around $T_c$

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(With E. Shuryak, in preparation)

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Outline

1 Motivation
   Lattice QCD Equation of State
   Magnetic Scenario of QCD
   Goals

2 Methods

3 Effective Quantum Model for Monopoles

4 Conclusions and Outlook
Lattice QCD Equation of State

Results adapted from HotQCD Collaboration (2014)
Motivation

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3. Effective Quantum Model for Monopoles

4. Conclusions and Outlook
Dual Superconductivity Model of the QCD Vacuum


- Electric quasiparticles (quarks and gluons) and magnetic quasiparticles (monopoles, etc.)
- Confinement is due to the Bose-Einstein condensation (BEC) of magnetic (quasi)particles
- Lattice studies have identified electric flux-tubes and monopole currents

Can we make an effective model for the magnetic component of QCD above $T_c$?
Magnetic Scenario of QCD

- Condition for making Dirac strings invisible: \( \frac{eg}{4\pi} = \text{integer} \) (Dirac 1931)
  - \( \alpha_s = \frac{e^2}{4\pi} \) in QCD runs with \( T \) and \( \mu \)
  - \( \alpha_m = \frac{g^2}{4\pi} \) runs oppositely

- Classical studies of electric-magnetic plasma have seen this behavior of the coupling (Liao and Shuryak 2007)
Monopoles on the Lattice

Antimonopole-monopole correlations **stronger** at higher temperatures

![Graph showing correlations at different temperatures](image)

D’Alessandro and D’Elia (2007)
Monopoles on the Lattice

Density: \( \rho_m / T^3 \sim \log(T)^{-2} \rightarrow \text{Monopoles important near } T_c \)

\[ \rho / T^3 \]

\[ T / T_c \]

D’Alessandro and D’Elia (2007)
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4 Conclusions and Outlook
• Classical monopoles $\rightarrow$ Quantum monopoles (BEC)

• Reproduce behavior of lattice monopoles, without other degrees of freedom

• Identify the contribution of magnetic monopoles to the thermodynamics of QCD around $T_c$

• Quantify BEC critical temperature behavior of one- and two-component quantum Coulomb systems
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   Critical Temperature Analysis

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Path-Integral Monte Carlo

- Density matrix at finite temperature from path integrals:
  
  Feynman (1953), Matsubara (1951)

  \[
  \rho(x_i, x_j, \beta) = \int_0^\beta Dx(\tau) \exp \{-S_E[x(\tau)]\}
  \]

  where \( \beta = 1/T \)

- Partition function: \( Z = \text{Tr}[\rho] \rightarrow \text{periodic paths} \)

- Discretize paths and sample configurations using Markov chain Monte Carlo (MCMC)
  
  Ceperley (1995) (Review w/ focus on application to liquid helium)

  - The configuration weight is given by the Euclidean action,
    
    \[
    \pi(\{\vec{x}\}) = e^{-S_E(\{\vec{x}\})}
    \]

  - Sample the partition function \( \rightarrow \) thermodynamics
Path-Integral Monte Carlo

- The imaginary-time paths of bosons can be permuted.
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Finding $T_c$, Method 1: Permutation Cycles

- Measure $n_k(T)$, the probability of finding a particle in a $k$-permutation cycle as a function of temperature.

- Densities of $k$-cycles go as

$$\rho_k(T) \equiv n_k(T)/k = \exp(-\hat{\mu}(T)k) \times f(k),$$

where $f(k)$ is some decreasing function $\sim 1/k^\alpha$.

- Critical temperature is where permutation cycles are no longer exponentially suppressed, i.e. $\hat{\mu}(T) = 0$

D’Alessandro, D’Elia, and Shuryak (2010)
Finding $T_c$, Method 2: Superfluid Fraction

- The superfluid fraction of a condensate is related to the spatial winding number distribution

$$\frac{\rho_s}{\rho} = \frac{\langle W^2 \rangle}{2\lambda \beta N}$$

- Thermodynamic limit $\to$ expect the superfluid fraction to go from 0 (above $T_c$) to 1 (below $T_c$)

- Study finite-size scaling of this quantity $\to$ determine critical point by the intersection of the superfluid fraction lines

Pollock and Ceperley (1987)
Finding $T_c$: Examples

Permutation-cycle calculation

Finite-size scaling of the superfluid fraction
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   Mapping Parameters
   Monopole Contribution to QCD Thermodynamics

4 Conclusions and Outlook
• Find the dependence of \( T_c \) on \( \alpha \), the coupling, with interaction potential:

\[
V_{\text{int}}(r_{ij}) = \alpha \frac{q_i q_j}{r_{ij}}
\]

• Critical temperature of an ideal Bose gas:

\[
T_0 = \left( \frac{2\pi \hbar^2}{mk_B} \right) \left( \frac{n}{\zeta\left(\frac{3}{2}\right)} \right)^{\frac{2}{3}}
\]

• \(|q| = m = n = 1 \rightarrow \) scale is fixed by the temperature \( T \)

• Ideal gas BEC temperature in these units:

\[
T_0 = 2\pi \left( \frac{1}{\zeta\left(\frac{3}{2}\right)} \right)^{\frac{2}{3}} = 3.3125
\]
One-Component Coulomb System

Same behavior seen by varying density of hard spheres  
Grüter, et al. (1997)

Analytically explained by  
M. Holzmann, G. Baym, J. P. Blaizot, and F. Laloe (2001)
Same qualitative behavior as one-component case, but the increase in $T_c$ at moderate coupling is slightly smaller.
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Matching Radial Distribution Functions

Lattice data from D’Alessandro and D’Elia (2007)
Fixing Parameters

- Scaling relation for $\alpha$, the simulation coupling, to match lattice results:
  \[
  \alpha(T/T_c) \approx 3.4 \rho_m^{1/3}(T/T_c)
  \]

- $\rho_m(T)$ is the monopole density (in fm$^{-3}$)
  D’Alessandro and D’Elia, 2007

- One unit of length in our simulations is $\rho_m^{-1/3}(T)$ (in fm)

- $T_c = 155$ MeV $\approx 3.45$ in our units of temperature
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Energy Per Monopole on the Physical Line

\[
\frac{E}{N} = \frac{\rho(T)}{T} \left( \frac{T}{T_c} \right)^{2/3}
\]

\[
\alpha = 2 \quad \star \quad \alpha = 10
\]

\[
\alpha = 5 \quad \diamond \quad \alpha = 3.4\rho_m(T)^{1/3}
\]
Equation of State

- Studies on the lattice found a large entropy density and trace anomaly at 170-250 MeV

- May not all be from quarks and gluons → other contributions to the thermodynamics of QCD-like theories at $T_c$ (hadrons, instantons, etc.)

- What portion comes from monopoles?

Wuppertal-Budapest Collab. (2013)
HotQCD Collab. (2014)
etc.
Monopole Contribution to Equation of State

\[ \frac{s}{T^3} \]
\[ \frac{\epsilon}{T^4} \]
\[ \frac{(\epsilon - 3p)}{T^4} \]

\( T/T_c \)
Comparison to Lattice QCD

Lattice results from HotQCD Collaboration (2014)
Conclusions and Outlook

• Found parameters of a two-component Coulomb quantum Bose gas that give an effective model of QCD monopoles

• Identified the contribution of monopoles to QCD equation of state

• Found the effect of coupling on the critical temperature of Coulomb quantum Bose gases

• Can we use the parameters of this model to simulate real-time properties of QGP, such as viscosity, jet quenching, and heavy quark propagation?
Backup Slides
One-Component Radial Distribution Functions

\[ g(r) \]

- \( T = 1.6 \)
- \( T = 2.4 \)
- \( T = 3.3 \)
- \( T = 4.2 \)
- \( T = 5.1 \)

\( \alpha = 1 \)
\( \alpha = 5 \)
\( \alpha = 10 \)
\( \alpha = 20 \)
\( \alpha = 50 \)
\( \alpha = 100 \)

\[ \alpha \, \square \]

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Two-Component Radial Distribution Functions

\[ g_{++}(r) \]

- \( \alpha = 0.5 \)
  - \( T = 0.4 \)
  - \( T = 3.9 \)
  - \( T = 1.5 \)
  - \( T = 5.1 \)
  - \( T = 2.7 \)

- \( \alpha = 1 \)
  - \( T = 0.4 \)
  - \( T = 3.9 \)
  - \( T = 1.5 \)
  - \( T = 5.1 \)
  - \( T = 2.7 \)

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- \( \alpha = 10 \)
  - \( T = 0.4 \)
  - \( T = 3.9 \)
  - \( T = 1.5 \)
  - \( T = 5.1 \)
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Two-Component Radial Distribution Functions

\[ g_+(r) \]

\[ g_+(r) \]

\[ \alpha = 0.5 \]

\[ \alpha = 1 \]

\[ \alpha = 5 \]

\[ \alpha = 10 \]

\[ T = 0.4 \]

\[ T = 3.9 \]

\[ T = 1.5 \]

\[ T = 5.1 \]

\[ T = 2.7 \]
Critical Temperature of Hard Spheres

PIMC results for hard spheres from Grüter, et al. (1997)
Two-Component Critical Temperature (incl. core)

![Graph showing critical temperature behavior](image)
Two-Component Internal Energy

![Graph showing the relationship between E/N and T for different values of α (0.5, 1, 2, 5, 10). The graph has labels for each curve representing α values.]
In the primitive approximation, the density matrix between time slice $i$ and $i+1$ is given by

$$\rho(R_i, R_{i+1}, \tau) = \langle R_i | e^{-\tau \hat{T}} e^{-\tau \hat{V}} | R_{i+1} \rangle = \langle R_i | e^{-\tau \hat{T}} | R_{i+1} \rangle e^{-\tau V}$$

The kinetic matrix element for $N$ particles can be computed using the eigenfunction expansion of the kinetic operator,

$$\langle R_i | e^{-\tau \hat{T}} | R_{i+1} \rangle = \frac{1}{(4\pi \lambda \tau)^{3N/2}} \exp \left\{ -\frac{(R_i - R_{i+1})^2}{4\lambda \tau} \right\}$$

where $\lambda = (2m)^{-1}$
Partition Function

- For periodic paths, such that \( x_i = x_j \),

\[
Z = \sum_n e^{-\beta E_n} = \text{Tr}[e^{\beta \hat{H}}] = \text{Tr}[\hat{\rho}]
\]

- The expectation value of the operator \( \mathcal{O} \) is

\[
\langle \mathcal{O} \rangle = \frac{\text{Tr}[\mathcal{O} \hat{\rho}]}{\text{Tr}[\hat{\rho}]} = \frac{\text{Tr}[\mathcal{O} \hat{\rho}]}{Z}
\]

- For \( N \) bosons in the primitive approximation,

\[
Z = \frac{1}{N!(4\pi \lambda \tau)^{3N/2}} \prod_{i=1}^{M-1} \prod_{n=1}^{N} \sum_P \int dx_{i,n} \\
\times \exp \left\{ - \frac{(x_{i,n} - x_{P,i+1,n})^2}{4\lambda \tau} - \tau V(x_{i,n}) \right\}
\]
The general partition function is

$$Z = \frac{1}{N!} (4\pi \lambda \tau)^{-3NM/2} \int DR \exp \left\{ -\tau \left[ \frac{(R_i - R_{i+1})^2}{4\lambda \tau^2} + V \right] \right\}$$

The thermodynamic estimate for the energy is

$$\langle E \rangle = \frac{1}{Z} \frac{\partial}{\partial \beta} Z$$

$$= \left\langle \frac{3N}{2\tau} - \frac{1}{M} \sum_{i=0}^{M-1} \frac{(R_i - R_{i+1})^2}{4\lambda \tau^2} + V + \tau \frac{\partial}{\partial \tau} V \right\rangle$$

The last potential term vanishes in the primitive approximation.
Finding $T_c$: Permutation Cycles

As done by D’Alessandro, et al. (2010) on the lattice, we can use permutation cycles to study the transition temperature in PIMC

- The partition function of a *non-interacting* ideal gas of bosons can be broken up into a product of contributions of $k$-cycles – for example, the permutation $(1, 2, 3) \rightarrow (2, 3, 1)$ is considered a 3-cycle

$$Z = \frac{1}{N!} \sum_{P} \prod_{k} z_{k}^{n_{k}},$$

where $n_{k}$ is the number of $k$-cycles present in the system

- Expanding these contributions,

$$z_{k}(T) = \int dy_{1}..dy_{k} \langle y_{2}, y_{3}, \ldots, y_{k}, y_{1} | e^{-\beta \hat{H}} | y_{1}, y_{2}, \ldots, y_{k} \rangle$$

$$= \int dy_{1} \langle y_{1} | e^{-k\beta \hat{H}} | y_{1} \rangle \equiv z_{1}(T/k)$$
• Using the partition function for a non-relativistic free particle in a box,

\[ z_k(T) = \frac{V}{\lambda_B^3 k^{3/2}} , \]

where \( \lambda_B \) is the thermal de Broglie wavelength and \( V \) is the volume of the box

• The total partition function is

\[ Z = \frac{1}{N!} \sum_P \sum_k \left( \frac{V}{\lambda_B^3 k^{3/2}} \right)^{n_k} \]

• This quantity is not easily computed for fixed particle number, but this problem is avoided if we go to the Grand Canonical ensemble,

\[ Z = \prod_k \left( \frac{V e^{\mu k/T}}{\lambda_B^3 k^{5/2}} \right) \]
• From this partition function, we can extract the density of $k$-cycles

$$\rho_k \equiv \frac{\langle n_k \rangle}{V} = \frac{e^{\mu k/T}}{\lambda_B^3 k^{5/2}}$$

• The total particle density is

$$\frac{N}{V} = \sum_k k \rho_k = \sum_k \frac{e^{\mu k/T}}{\lambda_B^3 k^{5/2}},$$

which has an upper limit of $\mu = 0$. This approach is fully valid for any non-interacting gas above $T_c$.

• Even though we assumed non-interaction, we expect the densities of cycles to go roughly as

$$\rho_k = e^{-\hat{\mu} k} f(k),$$

where $\hat{\mu} = -\mu/T$ and $f(k)$ is some decreasing function of $k$ of the form $f(k) \sim 1/k^\alpha$. 
Finding $T_c$, Test of Method 1: Liquid $^4$He

1% accuracy to experimentally-determined $T_c$!