Modeling the hadronization processes in HIC (based on the Nambu Jona-Lasinio Lagrangian)

in collaboration with
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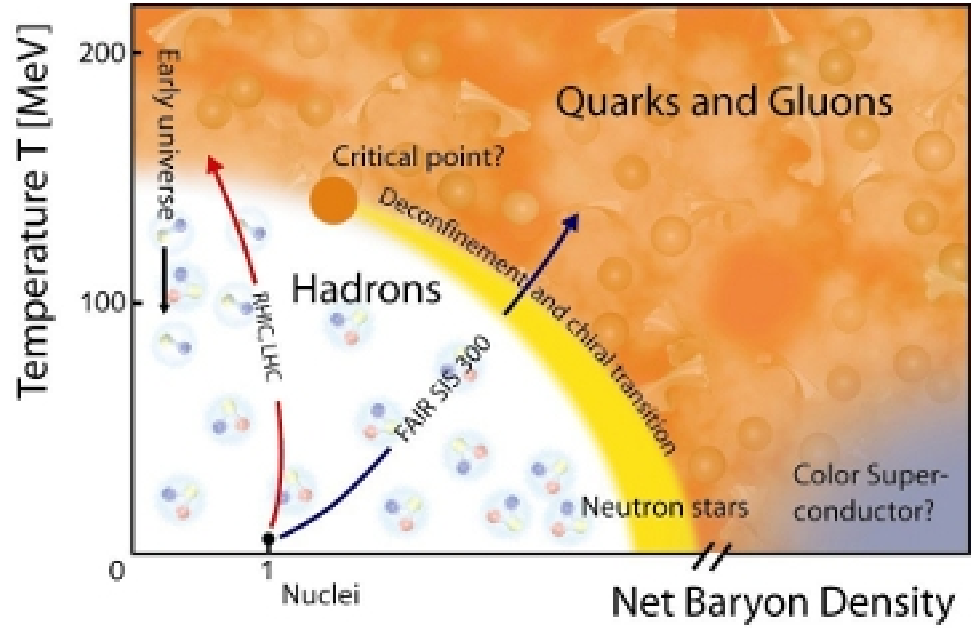
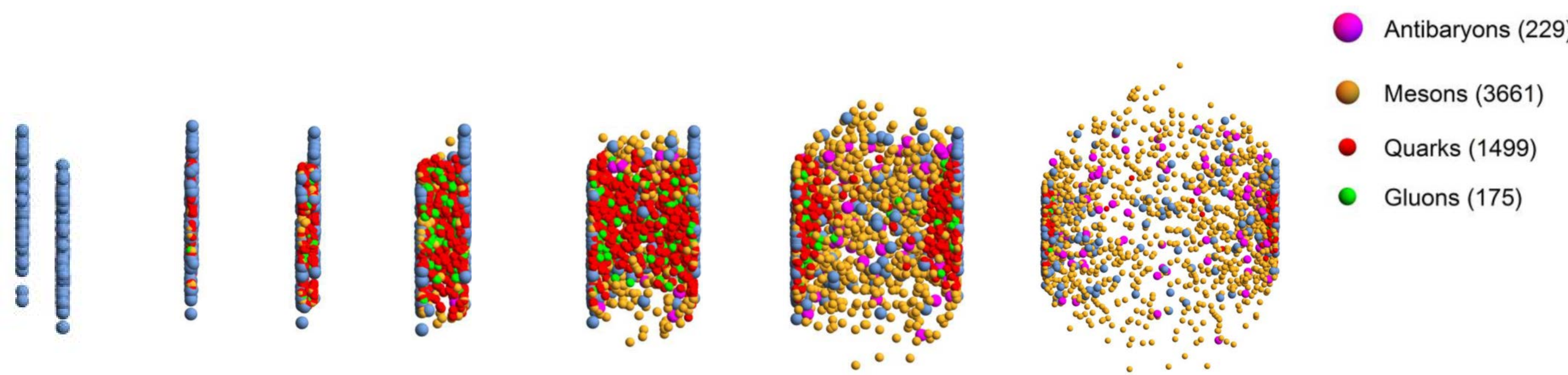
- How one obtains the NJL Lagrangian
- How to construct mesons Mesons (and Baryons)
- Cross section for elastic scattering and hadronisation
- Expanding plasma: How quarks hadronize
- Realistic simulations
- Advantage of this approach and perspectives

(For the experts: We use PNJL in these calculations) circumstantial evidence:

For beam energies $> \approx 100$ AGeV
a plasma of quark and gluons (QGP)
is formed

The challenge:
How to come from quarks to hadrons





As PHSD calculations see a heavy ion reaction
is there local equilibrium?

Courtesy:
P. Moreau 2015

We want to conserve the symmetries

$$L_{QCD}(x) = \bar{\psi}(x)\left(i\gamma^{\mu}\left[\partial_{\mu} - igt^{a}A_{\mu}^{a}\right] - \hat{M}^{0}\right)\psi(x) - \frac{1}{4}G_{\mu\nu}^{a}(x)G^{\mu\nu a}(x)$$

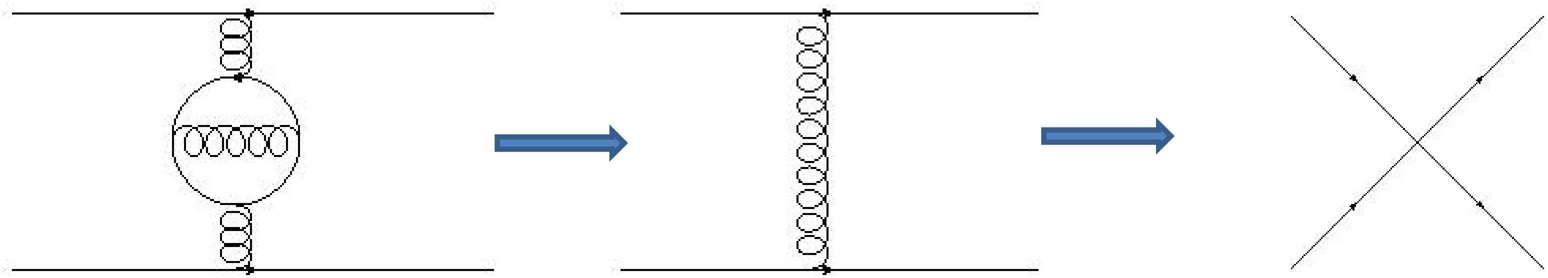
1) local *SUc*(3) color gauge transformation (by construction)

2) global SU_f (3) flavor symmetry

3) for massless quarks ONLY:
   chiral invariance of QCD Lagrangian: $SU_f(3)_V \times SU_f(3)_A$

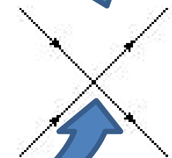
However, chiral symmetry is a spontaneously broken since quarks have non-zero masses.

⇒ To explore more simple *effective Lagrangians* with the same symmetries for the quark degrees of freedom, however, discarding the gluon dynamics completely.



NJL Langrangian

$$\mathscr{L}_{int} = -\mathrm{G_c^2} \ [\bar{\Psi}_{\mathrm{i}}\gamma^{\mu}\mathrm{T^a}\delta_{\mathrm{ij}}\Psi_{\mathrm{j}}] \ [\bar{\Psi}_{\mathrm{k}}\gamma_{\mu}\mathrm{T^a}\delta_{\mathrm{kl}}\Psi_{\mathrm{l}}]$$



$i, j = 1...N_f = 3$ flavor index ;

T^a : color generatorss $a = 1...N_c^2 - 1 = 8(N_c = 3)$.

More formal derivation: Lectures at SQM 2015 in Dubna

Symmetries of the massless NJL Lagrangian:

Yoichiro Nambu
1921

$$\mathrm{SU_V(3) \otimes SU_A(3) \otimes U_V(1) \otimes U_A(1)}$$

U_A (1) symmetry not realized in nature
(η and η’ would have the same mass)

Remedy: t’Hooft 6-point interaction

Giovanni Jona-Lasinio
1932

For reviews see Vogl and Weise (1991), Klevansky (1992), Ebert, Reinhardt and Volkov (1994), Hatsuda and Kunihiro (1994), Buballa (2004)...

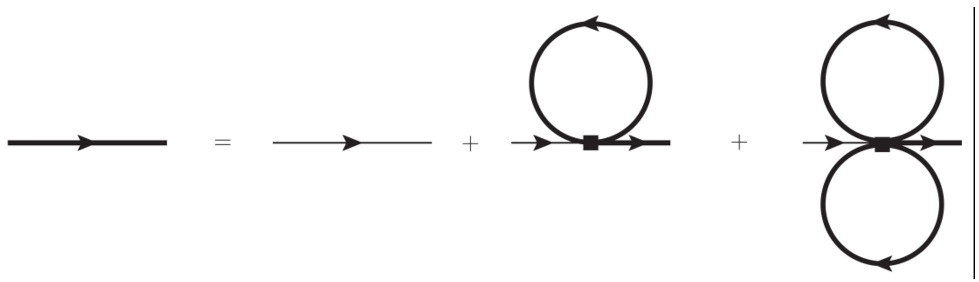
NJL Lagrangian

$$\mathscr{L}_{NJL} = \bar{\Psi}_i(i\gamma_\mu\partial^\mu - \hat{M}_0)\Psi_i - G_c^2 \; [\bar{\Psi}_i\gamma^\mu T^a\delta_{ij}\Psi_j] \; [\bar{\Psi}_k\gamma_\mu T^a\delta_{kl}\Psi_l]$$

$$+ \; H \det_{ij} [\bar{\Psi}_i(1-\gamma_5)\Psi_j] - H \det_{ij} [\bar{\psi}_i(1+\gamma_5)\psi_j]$$

$\mathscr{L}_{NJL}$: Shares the symmetries with the QCD Lagrangian (color we discuss later)

Allows for calculating effective quark masses:



$$M = \hat{M}_0 - 4G < \bar{\psi}\psi > +2H < \bar{\psi}'\psi' >< \bar{\psi}''\psi'' >$$

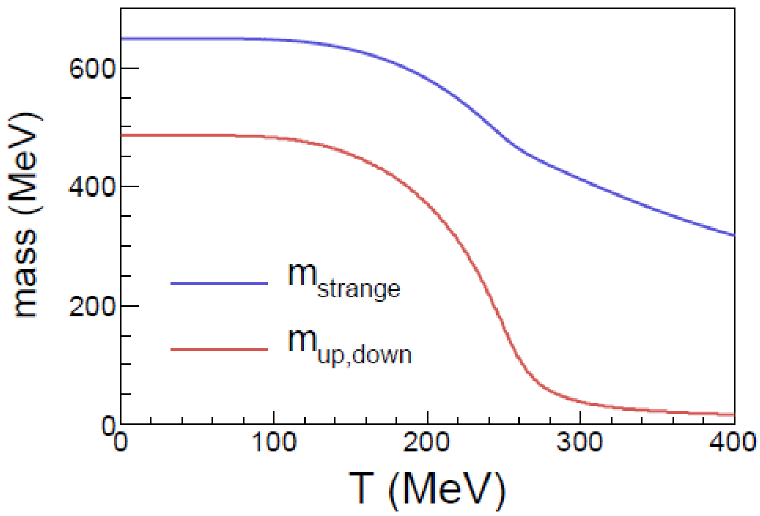
But contains only quarks

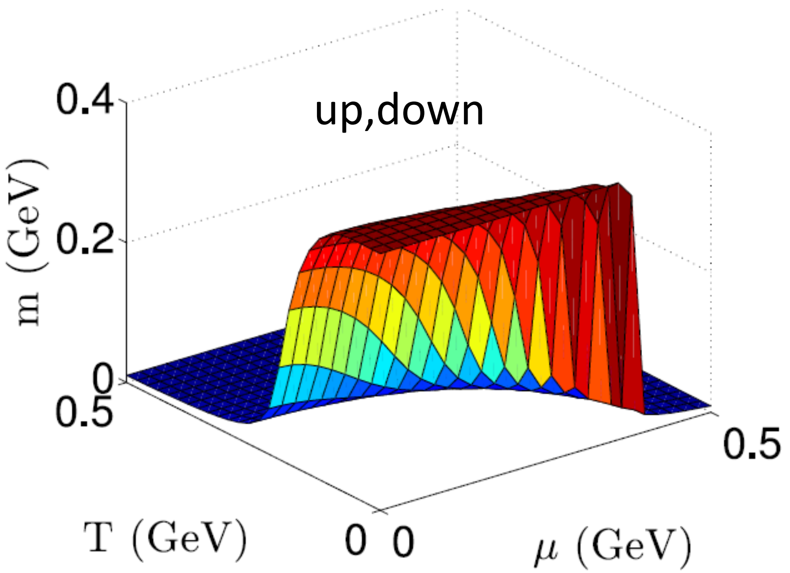
no gluons and

no hadrons

So not very obvious how of use for hadronisation.

First results: Quark masses





How can we get mesons?

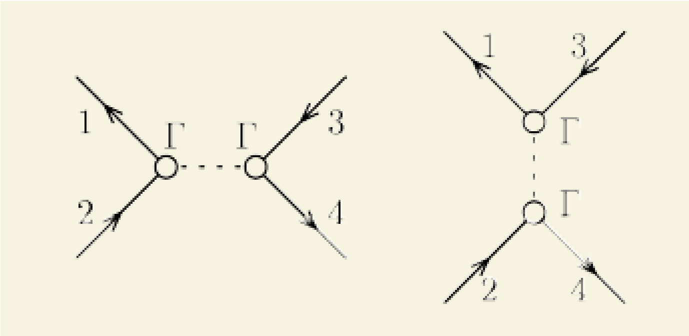
Trick : Fierz transformation of the original Lagrangian

Fierz Transformation allows for a reordering of the field operators in 4 point contact interactions. It is simultaneously applied in Dirac, colour and flavour space

Example in Dirac space:

$$(\bar{\chi}\gamma^{\mu}\psi)\,(\bar{\psi}\gamma_{\mu}\chi) = (\bar{\chi}\chi)\,(\bar{\psi}\psi) - \frac{1}{2}\,(\bar{\chi}\gamma^{\mu}\chi)\,(\bar{\psi}\gamma_{\mu}\psi) - \frac{1}{2}\,(\bar{\chi}\gamma^{\mu}\gamma_5\chi)\,(\bar{\psi}\gamma_{\mu}\gamma_5\psi) - (\bar{\chi}\gamma_5\chi)\,(\bar{\psi}\gamma_5\psi)$$

Scalar vector peudovector pseudoscalar



How to get mesons? I

$$\mathscr{L}_{int} = -\mathbf{G_c^2} \; [\bar{\Psi}_i \gamma^\mu \mathbf{T^a} \delta_{ij} \Psi_j] \; [\bar{\Psi}_k \gamma_\mu \mathbf{T^a} \delta_{kl} \Psi_l]$$

Fierz transformation transforms original Lagrangian to one for mesons

$$\mathscr{L}_{\text{Pseudo scalar}} = \mathbf{G} \; (\bar{\Psi}_i \; \tau^a_{il} \; \mathbb{1}_c i\gamma_5 \; \Psi_l) \; (\bar{\Psi}_k \; \tau^a_{kj} \; \mathbb{1}_c i\gamma_5 \; \Psi_j) \; ; \qquad \mathbf{G} = (\mathbf{N_c^2} - 1)/\mathbf{N_c^2 g}$$

Singulet in color mixing of flavour

Similar terms can be obtained for

Vector mesons γ_μ

Scalar Mesons 1

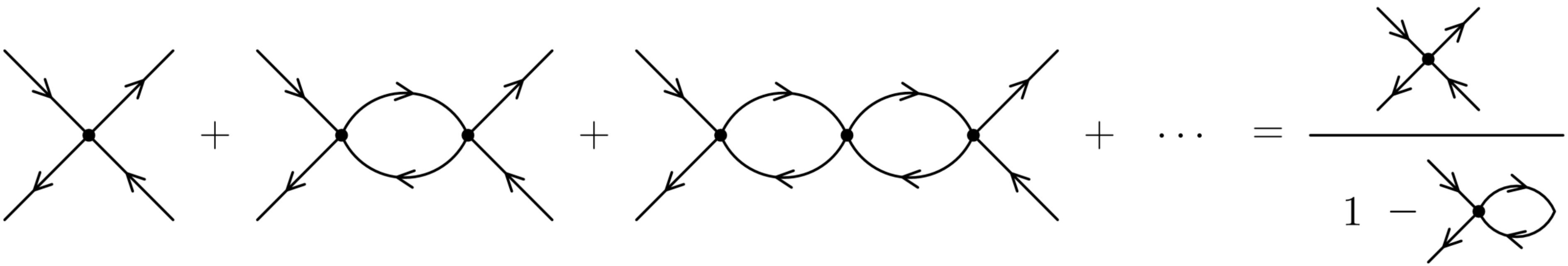
Pseudovector mesons $\gamma_\mu \gamma_5$

How to get mesons? II

$$\boldsymbol{\Omega} = \mathbb{1}_c \otimes \tau^a \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\} \qquad \mathscr{K} = \boldsymbol{\Omega}\; 2G_{eff}\; \bar{\boldsymbol{\Omega}}$$

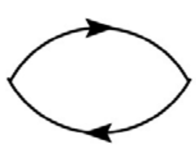
and use $\mathscr{K}$ as a kernel for a Bethe-Salpeter equation (relativistic Lippmann-Schwinger eq.)

$$T(p) = \mathscr{K} + i\int \frac{d^4k}{(2\pi)^4}\, \mathscr{K}\, S\left(k+\frac{p}{2}\right) S\left(k-\frac{p}{2}\right) T(p)$$



In (P)NJL one can sum up this series analytically:

$$T(p) = \frac{2G_{eff}}{1 - 2G_{eff}\,\Pi(p)}\,, \qquad \Pi(p_0, \mathbf{p}) = -\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, \boldsymbol{\Omega}\, S\left(k+\frac{p}{2}\right) \boldsymbol{\Omega}\, S\left(k-\frac{p}{2}\right)$$

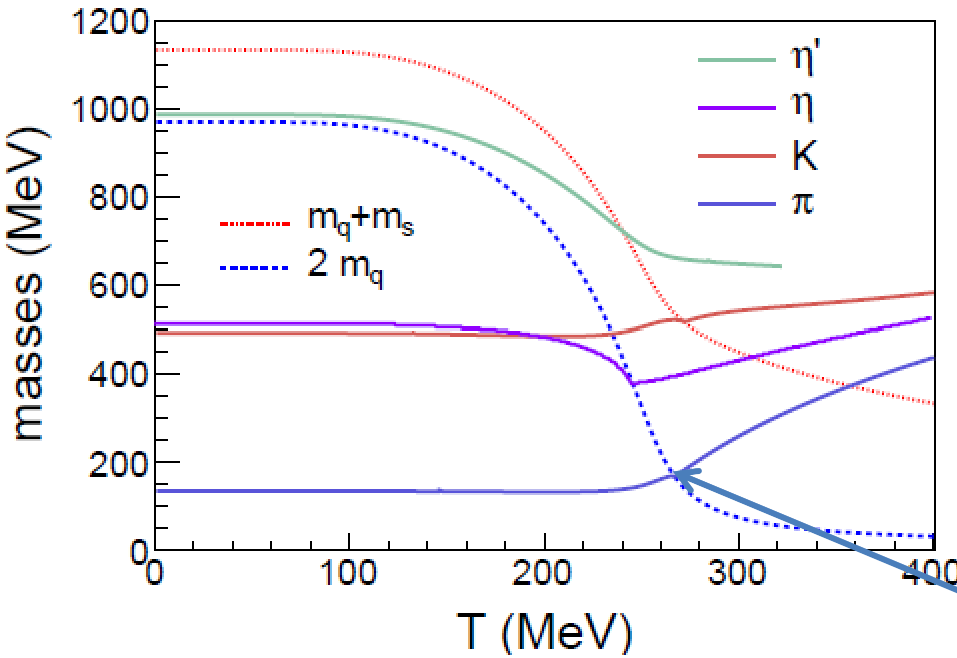


$\equiv \Pi$

How to get mesons? III

The meson pole mass and the width one obtains by solving:

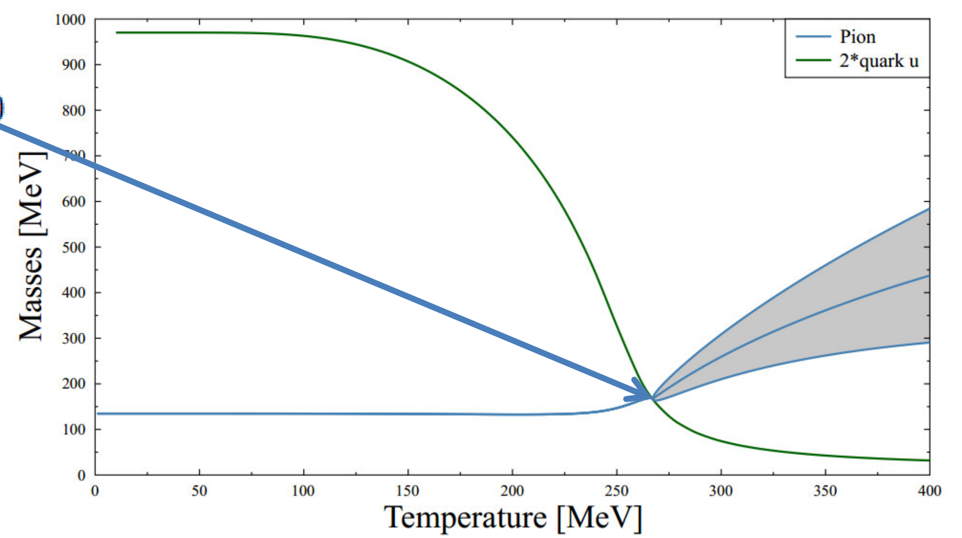
$$1 - 2G_{eff}\ \Pi(p_0 = M_{meson} - i\Gamma_{meson}/2, p = 0) = 0$$



masses of pseudoscalar mesons and of quarks at μ = 0

At T=0 physical and calculated mass agree quite well

When π's become unstable they develop a width



Looking back

We have seen that the NJL model describes quite well meson properties
For this one has to fix the 5 parameters of the model

Λ = upper cut off of the internal momentum loops
g = coupling constant
M_0 = bare mass of u,d and s quarks
H= coupling constant 't Hooft term

These parameters have been adjusted to reproduce

Masses of π and K in the vacuum , as well as the η-η' mass splitting
π decay constant, $q\bar{q}$ condensate (-241 MeV)^3

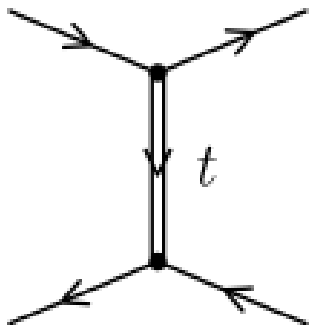
Therefore:
All properties of masses, cross sections etc. at finite μ and T
follow without any new parameters from vacuum observables.

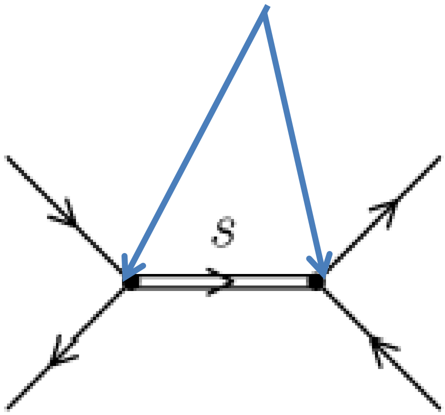
Cross sections

Having the Lagrangian we can derive in the usual way the Feynman rules and can calculate cross setions

Example: $u\bar{u} \rightarrow u\bar{u}$ matrixelements

original 4-point interaction





But also

elastic cross sections like $uu \rightarrow uu$

hadronization cross sections $q\bar{q} \rightarrow M\,M$ M=π,K,η,η’,ρ …

hadronization cross sections Diq Diq -> baryons +q etc

$u\bar{u} \to u\bar{u}$ Cross sections

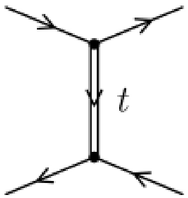
Phys.Rev. C53 (1996) 410-429

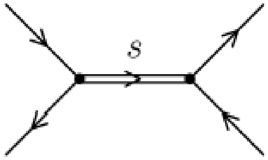
$$-i\mathcal{M}_s = \delta_{c1,c2}\delta_{c3,c4}\bar{v}(p_2)Tu(p_1)\left[i\mathcal{D}_s^S(p_1+p_2)\right]\bar{u}(p_3)Tv(p_4)$$
$$+\delta_{c1,c2}\delta_{c3,c4}\bar{v}(p_2)(i\gamma_5 T)u(p_1)\left[i\mathcal{D}_s^P(p_1+p_2)\right]\bar{u}(p_3)(i\gamma_5 T)v(p_4)$$

$$-i\mathcal{M}_t = \delta_{c1,c3}\delta_{c2,c4}\bar{u}(p_3)Tu(p_1)\left[i\mathcal{D}_t^S(p_1-p_3)\right]\bar{v}(p_2)Tv(p_4)$$
$$+\delta_{c1,c3}\delta_{c2,c4}\bar{u}(p_3)(i\gamma_5 T)u(p_1)\left[i\mathcal{D}_t^P(p_1-p_3)\right]\bar{v}(p_2)(i\gamma_5 T)v(p_4) \quad .$$

D= meson propagator

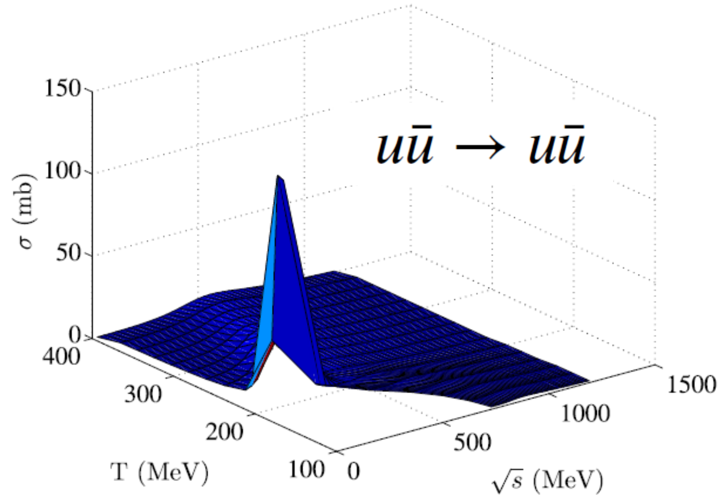
$$D(p_0,\vec{p}) \propto \frac{2G}{1-2G\Pi(p_0,\vec{p})}$$





Cross section up to 100 mb
close to cross over
due to resonant s-channel

otherwise small (5-10 mb)



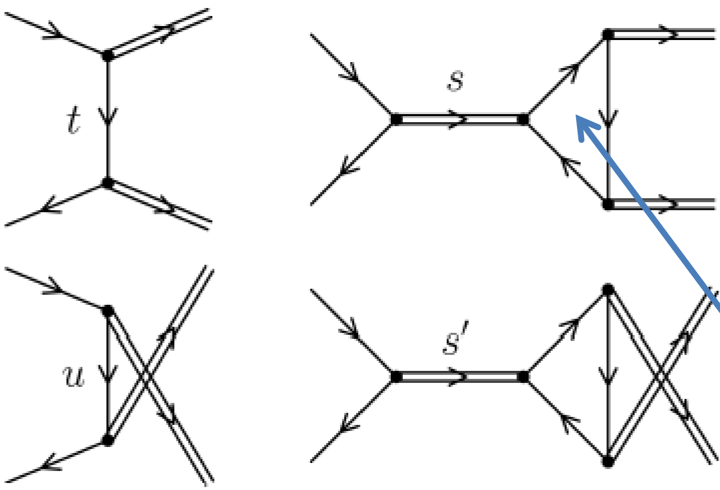
Hadronization cross sections

$$q\bar{q} \rightarrow M\,M$$

$$-iM_s = g^2_{Mqq'} f_s \bar{v}_2 u_1 \Gamma_\nu (i\mathscr{D}^{\mathcal{S}}{}_M) \Gamma^\nu_{q_1 q_2 q_3} \; +..$$

$$-iM_t = g^2_{Mqq'} f_t \bar{v}_2 \Gamma_\nu \frac{i(\not{p}_1 - \not{p}_3 + m_t)}{(p_1 - p_3)^2 - m_t^2} \Gamma^\nu u_1$$

$$-iM_u = g^2_{Mqq'} f_u \bar{v}_2 \Gamma_\nu \frac{i(\not{p}_1 - \not{p}_4 + m_t)}{(p_1 - p_4)^2 - m_t^2} \Gamma^\nu u_1$$

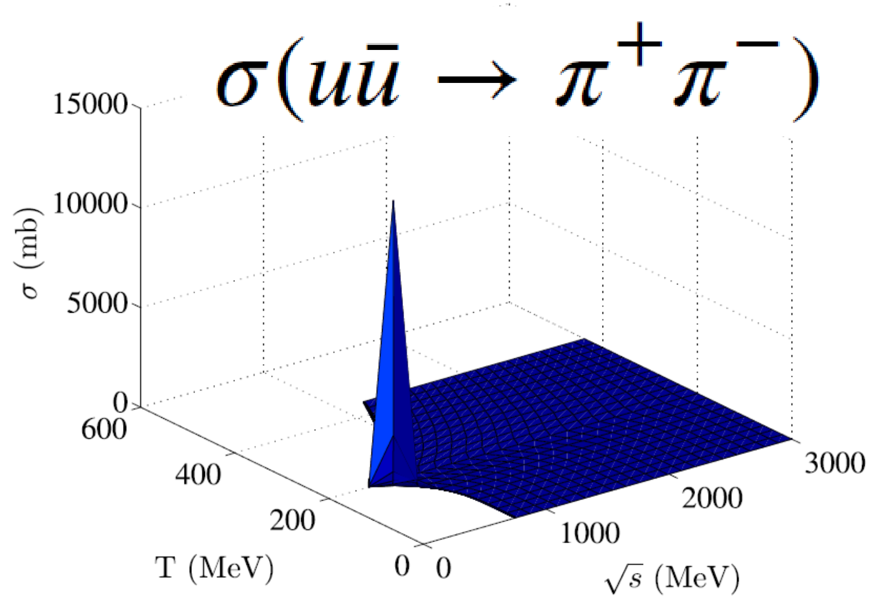


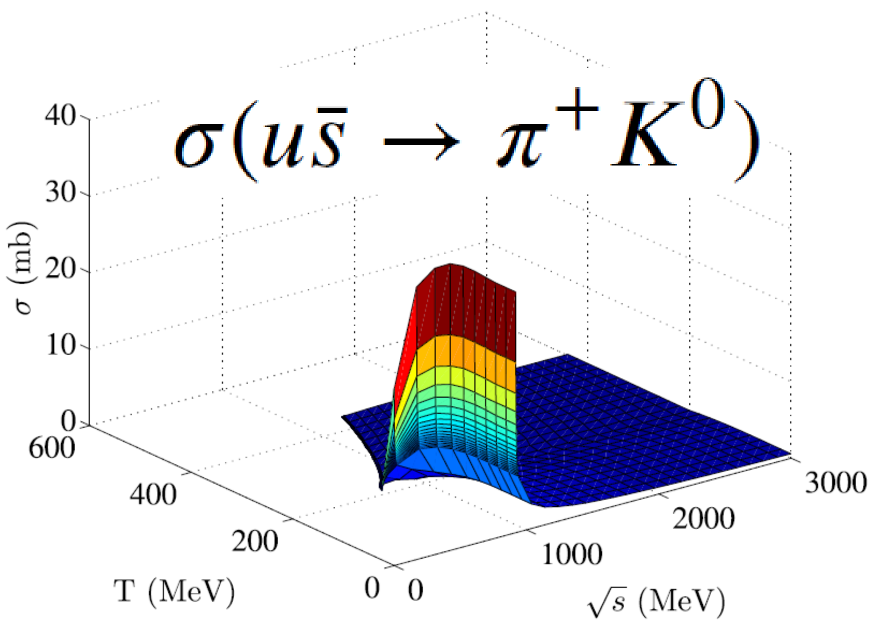
$\Gamma_{q_1 q_2 q_3}$ triangle vertex

Γ_ν appropriate γ matrix

Hadronization cross sections

These s-channel resonances create as well very large hadronization cross section close to T_c





Consequence:
If an expanding plasma comes to T_c quarks are converted into hadrons

despite of the NJL Lagragian does not contain confinement

How to make a transport theory out of NJL

Using 7 parameters fitted to ground state properties of mesons and baryons

the NJL model allows for calculating

Quark masses (T,μ)
Hadron masses (T,μ)
Elastic cross sections (T,μ)
Hadronization cross sections (T,μ)

So we have all ingredients for a transport theory

Problem:
With a mass of 2 MeV and temperatures $>$ 200 MeV the quarks move practically with the speed of light.

So we have to construct a fully relativistic transport theory

Relativistic Transport theory

Hamiltonian formulation

Hamilton-Jacobi eqs. : Eqs. for the time evolution of a particle in phase space (**p,q**)

$$\frac{d\mathbf{q}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

with Hamiltonian $\mathcal{H}(\mathbf{q}, \mathbf{p})$

On the trajectory of the particle the energy is conserved

Time evolution of observables A(p,q) :

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, \mathcal{H}\}$$

with

$$\{A, B\} = \sum_k^N \frac{\partial A}{\partial \mathbf{q}_k}\frac{\partial B}{\partial \mathbf{p}_k} - \frac{\partial A}{\partial \mathbf{p}_k}\frac{\partial B}{\partial \mathbf{q}_k}$$

$$\frac{d\mathbf{q}}{dt} = \{\mathbf{q}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \qquad \frac{\mathbf{dp}}{\mathbf{dt}} = \{\mathbf{p}, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

Problem: Hamilton eqs. cannot be extended to a relativistic approach:

$\mathbf{q}, \mathbf{p}, \mathcal{H}$ are components of 4 - vectors

eqs. cannot be Lorentz transformed

Relativistic Transport theory II

What we can extend to 4-vectors is

a) the Poison bracket:

$$\{A,B\} = \sum_{k=1}^{N} \frac{\partial A}{\partial q_k^\mu}\frac{\partial B}{\partial p_{k\mu}} - \frac{\partial A}{\partial p_k^\mu}\frac{\partial B}{\partial q_{k\mu}} \quad : \quad \{q_a^\mu, q_b^\nu\} = \{p_a^\mu, p_b^\nu\} = 0, \quad \{q_a^\mu, p_b^\nu\} = \delta_{ab}\, g^{\mu\nu}$$

b) the geometrical interpretation that

$$\frac{d\mathbf{q}}{dt} = \{\mathbf{q}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \quad \frac{\mathbf{dp}}{\mathbf{dt}} = \{\mathbf{p}, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}$$

describes the trajectory in the (p,q) phase space on which the Hamiltionain $\mathcal{H}(\mathbf{q},\mathbf{p})$ is conserved:

$$\boxed{\frac{dq^\mu(\tau)}{d\tau} = \lambda\{q^\mu(\tau), K\} \quad ; \quad \frac{dp^\mu(\tau)}{d\tau} = \lambda\{p^\mu(\tau), K\}}$$

describes the trajectory in the 8-dim phase space on which the Lorentz inv. quantity K is conserved.

τ is not a time but a parameter which characterizes the trajectory

Relativistic Transport theory III

Example: One free particle:
We need a trajectory in the 6+1 dimensional phase space (q,p,t)

Starting point : Choose 2 Lorentz invariant constraints $K = p_\mu p^\mu = m^2$ and $\chi(p_\mu, q_\mu, \tau) = 0$

$$\cancel{\frac{d\chi}{d\tau}} = \frac{\partial\chi}{\partial\tau} + \lambda\{\chi(\tau), K\} = 0 \Longrightarrow \lambda = -\frac{\partial\chi}{\partial\tau}\{\chi, K\}^{-1}$$

constraint

$$\frac{df}{d\tau} = \frac{\partial f}{\partial\tau} + \lambda\{f, K\} \Longrightarrow \frac{dq^\mu}{d\tau} = -\frac{\partial\chi}{\partial\tau}\frac{\{q^\mu, K\}}{\{\chi, K\}}$$

Free particle

$$\frac{dp^\mu(\tau)}{d\tau} = 0$$

All depends now on χ

$$a)\ \chi = q^0 - \tau = 0 \quad \rightarrow \frac{dp^\mu(\tau)}{d\tau} = \frac{p^\mu}{p^0}$$

$$b)\ \chi = x_\mu p^\mu - m\tau = 0 \rightarrow \frac{dp^\mu(\tau)}{d\tau} = \frac{p^\mu}{m}$$

Diff. χ → diff. eqs. of motion; τ is not time but parameter of trajectory

Before fixing constraints: rel. dynamics is incomplete

Relativistic Transport theory IV

This concept can be extended to N interacting particles (PRC87,034912)

$$K_i = p_{i\mu}p_i^\mu - m^2 + \sum_j V_{ij} = 0 \quad ; \quad \chi_{i<N}(q_1,..,q_N,p_1,..,p_N) \quad ; \quad \chi_N(q_1,..,q_N,p_1,..,p_N,\tau)$$

Synchronizes τ with q_i^0

with the eqs. of motion

$$\frac{dq_i^\mu(\tau)}{d\tau} = \{q_i^\mu(\tau), K_j\}S_{lj}\frac{d\chi_l}{d\tau}$$
$$\frac{dp_i^\mu(\tau)}{d\tau} = \{p_i^\mu(\tau), K_j\}S_{lj}\frac{d\chi_l}{d\tau}$$

with

$$S_{ij} = \{\chi_j, K_i\}^{-1}$$

The 2N constraints K_i, χ_i reduce the phasespace

$$8N\dim(q_\mu, p_\mu) \rightarrow (6N+1)\dim\,(\mathbf{q}, \mathbf{p}, \tau)$$

reduction not unique→ different eqs. of motions → different trajectories

Relativistic Transport theory IV

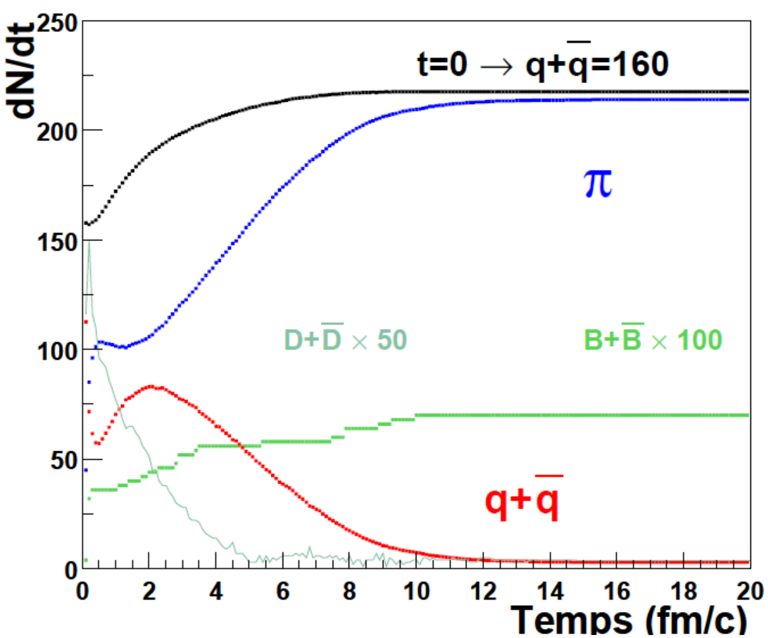
Eqs. of motion with the constraints (which assures cluster separability):

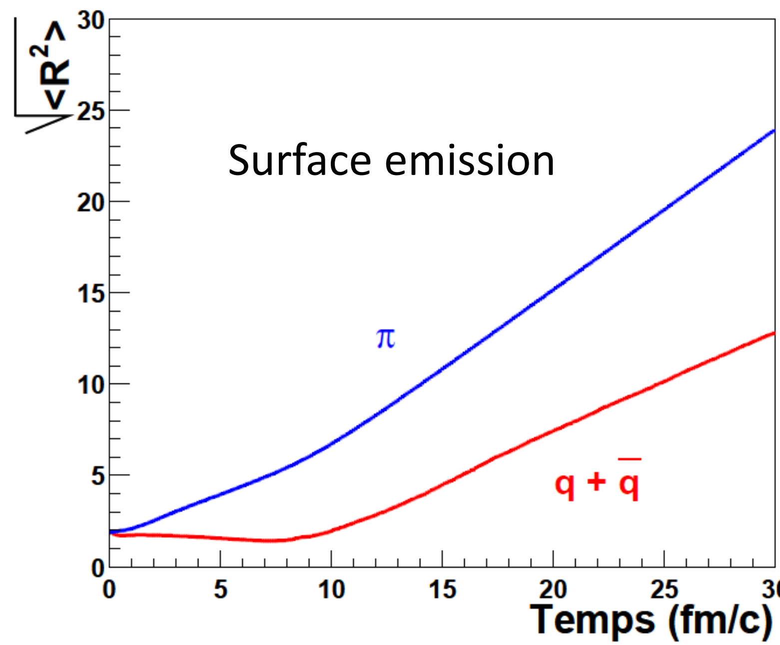
$$K_i = p_i^\nu p_{i\nu} - m_i^2 + V_i({q_T}^2) = 0 \qquad {q_T}^\mu_{ij} = q^\mu_{ij} - [(q_{ij})_\sigma u^\sigma_{ij}]u^\mu_{ij}$$

$$\chi_i = \frac{\sum_{j\neq i} q^\nu_{Tij}}{N} U_\nu = 0 \qquad U^\mu = \frac{P^\mu}{\sqrt{P^2}} \qquad u^\mu_{ij} = \frac{p^\mu_{ij}}{\sqrt{p^2_{ij}}}$$

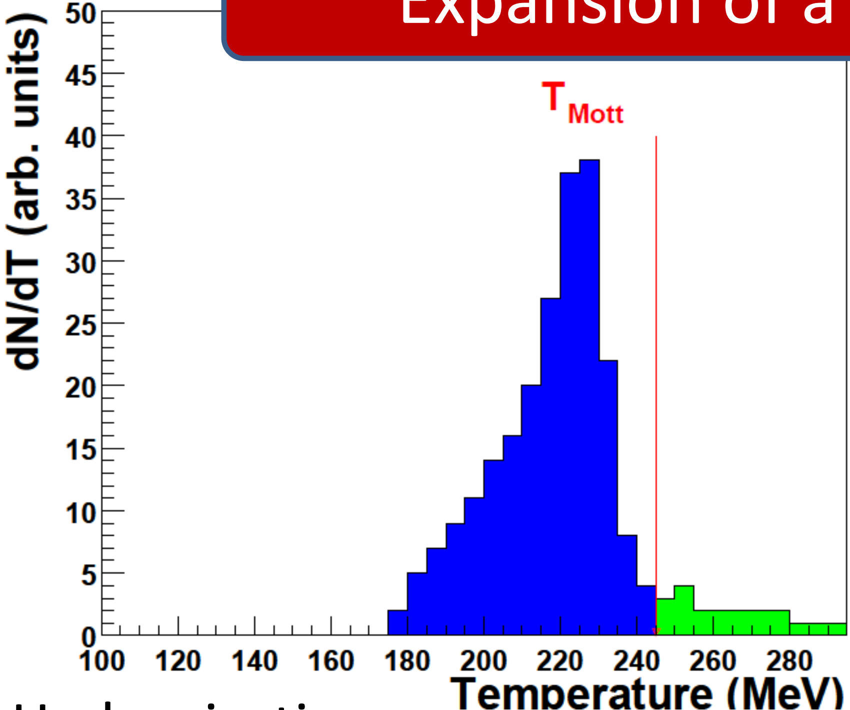
$$\chi_N = \frac{\sum_j q^\nu_{Nj}}{N} U_\nu - \tau = 0$$

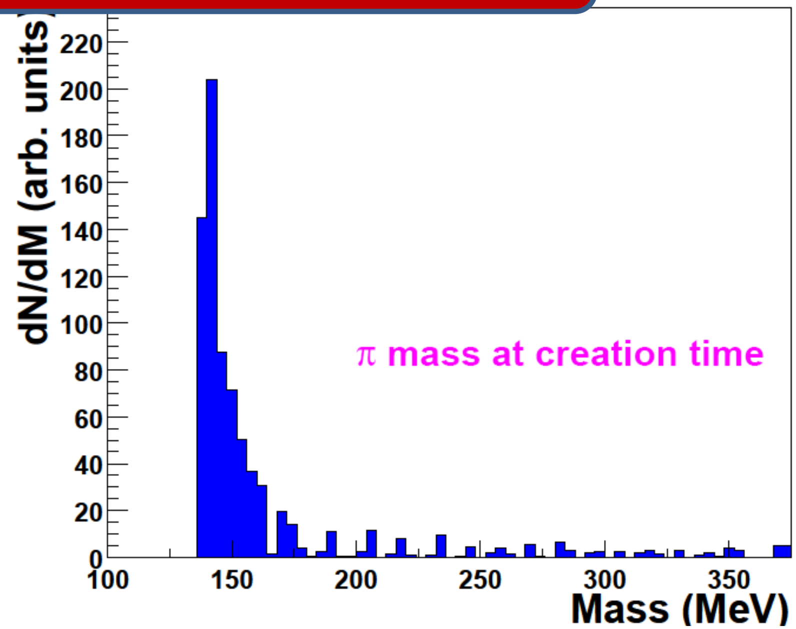
Time evolution of a thermal mini plasma





Expansion of a equilibrated plasma



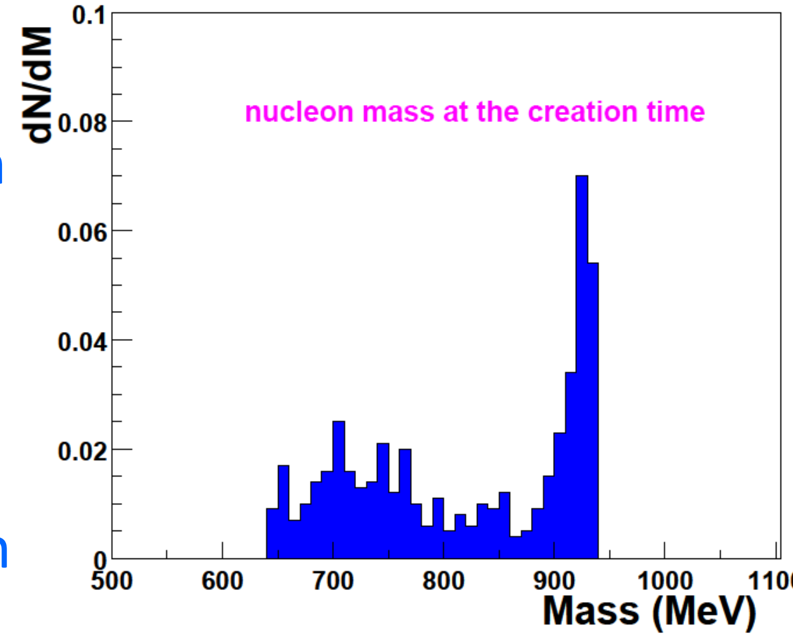


Hadronization:

Not at a fixed T but broad T distribution

Particles are produced over a wide mass range

Come to vacuum mass during expansion

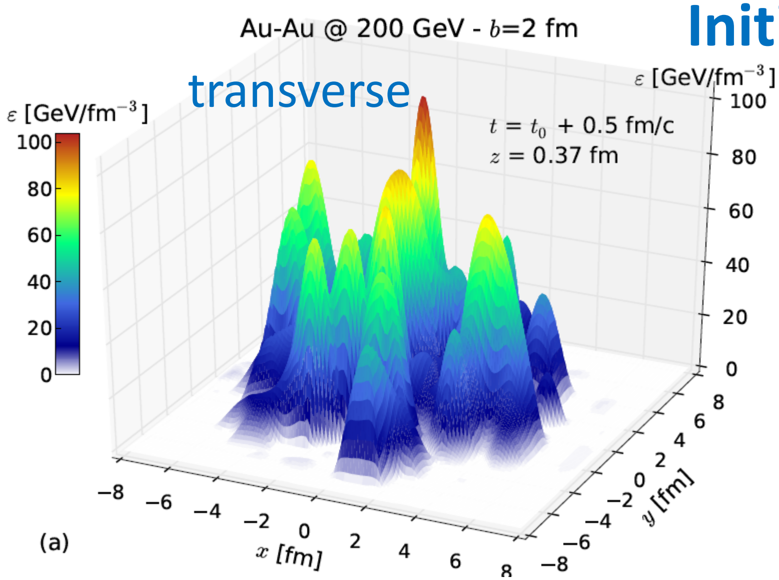


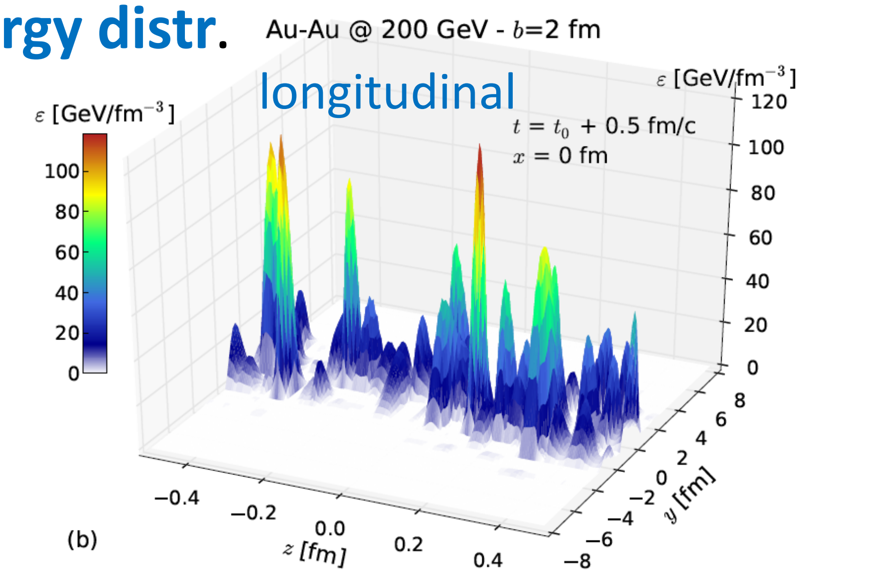
Expansion of a plasma

For realistic calculations we use the initial configuration of the PHSD approach and compare NJL with PHSD calculations

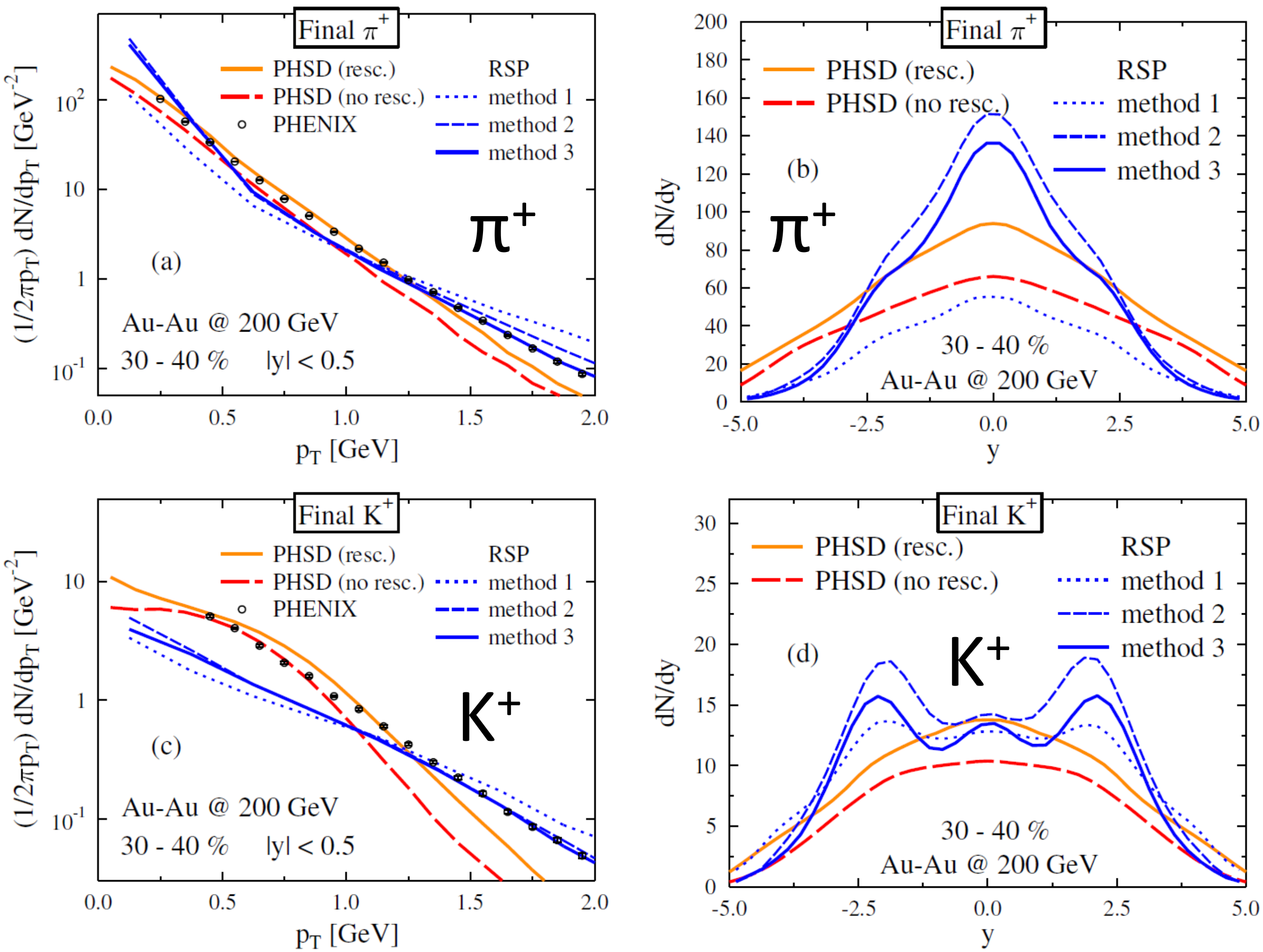
NJL	PHSD
$m_q \approx 5$ MeV	400 MeV $\leq m_q \leq$ 800 MeV
no gluons	gluons
g fix	g running
Hadronization by cross section $q\bar{q} \rightarrow m_1 + m_2$	$q\bar{q} \rightarrow m$ (or "string"); $qqq \rightarrow b$ (or "string")

Initial energy distr.



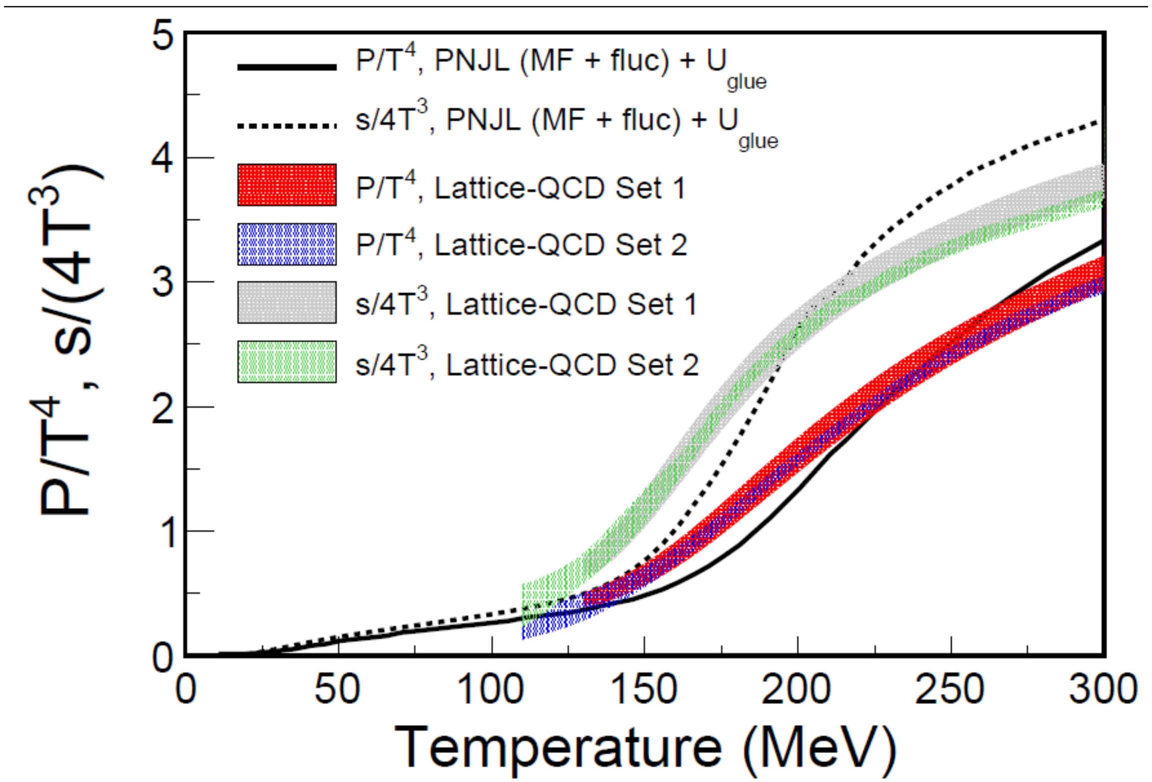


Expansion of a plasma with PHSD initial cond. I



Remember: Only parameters in the model are 7 well determined vacuum values

QCD-thermodynamics is not perfectly but reasonable well reproduced



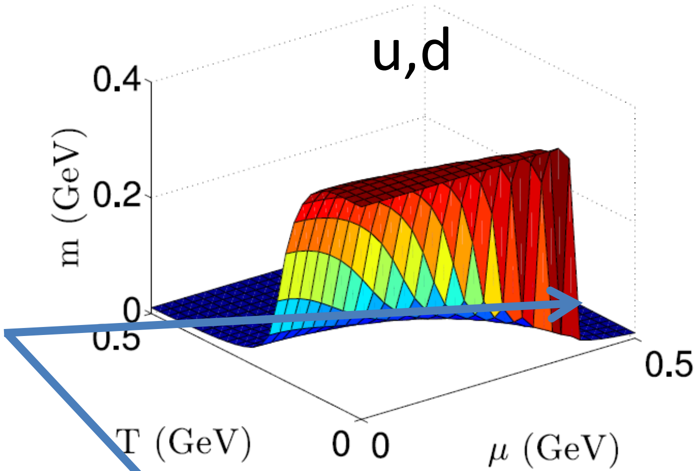
Perspectives

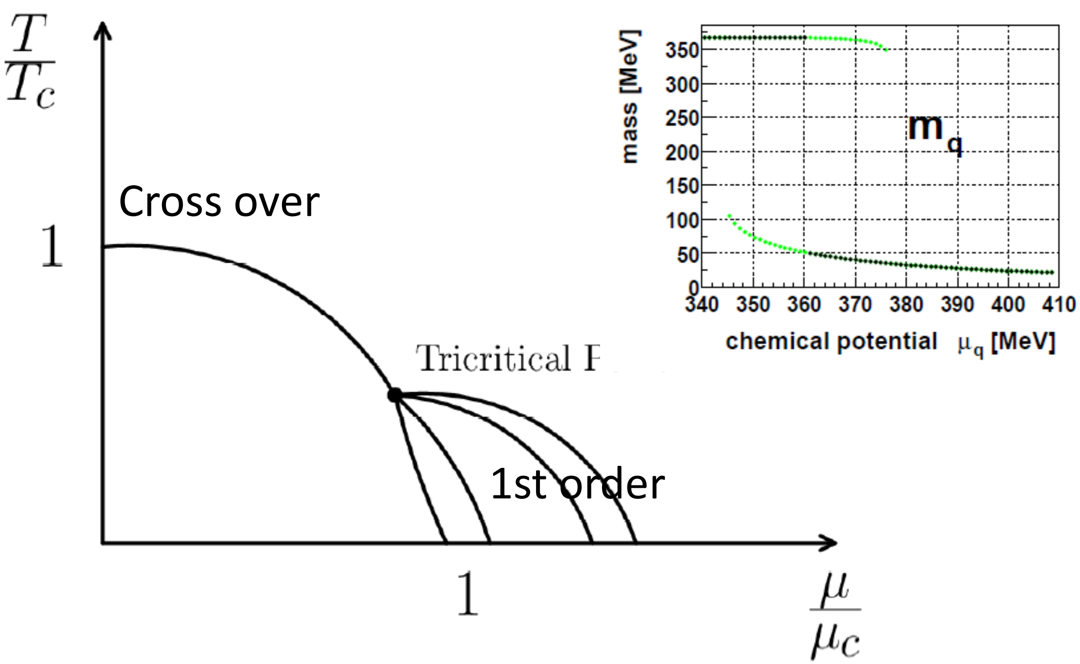
NJL Lagrangian:
transition between quarks and hadrons

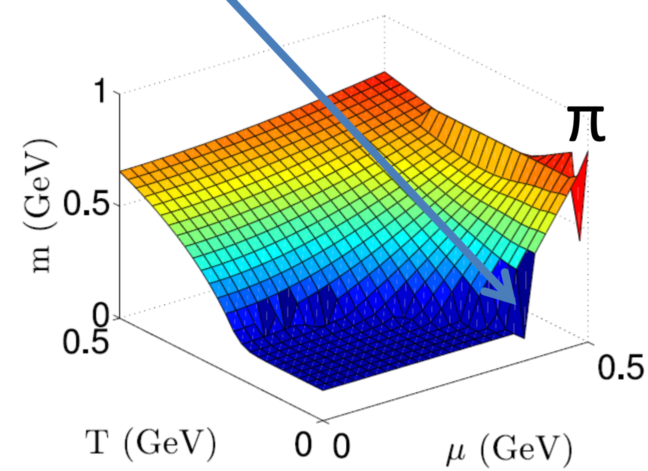
Cross over at μ = 0

1st order transition μ >>0

sudden change of q and meson mass

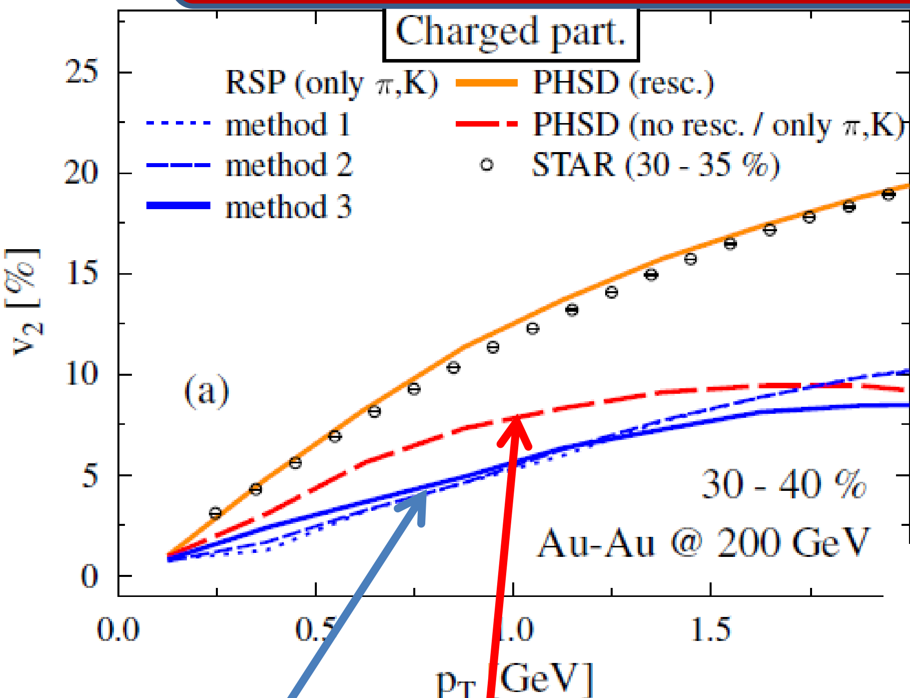






Details have not been explored yet

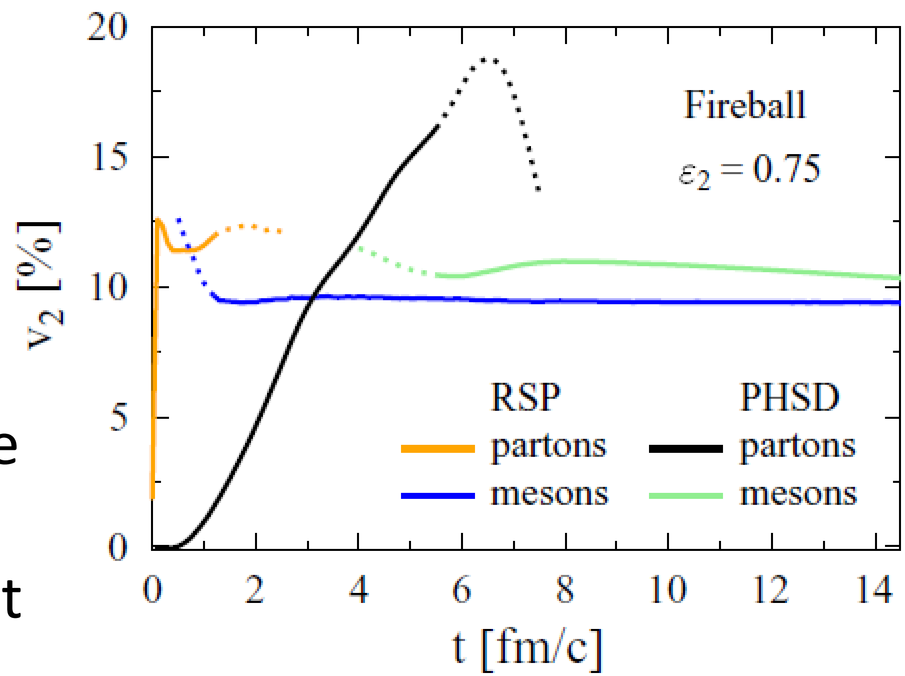
Expansion of a plasma with PHSD initial cond. I



Expanding almond shaped fireball as initial condition

NJL (RSP) has no hadronic rescattering

without rescattering
NJL (RSP) and PHSD have about the same v_2
Time evolution completely different



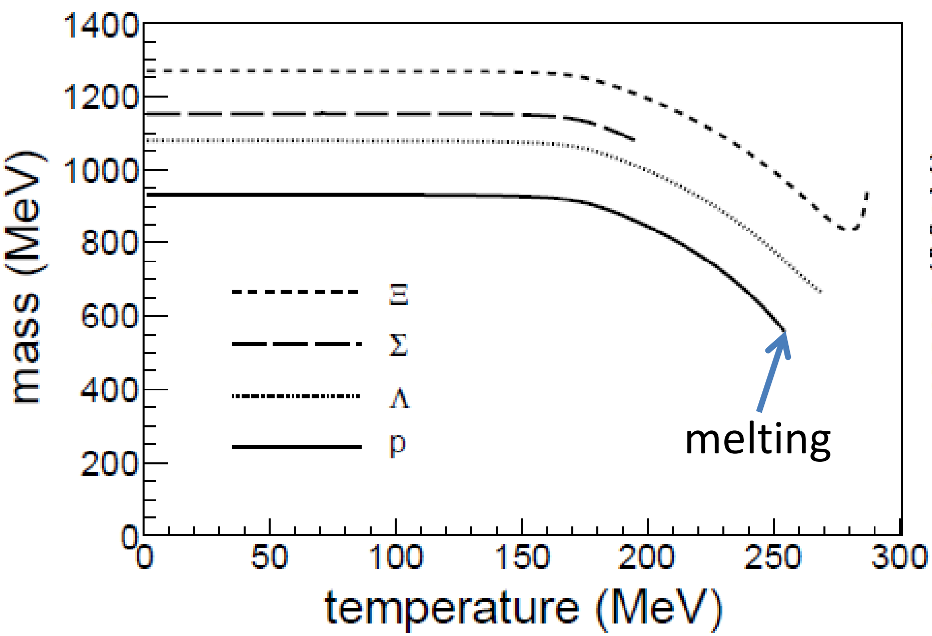
Baryons II

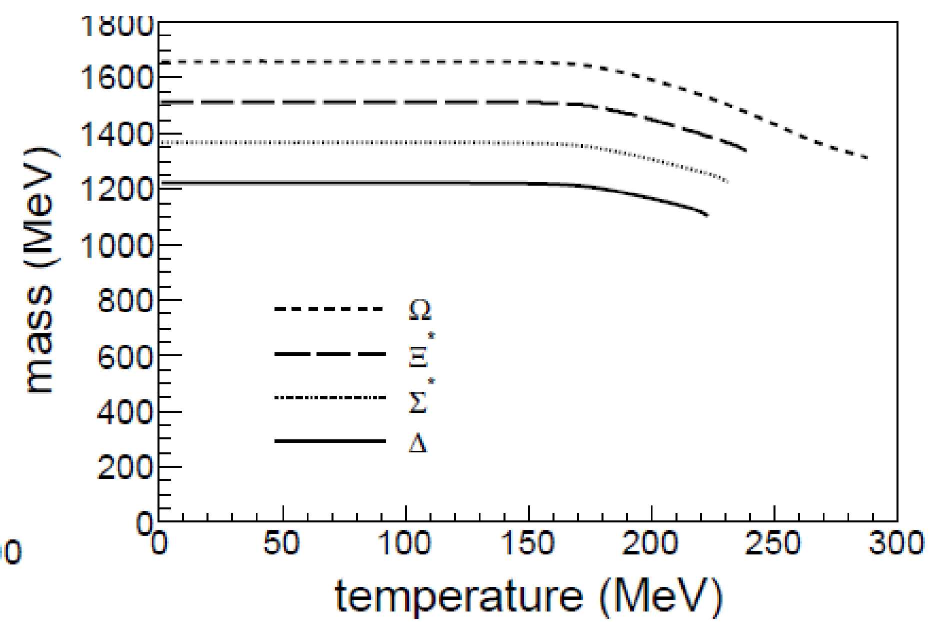
Omitting Dirac and flavour structure :

$$\left[1-\frac{2}{m_{quark}}\frac{1}{\beta}\sum_n\int\frac{d^3q}{(2\pi)^3}S_q(i\omega_n,q)\;t_D(i\nu_l-i\omega_n,-q)\right]\Bigg|_{i\nu_l\to P_0+i\epsilon=M_{Baryon}}=0$$

where we approximated the quark propagator for the exchanged quark by:

$$S_q(q)=\frac{1}{\not{q}-m_{quark}}\to-\frac{1\!\!1_{Dirac}}{m_{quark}}$$ 5% error (Buck et al. (92))





The more strange quarks the higher the melting temperature

Baryons II

Hadron	PDG mass (MeV)	PNJL mass (T=0) (MeV)	NJL T_c (MeV)	PNJL T_c (MeV)
π	136	135	267	282
K	495	492	271	286
p	939	932	234	254
Λ	1116	1078	252	269
Σ	1193	1152	156	195
Ξ	1318	1269	272	287
Δ	1232	1221	200	223
Σ^*	1383	1366	211	231
Ξ^*	1533	1512	219	239
Ω	1672	1658	275	288

With 5 parameters fixed to mesonic vacuum physics (+ 2diquark coupling const. for baryons)

(P)NJL can describe

the vacuum masses of all pseudoscalar mesons + all octet and decouplet baryons
with a precision of less than 5%

The T and μ dependence of all these hadrons

It predicts : melting temperature depends on the hadrons species

Polyakov NJL: gluons on a static level

Eur.Phys.J. C49 (2007) 213-217

It is not possible to introduce gluons as dynamical degrees of freedom without spoiling the simplicity of the NJL Lagrangian which allows for real calculations

but

one can introduce gluons through an effective potential for the Polyakov loop

$$\frac{\mathbf{U}(\mathbf{T},\Phi,\bar{\Phi})}{\mathbf{T}^4} = -\frac{\mathbf{b}_2(\mathbf{T})}{2}\bar{\Phi}\Phi - \frac{\mathbf{b}_3}{6}\left(\Phi^3 + \bar{\Phi}^3\right) + \frac{\mathbf{b}_4}{4}(\bar{\Phi}\Phi)^3$$

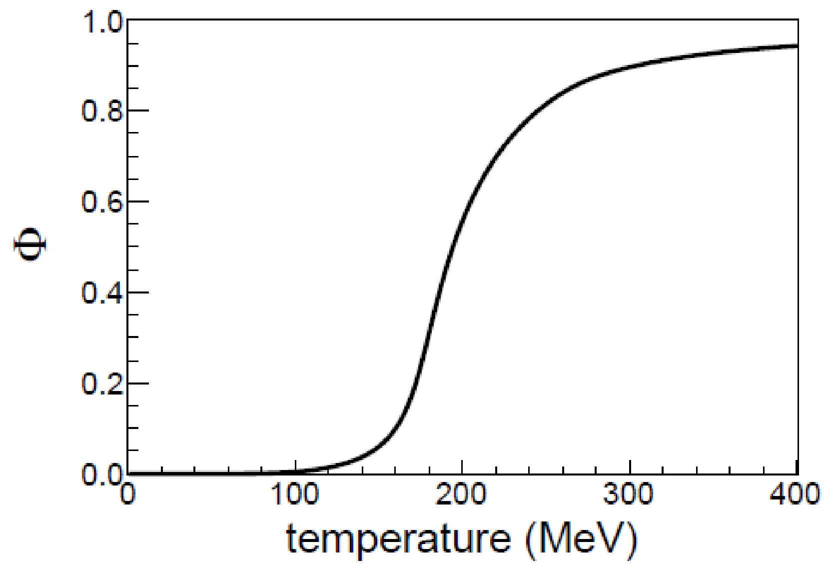
$$\mathbf{b}_2(\mathbf{T}) = \mathbf{a}_0 + \mathbf{a}_1\frac{\mathbf{T}_0}{\mathbf{T}} + \mathbf{a}_2\left(\frac{\mathbf{T}_0}{\mathbf{T}}\right)^2 + \mathbf{a}_3\left(\frac{\mathbf{T}_0}{\mathbf{T}}\right)^3$$

$$\mathbf{a}_0 = 6.75, \mathbf{a}_1 = -1.95, \mathbf{a}_2 = 2.625, \mathbf{a}_3 = -7.44, \mathbf{b}_3 = 0.75, \mathbf{b}_4 = 7.5$$

Parameters-> right pressure in the SB limit

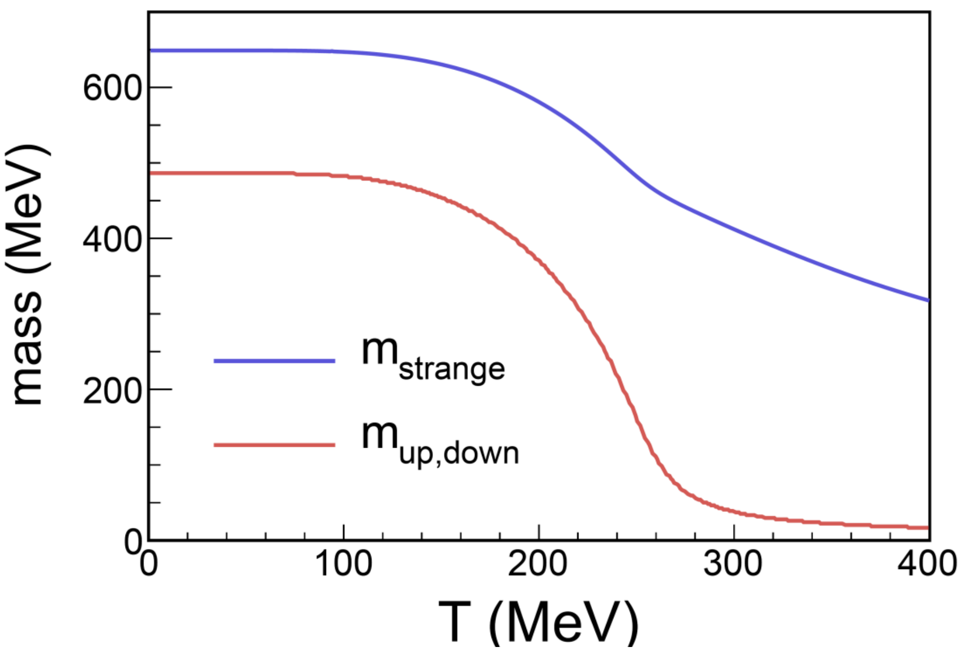
Φ is the order parameter of the deconfinement transition

$$\Phi = \frac{1}{\mathbf{N}_c}\,\mathrm{Tr}_c\,\left\langle \mathbf{P}\exp\left(-\int_0^\beta \mathbf{d}\tau\mathbf{A}_0(\mathbf{x},\tau)\right)\right\rangle$$

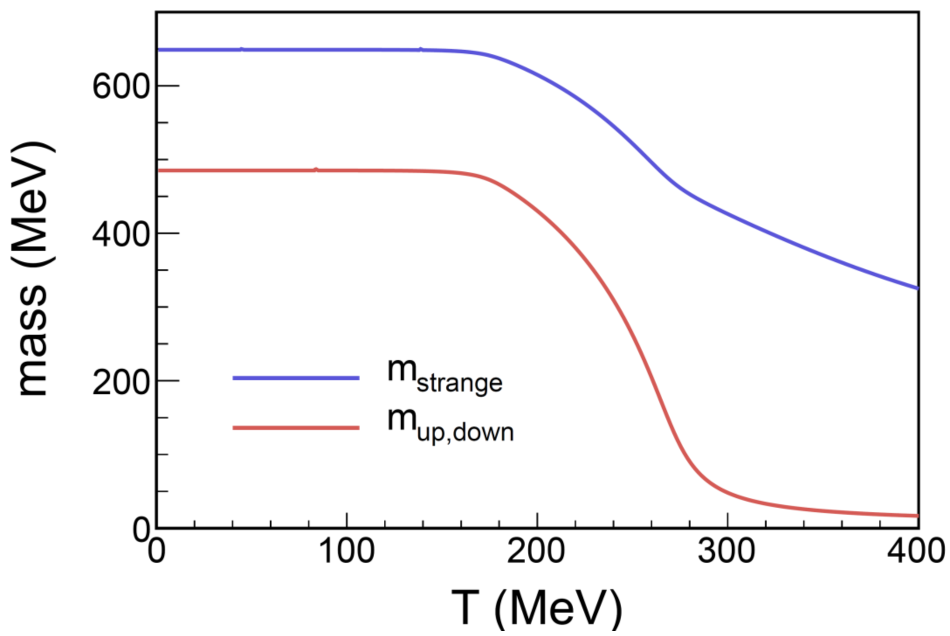


Quark Masses in NJL and PNJL

NJL



PNJL



In PNJL the transition is steeper than in NJL

From QCD to the NJL Langrangian I

$$\boxed{L_{QCD}(x) = \bar{\psi}(x)\left(i\gamma^{\mu}\left[\partial_{\mu} - igt^{a}A^{a}_{\mu}\right] - \hat{M}^{0}\right)\psi(x) - \frac{1}{4}G^{a}_{\mu\nu}(x)G^{\mu\nu\, a}(x)}$$

- Euler-Lagrange equations:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_{\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi)}\right] = 0} \qquad (4)$$

for any field φ (the same equation for $\bar{\varphi}$): e.g. $\varphi = \Psi(x)$ or $A^{a}_{\mu}(x)$.

1) Consider quark field $\bar{\Psi}(x)$

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_{\mu}\left[\cancel{\frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\bar{\Psi})}}\right] = 0 \qquad (5)$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = 0,$$

since the second term in Eq.(4) is equal to zero while no terms with $\partial^{\mu}\bar{\Psi}$.

From eqs.(1,5) follows that

$$(i\gamma^{\mu}\partial_{\mu} - \hat{M}^{0})\Psi_{q}(x) = -g\gamma^{\mu}t^{a}A^{a}_{\mu}(x)\Psi_{q}(x)\ . \qquad (6)$$

From QCD to the NJL Langrangian II

$$\boxed{L_{QCD}(x) = \overline{\psi}(x)\left(i\gamma^{\mu}\left[\partial_{\mu} - igt^{a}A_{\mu}^{a}\right] - \hat{M}^{0}\right)\psi(x) - \frac{1}{4}G_{\mu\nu}^{a}(x)G^{\mu\nu\, a}(x)} \qquad (1)$$

2) Consider field $A_\nu^a(x)$:

Euler-Lagrange equation for gluon field:

$$\frac{\mathcal{L}}{\partial A_\nu^a(x)} - \partial_\mu \left[\frac{\mathcal{L}}{\partial(\partial^\mu A_\nu^a(x))}\right] = 0. \qquad (7)$$

- Using (1) → first term in eq. (7): $$\frac{\mathcal{L}}{\partial A_\nu^a(x)} = g\bar{\Psi}\gamma_\nu t^a \Psi + \prod\nolimits_g, \qquad (8)$$

  where Π_g is the ‚self-energy' of gluons: $$\prod\nolimits_g = \frac{\partial}{\partial A_\nu^a}\left[-\frac{1}{4}G_{\mu\nu}^a(x)G^{\mu\nu\, a}(x)\right] \qquad (9)$$

$$G_{\mu\nu}^a(x) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b(x)A_\nu^c$$

- Using (1) → second term in eq. (7): $$\frac{\mathcal{L}}{\partial(\partial^\mu A_\nu^a(x))} = \partial^\mu A_\nu^a(x) \qquad (10)$$

- Substitute (8), (10) into (7): $$\partial_\mu \partial^\mu A_\nu^a(x) = -g\bar{\Psi}\gamma_\nu t^a \Psi - \prod\nolimits_{g,\nu} \qquad (11)$$

From QCD to the NJL Langrangian III

- Approximation: scalar terms dominates and is positive: $\Pi_{g,\nu} = M_g^2$

constinuent gluon mass $\neq 0$ due to self-interactions of gluons.

Then from eq. (11) ➔

$$\partial_\mu \partial^\mu A_\nu^a(x) = -g\bar{\Psi}\gamma_\nu t^a \Psi - \prod_{g,\nu}$$

$$\boxed{\left(\partial_\mu \partial^\mu + M_g^2\right) A_\nu^a(x) \approx -g\bar{\Psi}\gamma_\nu t^a \Psi,} \tag{12}$$

- Solution of eq. (12):

$$A_\nu^a(x) = -\int d^4x\, G(x-x')\, g\bar{\Psi}(x')\gamma_\nu t^a \Psi(x') \tag{13}$$

Green function:

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - M_g^2} \tag{14}$$

- Approximation: consider <u>low energy physics</u> : $\boxed{q^2 \ll M_g^2}$
  i.e. small momentum or large distance

In this limit:

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \left.\frac{e^{-iq(x-x')}}{q^2 - M_g^2}\right|_{q^2\to 0} \longrightarrow \approx \frac{1}{M_g^2} \underbrace{\int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')}}_{\delta(x-x')} \tag{15}$$

From QCD to the NJL Langrangian III

- Approximation: scalar terms dominates and is positive: $\Pi_{g,\nu} = M_g^2$

  constinuent gluon mass $\neq 0$ due to self-interactions of gluons.

Then from eq. (11) ➔

$$\partial_\mu \partial^\mu A_\nu^a(x) = -g\bar{\Psi}\gamma_\nu t^a \Psi - \prod_{g,\nu}$$

$$\boxed{\left(\partial_\mu \partial^\mu + M_g^2\right) A_\nu^a(x) \approx -g\bar{\Psi}\gamma_\nu t^a \Psi,} \qquad (12)$$

- Solution of eq. (12):

$$A_\nu^a(x) = -\int d^4x\, G(x-x')\, g\bar{\Psi}(x')\gamma_\nu t^a \Psi(x') \qquad (13)$$

Green function:

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - M_g^2} \qquad (14)$$

- Approximation: consider low energy physics : $\boxed{q^2 \ll M_g^2}$

  i.e. small momentum or large distance

In this limit: (15)

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - M_g^2}\bigg|_{q^2\to 0} \longrightarrow \approx \frac{1}{M_g^2} \underbrace{\int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')}}_{\delta(x-x')}$$

From QCD to the NJL Langrangian IV

From eq. (15) ➔ $$G(x-x') \Rightarrow \underbrace{\delta^4(x-x')}_{local\ \ interaction} \cdot \underbrace{M_g^{-2}}_{const}. \quad (16)$$

- Substitute (16) into (13): $$A_\nu^a(x) = -\frac{g}{M_g^2}\bar{\Psi}(x)\gamma_\nu t^a \Psi(x). \quad (17)$$

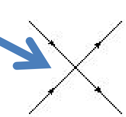
- Substitute (17) into (6):

$$(i\gamma^\mu\partial_\mu - \hat{M}^0)\Psi(x) - G_c^2\gamma^\mu t^a \underbrace{\left(\bar{\Psi}(x)\gamma_\mu t^a\Psi(x)\right)}_{\sim A_\mu^a(x)}\Psi(x) = 0 \quad (18)$$

where the low energy coupling constant: $$G_c^2 = g^2/M_g^2 \quad (19)$$

$$\mathcal{L}_{eff} = \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - \hat{M}^0)\Psi(x) - \underbrace{G_c^2\sum_{a=1}^{8}\left(\bar{\Psi}(x)\gamma^\mu t^a\Psi(x)\right)^2}_{local\ \ color\ \ current\ \ interaction}. \quad (20)$$

NJL Lagrangian



interaction between gluons -> approximated by a gluon mass M_g

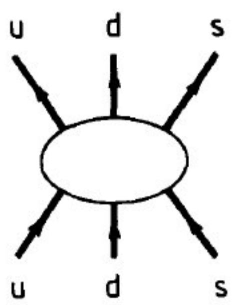
$q^2 < M_g$

U_A (1) symmetry breaking

U_A (1) symmetry is broken (quantum fluctuations violate axial current conservation)

The breaking of the U_A (1) symmetry can be obtained by adding the 't Hooft Lagrangian

$$\mathscr{L}_{'\mathbf{t\ Hooft}} = \mathbf{H} \det_{\mathbf{ij}} \, [\bar{\Psi}_\mathbf{i}(\mathbf{1} - \gamma_5)\Psi_\mathbf{j}] - \mathbf{H} \det_{\mathbf{ij}} \, [\bar{\psi}_\mathbf{i}(\mathbf{1} + \gamma_5)\psi_\mathbf{j}]$$



For N_f = 3:
Six point interaction
taking into account on the mean field level

H is determined by the experimental η-η'
mass gap

Brief survey on thermal field theory

How to calculate physical quantities at final temperature and final chemical potential ?

Imaginary time formalism (one introduces $0 \leq it \leq \beta = \frac{1}{T}$ $(T = temperature)$)

In all momentum space integrals replace

$$\mathbf{k_0 \to i\omega_n}, \qquad \int \frac{d^4k}{(2\pi)^4} \to iT \sum_{n\in Z} \int \frac{d^3k}{(2\pi)^3}$$

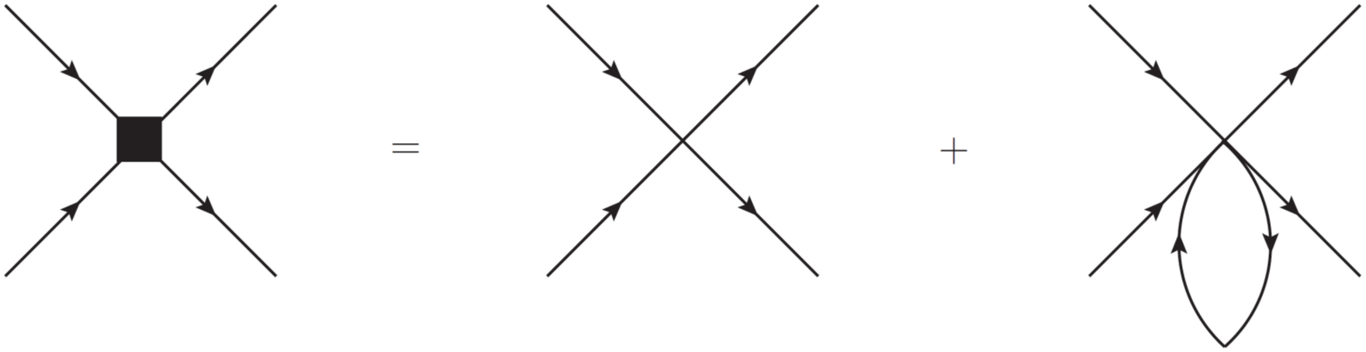
With the fermionic Matsubara frequencies $\mathbf{i\omega_n = i\pi T(2n+1)}$

A chemical potential can be introduced by the Lagragian

$$\mathscr{L}_\mu = \sum_{ij} \bar{\psi}_i \mu_{ij} \gamma_0 \psi_j \qquad\qquad \mu_{ij} = \mathrm{diag}\,(\mu_u, \mu_d, \mu_s)$$

How to get mesons? II

For calculations : Include the ‘t Hooft term to an effective coupling



$$G_{eff} = G + \frac{1}{2} H \langle \bar{s}s \rangle$$

e an eff. coupling constant

$$\mathscr{K} = \Omega \; 2G_{eff} \; \bar{\Omega}$$

e interaction kernel

n contains color, flavour and Dirac matrices

$$\Omega = \mathbb{1}_c \otimes \tau^a \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\}$$

Diquarks – the road to baryons I

The Fierz transformation produces also a term for scalar diquarks

$$\mathscr{L}_{qq} = G_{DIQ}\,(\bar{\Psi}_i \tau_A t_{A'} i\gamma_5 C \bar{\Psi}_k^T)(\Psi_j^T \tau_A t_{A'} C i\gamma_5 \Psi_l)\,, \qquad G_{DIQ} = (N_c + 1)g/(2N_c)$$

$C = i\gamma_0\gamma_2;\, t_a, \tau_a$: Antisymmetric SU(3) matrices in color and flavour

as well as for axial diquarks.
Mass is determined like for mesons (Bethe Salpeter equation with elementary interaction kernel)

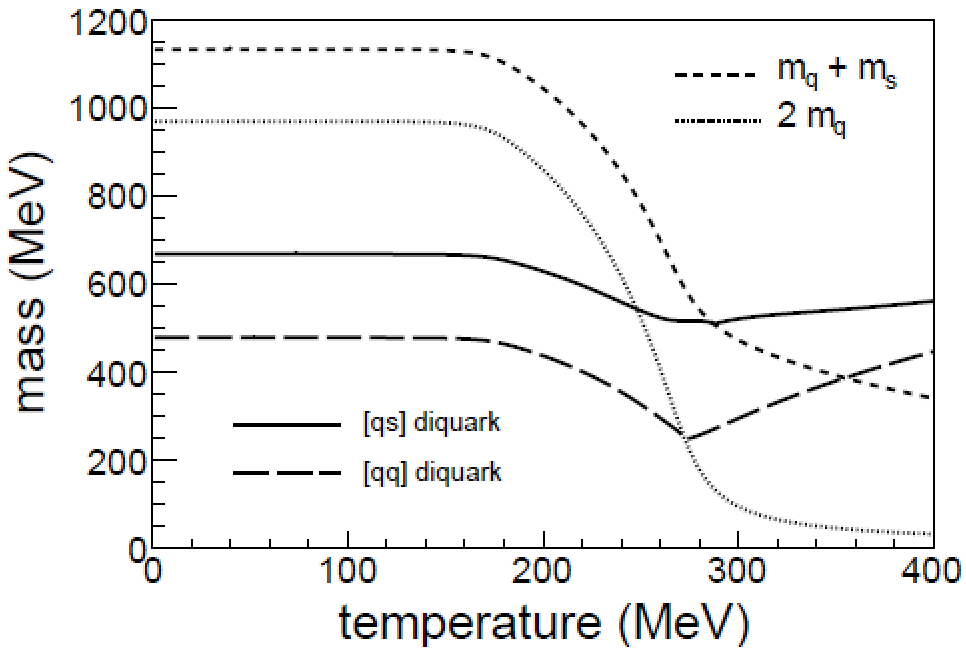
$$T(p) = \frac{2G_{DIQ}}{1 - 2G_{DIQ}\Pi(p)}$$

$$\Pi(p) = i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\bar{\Omega}\, S\left(k + \frac{p}{2}\right)\Omega S^T\left(\frac{p}{2} - k\right)\right]$$

$$\Omega = \text{color} \otimes \text{flavour} \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\}$$

Diquarks – the road to baryons II

Scalar diquarks



diquarks are bound

T_c [qq] = 256 MeV
T_c [qs] = 273 MeV

Strange diquarks melt at higher temperature

Diquarks form together with a quark the baryons

$$\mathbf{3} \otimes (\bar{\mathbf{3}} \oplus \mathbf{6}) = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$$

quark antitriplet scalar diquark sextet axial diquark

QCD: The theory which contains the solution

$$L_{QCD}(x) = \bar{\psi}(x)\left(i\gamma^{\mu}\left[\partial_{\mu} - igt^{a}A^{a}_{\mu}\right] - \hat{M}^{0}\right)\psi(x) - \frac{1}{4}G^{a}_{\mu\nu}(x)G^{\mu\nu a}(x)$$

$$\psi(\mathbf{x}) \rightarrow e^{i\alpha_{a}(x)t_{a}}\psi(\mathbf{x})$$

Gluonic field strength tensor:

$$G^{a}_{\mu\nu}(x) = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}(x)A^{c}_{\nu}$$

$\psi(x)$ - quark field

flavor space	***Dirac space***	***color space***
$q = u, d, s$	$\mu = 0,1,2,3$	$c = r, b, g$

In flavor space (3 flavors):

$$\psi(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Mass term:

$$\hat{M}^{0} = \begin{pmatrix} m^{0}_{u} & 0 & 0 \\ 0 & m^{0}_{d} & 0 \\ 0 & 0 & m^{0}_{s} \end{pmatrix}$$

3x3 diagonal matrix in flavor space with the bare quark masses on the diagonal