Study of Dynamic Aperture of Shielded Solenoids using Symplectic Maps

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Solenoid Fringe Fields

- Leakage of B field outside of solenoid
- Shield solenoid to reduce field
  - Quenching of neighbouring solenoids
- Affects beam dynamics…

Solenoid Dynamics

- Lagrangian $\mathcal{L} = -\gamma m_0 c^2 + q\vec{v} \cdot \vec{A}$
- Rotational symmetry
  - Vector potential $\vec{A}$ has the expansion
  - $A_z = A_r = 0; \qquad A_\phi = R(r^2)Z(z)$
  - R always same. Entirely specified by $Z(z)$
- Hamiltonian $H = \sqrt{m^2 + (p_x - qA_x)^2 + \left(p_y - qA_y\right)^2 + p_z^2}$
  - $p_x = \gamma m_0 \dot{x} + qA_x$
- Conservation of Canonical Angular momentum
  - $L_{canon} = xp_y - yp_x$

Solenoid Dynamics

- Expand H up to 4th order in x, y, p_x, p_y
- Many terms…
  - 2 terms depend on $b_2 = \frac{\partial^2 B}{\partial z^2}$
  - Sharp field yields high b_2
  - Shielding → greater nonlinear effects
  - Study beam stability

Method

- Use a transfer map to look at emittance growth
- Map must conserve expected quantities
  - Energy, canonical momentum…
  - Artificial emittance growth
- Can constrain map for each quantity
  - Causes unexpected behaviour
- Solution: Symplectic Maps

Symplectic Maps

- Any map that preserves Poisson brackets
  - $[Mf, Mg] = M[f, g]$
- Conserves Hamilton's eq.s of motion
  - No artificial energy growth
- $J^T \Omega J = \Omega$, M is Jacobian
  - $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \left[u^i, u^j\right]_{PB}$
- Large number of non-linear conditions on M

Symplectic Maps

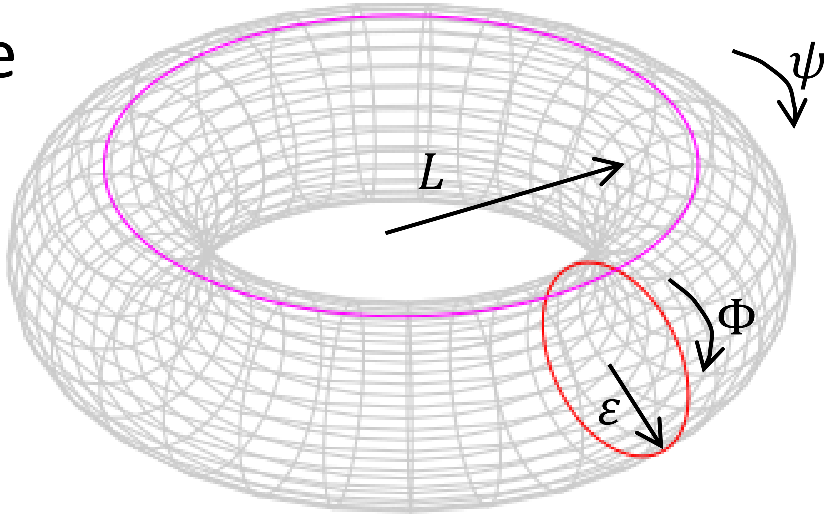
- Instead use Lie algebraic techniques (Dragt, 1988)
  - Symplectic Maps form a Lie Group
  - Can be generated through Lie algebraic elements
- Dragt shows that $\dot{M} =: -H: M$
  - $:f:g = [f,g]_{PB}$
  - Think of a “tangent map” at the identity

Lie Algebraic Techniques

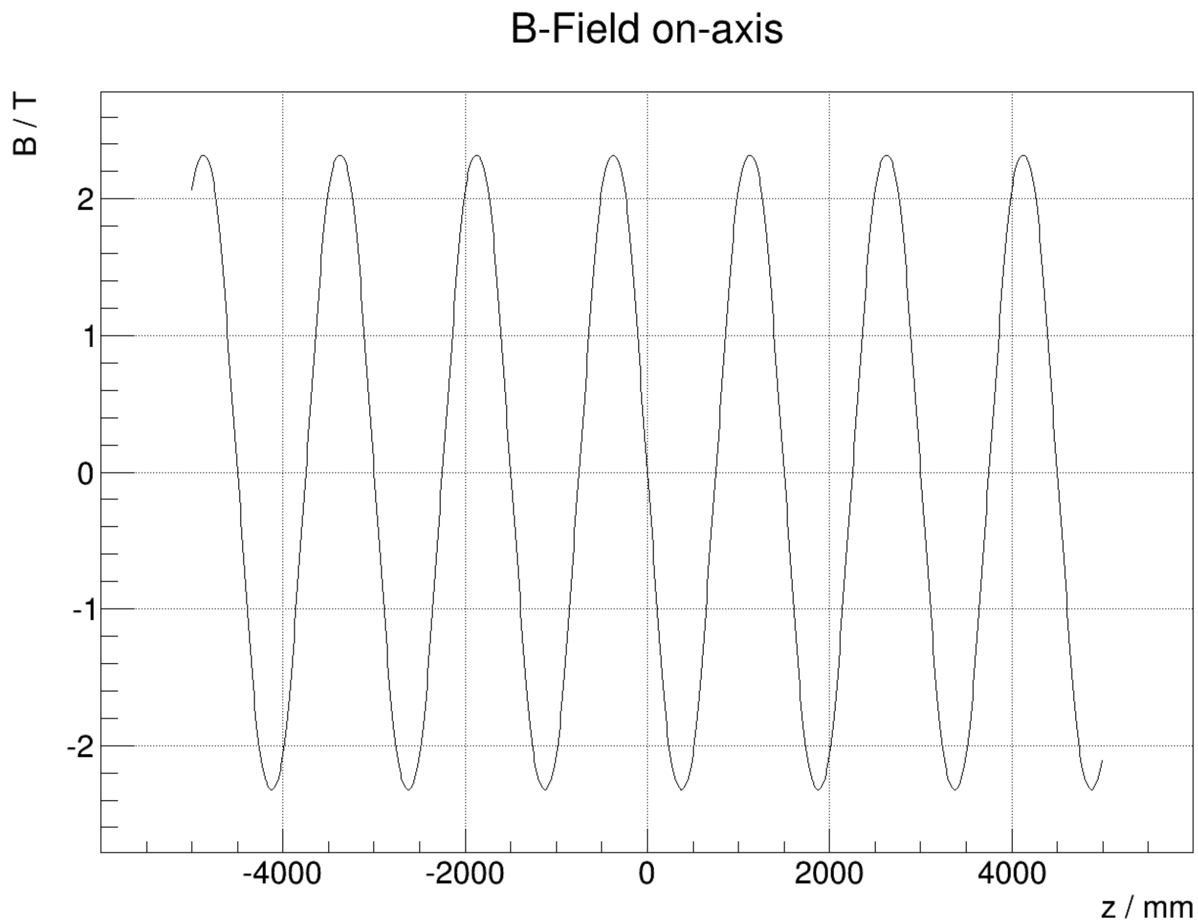
- exp : Lie algebra -> Lie group
- $\exp{:f:} = 1 + :f: + \frac{:f:^2}{2!} + \cdots$
- Use factorisation theorem
  - $M = \cdots M_3 M_2 M_1$
    $= \cdots \exp{:f_4:} \exp{:f_3:} \exp{:f_2:}$
- Homogenous polynomials f_i
- Related to Hamiltonian expansion
  - $H = H_2 + H_3 + H_4 + \cdots$

Expected Behaviour

- Perturbation on phase space ellipse
- Terms $1 + :f: + \frac{:f:^2}{2!} + \cdots$ include 1st, 3rd, 5th … order terms
- As 5th order terms come in, loose accuracy
  - Make plots in ϵ-Φ plane
  - Vary L

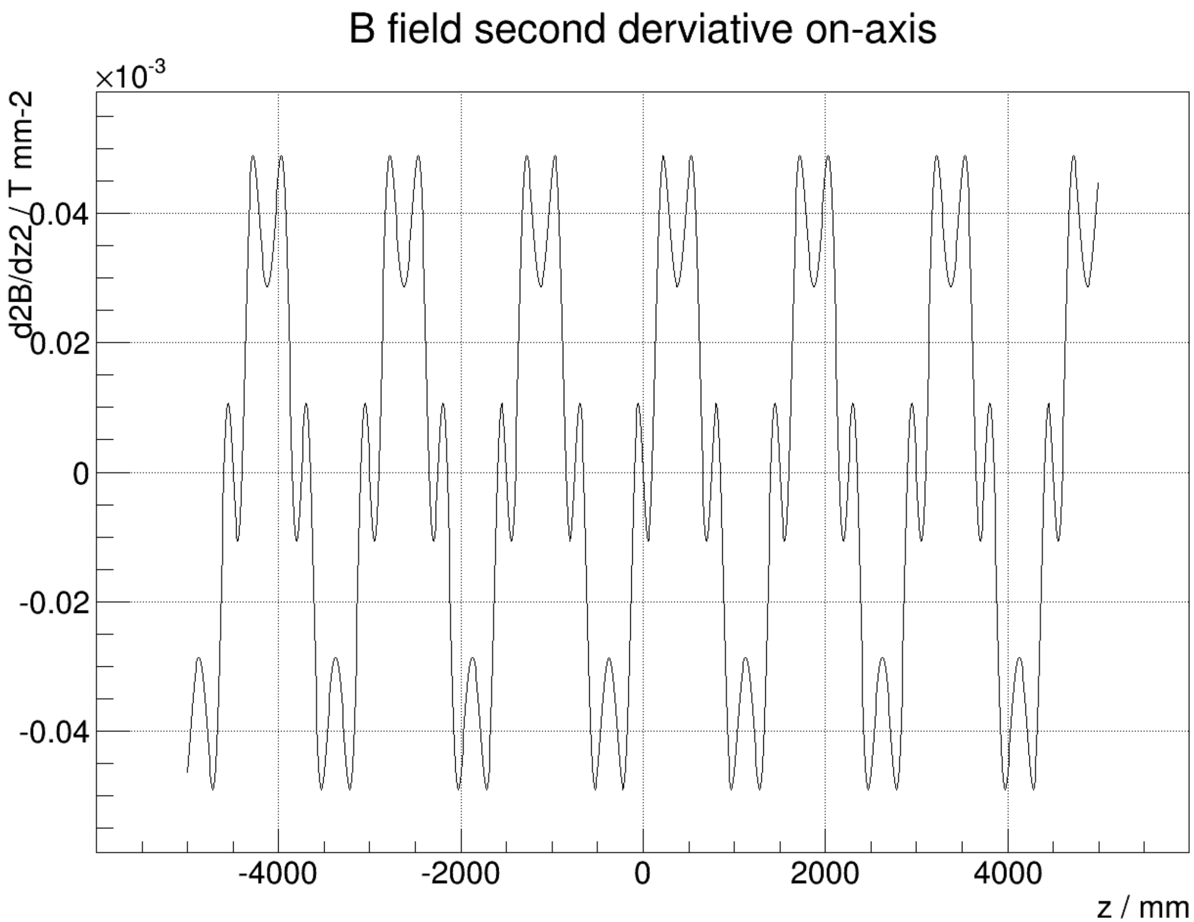


Lattice



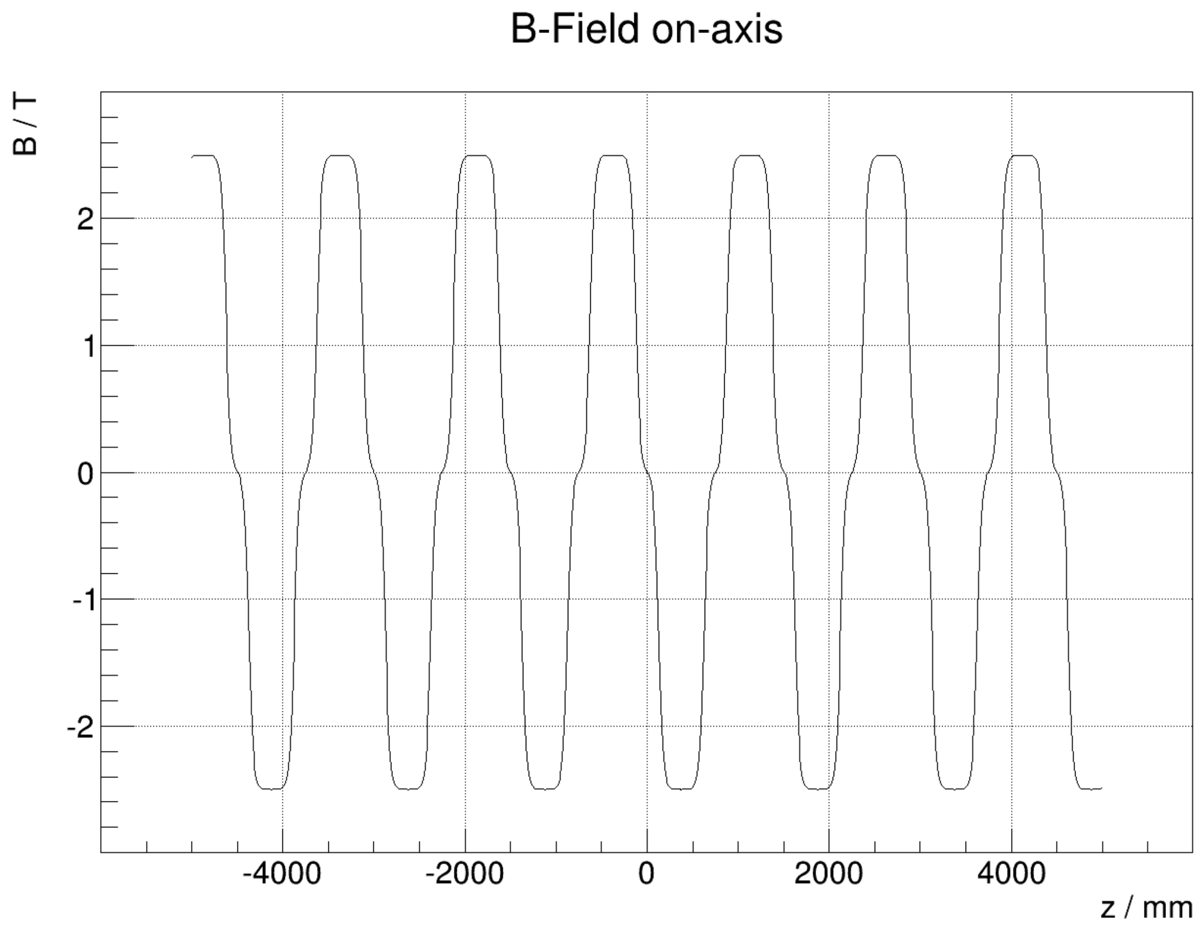
$$\lambda = 150\text{mm}$$

Lattice



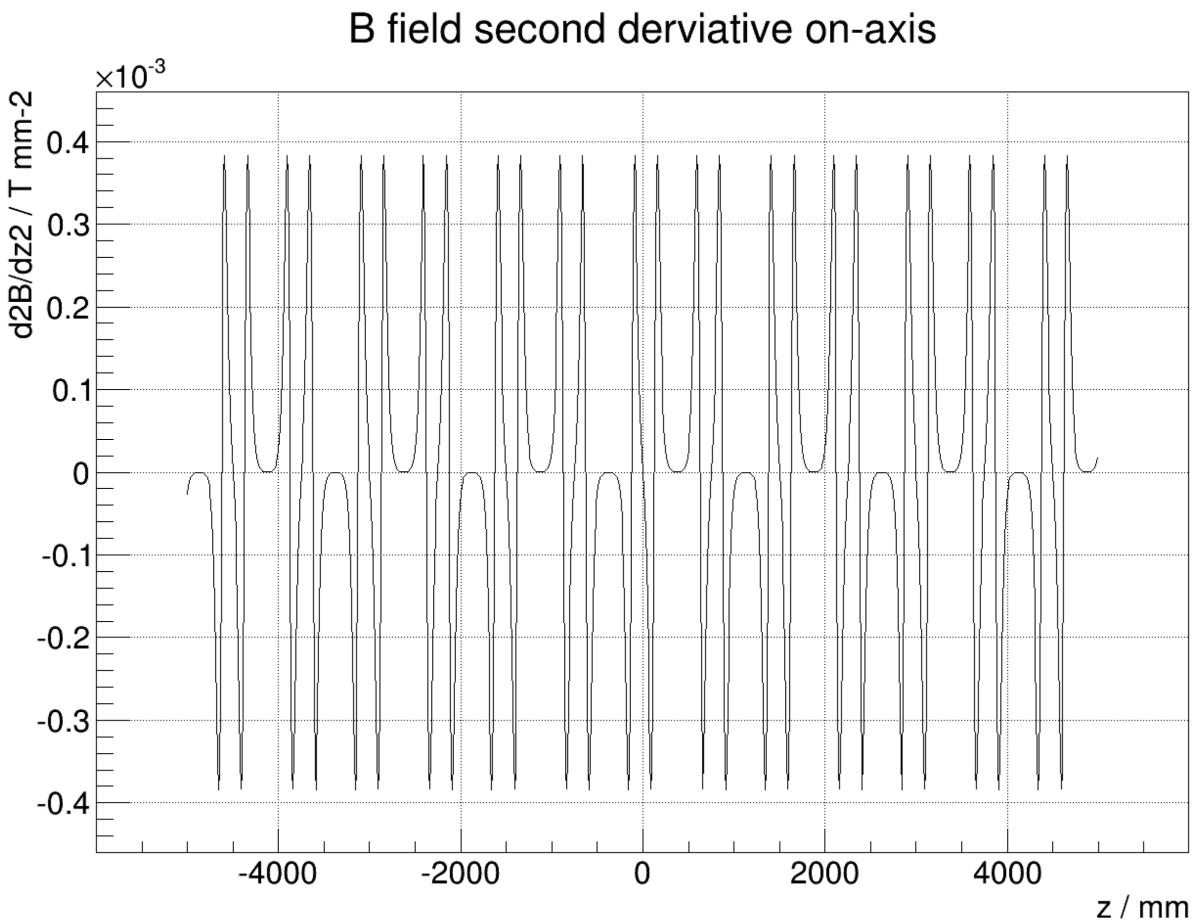
$\lambda = 150\text{mm}$

Lattice



$\lambda = 50\text{mm}$

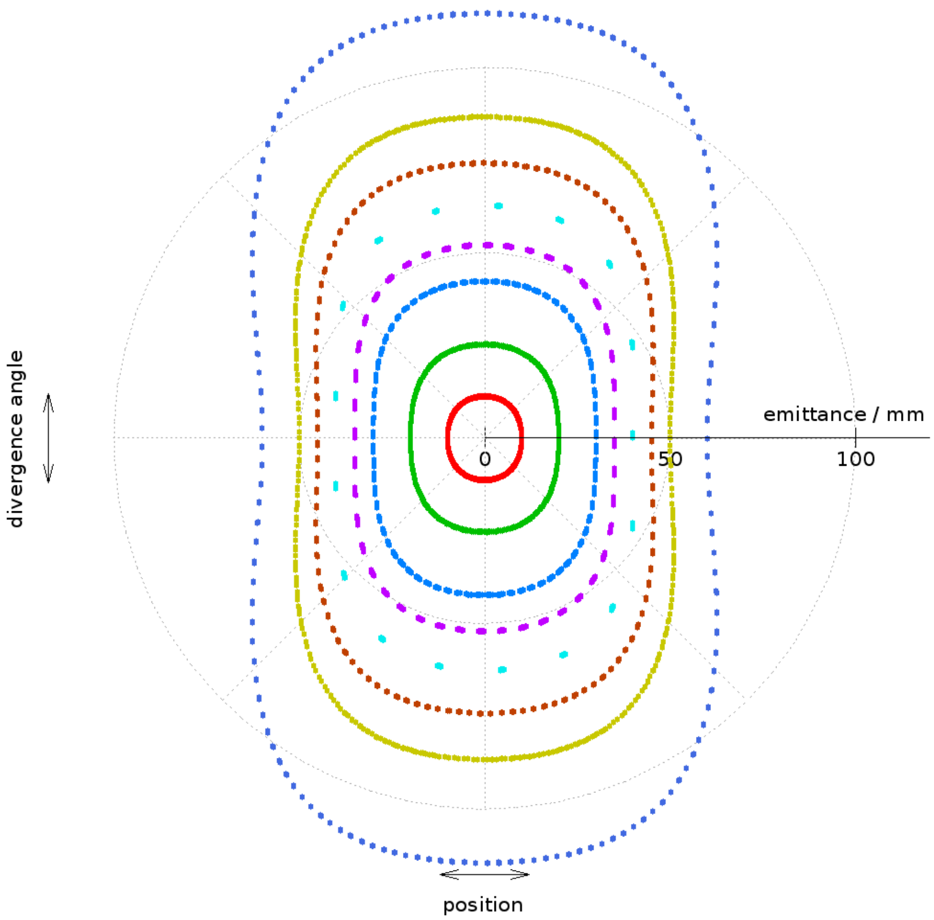
Lattice



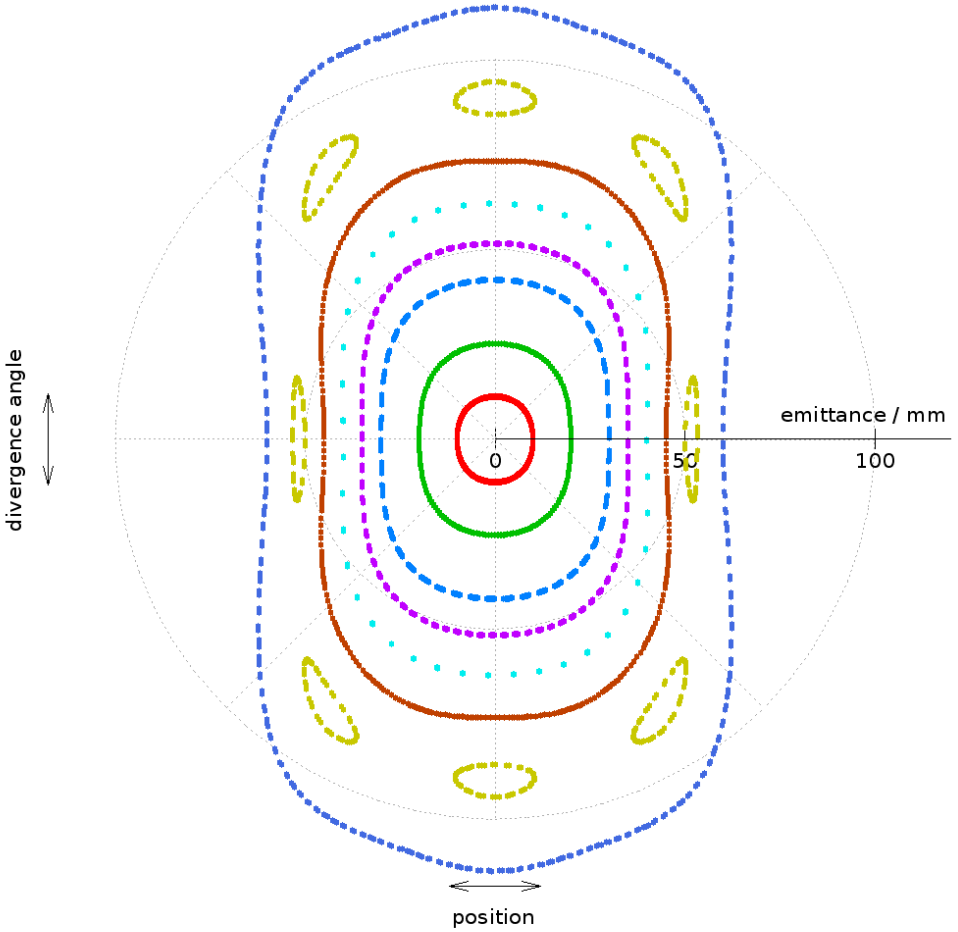
$\lambda = 50\text{mm}$

Particle Orbits

Tracking



Transfer Map

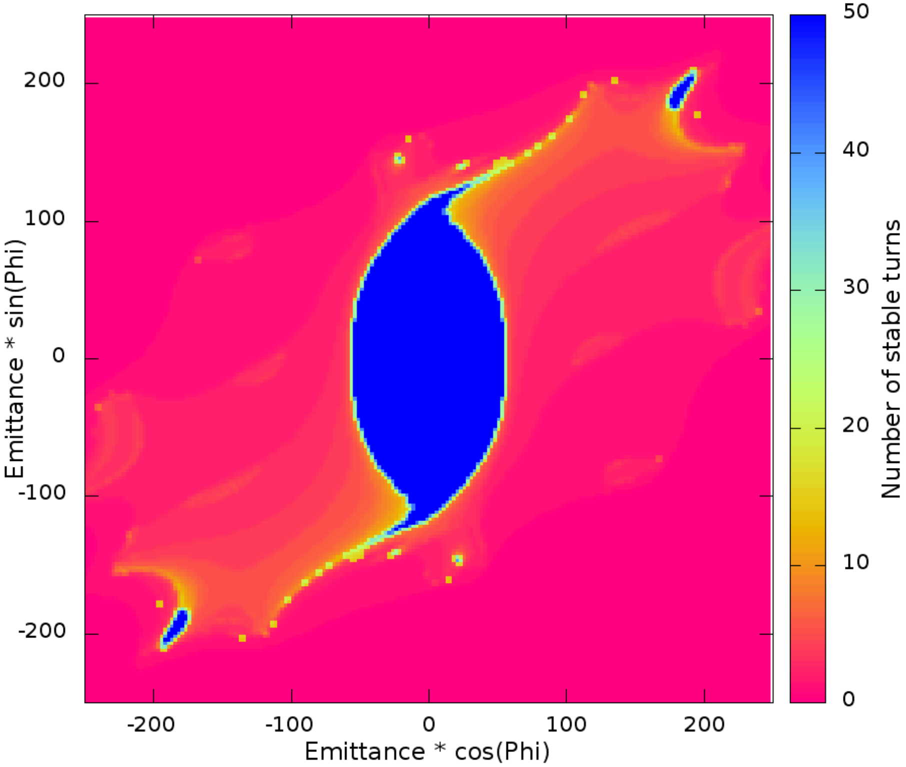


$\lambda = 500\text{mm}$, cell length $= 750\text{mm}, B_{\text{peak}} = 2.5\text{T}, L = 0, p_z = 200\text{MeV}/c$

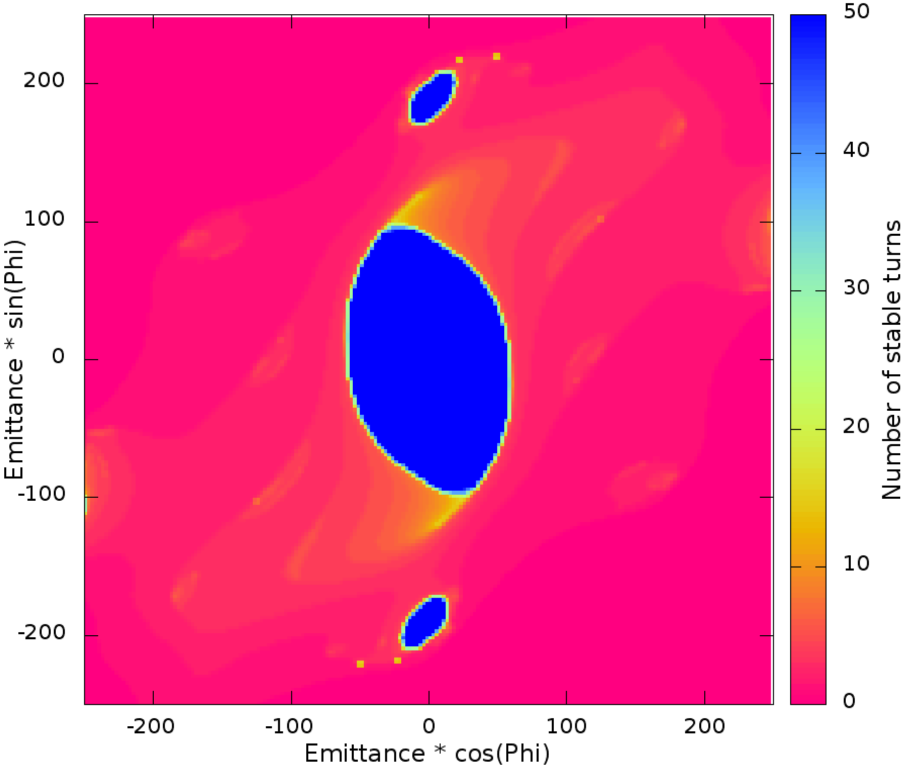
Higher Order Terms

- Does third order behaviour give accurate picture of stability?
  - Compare to tracking – slow
    - Approx 2,000,000 cells to track through
    - To be completed
  - For high b_2, instability comes in before 5^{th} order terms
  - Useful for studying shielding…

Stability

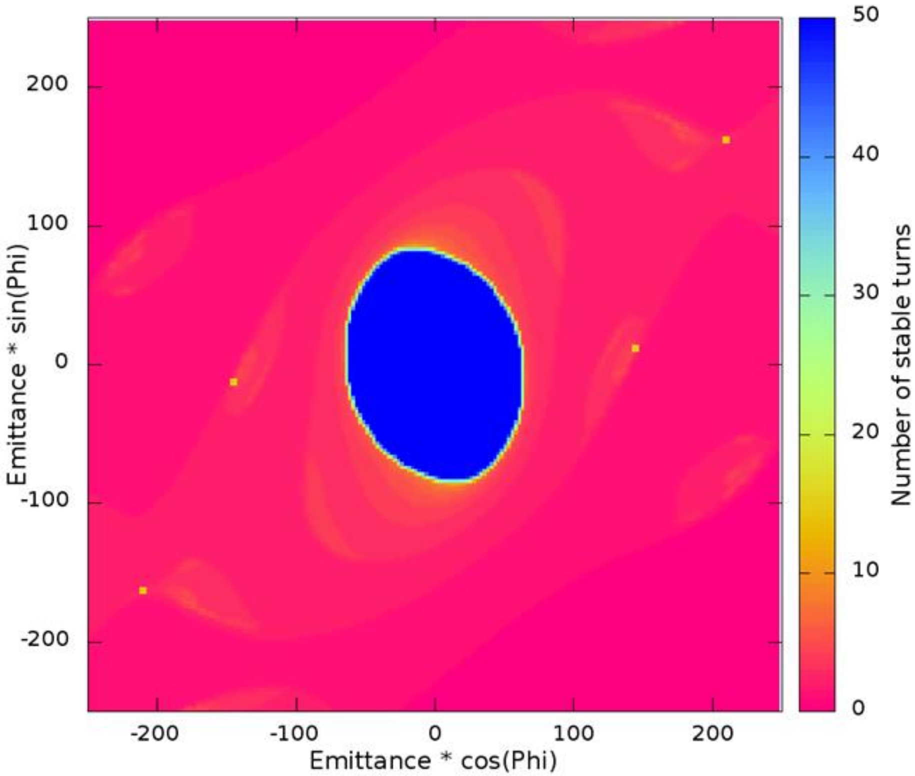


$\Delta\Phi = 0°$

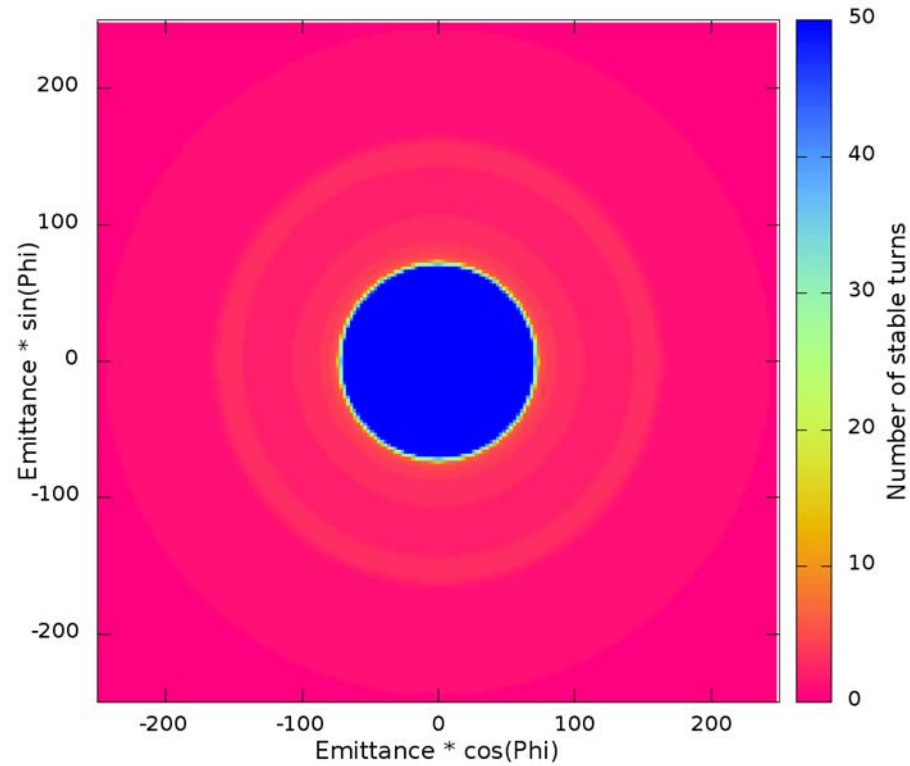


$\Delta\Phi = 45°$

Stability

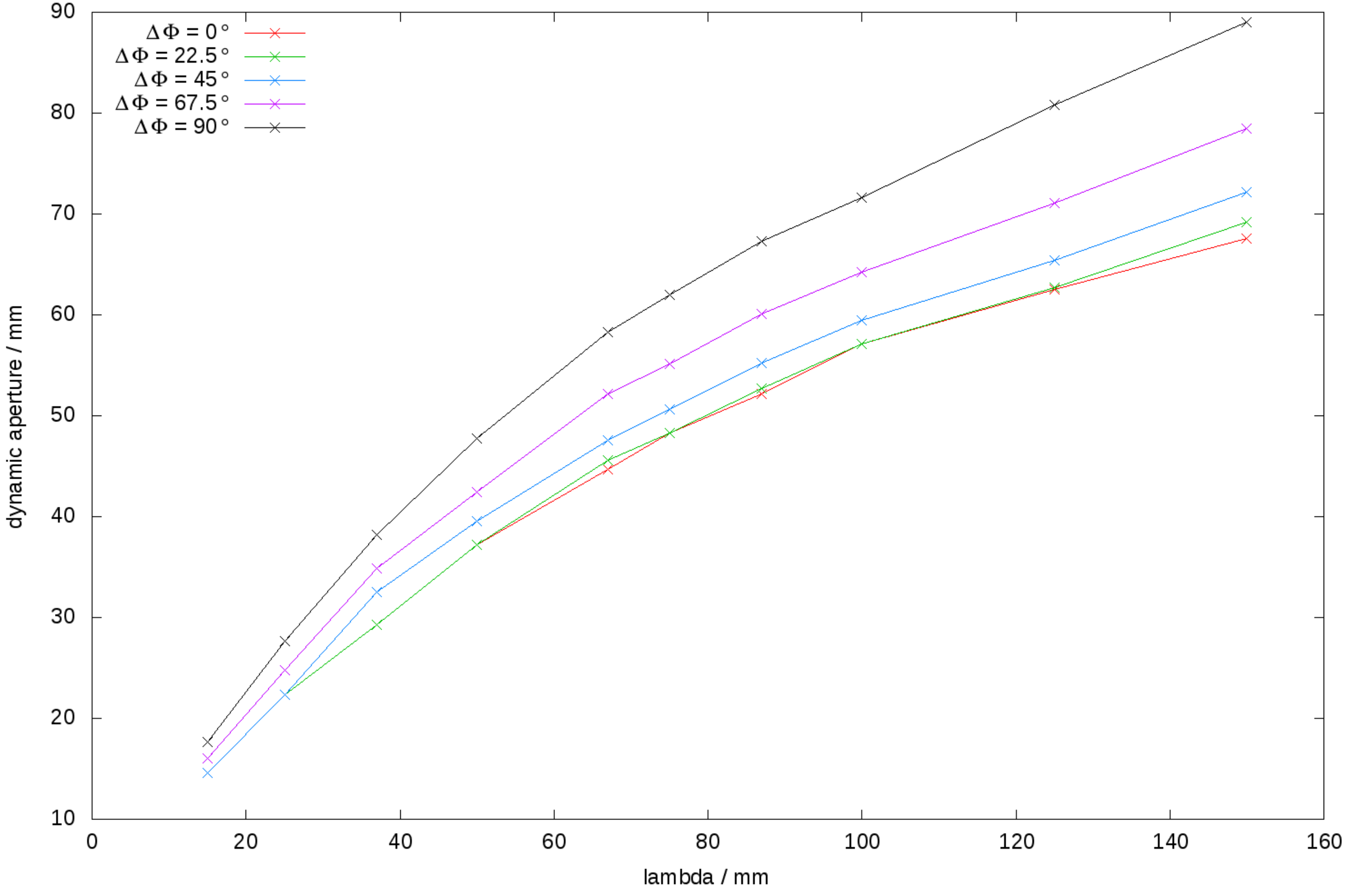


$\Delta\Phi = 67.5°$



$\Delta\Phi = 90°$

Effect of Shielding on Dynamic Aperture



Extension to Different Geometries

- Third order terms of order $\frac{x^3 b_2 q}{p_z}$
- In terms of λ lattice:
  - $\lambda_{unstable} \propto b_{peak} q \left(\frac{a^2 \epsilon^3 \beta^3}{p_z}\right)^{\frac{1}{4}}$
  - β Twiss parameter

Improvements

- Fifth order addition
  - Involved quadruple integrals
- Convergence of exponential
  - Polynomials $f_i \sim 10$
  - Split and use symplectic integrator?

Thanks!

Questions?