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Lensing and deflection angles in high precision cosmology

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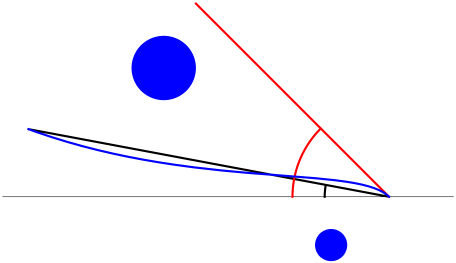
based on F, Gasperini, Marozzi, Veneziano, JCAP 1508 (2015) 08, 020

Contents

- What do weak lensing consist of?
- What is Geodesic Light-Cone (GLC) gauge?
- How can GLC help us in studying it?

Weak lensing and deflection angles

- We know that one of the most important relativistic effects which happens in light propagation is deviating light-like signals trajectories because of the content of matter along the traveling
- Finding the relation $\theta_s^a\left(\theta_o^b\right)$ is the key point of lensing theory

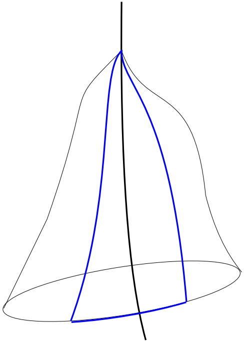


- It is useful defining the well known amplification matrix $\mathcal{A}_b^a = \frac{\partial \theta_s^a}{\partial \theta_o^b}$, because its connection with other physical observables like angular distance

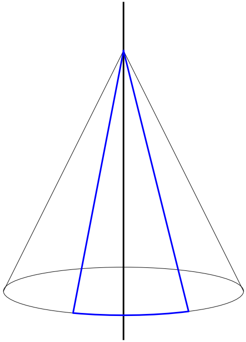
The GLC gauge (1)

The Geodesic Light-Cone (GLC) coordinates consist of a timelike coordinate τ (which can always be identified with the proper time of the synchronous gauge), of a null coordinate w and of two angular coordinates $\tilde{\theta}^a$ ($a = 1, 2$):

$$ds^2 = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$



Past light-cone in FRW coordinates



Past light-cone in GLC coordinates

The GLC gauge (2)

Fundamental properties:

- $w =$ constant defines the past light-cone of the observer (ourselves)
- $u_\mu = -\partial_\mu \tau$ describes a geodesic flow (related to SG)
- $k^\mu = \omega \Upsilon^{-1} \delta^\mu_\tau$ is the quadri-momentum of the photon (constant w and θ^a)
- These properties are really useful: they allow us to express easily redshift only as function of τ:

$$1 + z = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{\Upsilon\left(\tau_o, w, \tilde{\theta}^a\right)}{\Upsilon\left(\tau_s, w, \tilde{\theta}^a\right)}$$

- Really similar to FRW metric, but exact and non perturbative!!!

Weak lensing and GLC gauge

What is the advantage in using GLC gauge for studying deflection angles?

- First of all, $\tilde{\theta}^a = \theta^a_o$ by construction
- In this way, if $x^\mu = \left(\tau, w, \tilde{\theta}^a\right)$ and $y^\mu = (\eta, r, \theta^a)$, the crucial relation between angles can be found by a simple coordinates transformation:

$$g^{\mu\nu} = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} g^{\alpha\beta}_{GLC}$$

- Solving this set of equations will give us the relation:

$$\theta^a = \theta^a\left(\tau, w, \tilde{\theta}^b\right) = \theta^a\left(z, w, \tilde{\theta}^b\right)$$

Connecting Poisson gauge to GLC gauge (1)

- Let us evaluate the deflected angles in the well known Poisson gauge
- By defining $\eta^+ = \eta + r$, we adopt the following form for its third order expression:

$$ds^2 = g^{PG}_{\mu\nu} dy^\mu dy^\nu = a^2(\eta)\Big[-2d\eta^2(\Phi + \Psi) + (1 - 2\Psi)\left(d\eta^{+2} - 2d\eta d\eta^+\right)$$

$$+ (1 - 2\Psi)\left(\eta^+ - \eta\right)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big]$$

where $\Psi = \psi + \frac{1}{2}\psi^{(2)} + \frac{1}{6}\psi^{(3)}$ and $\Phi = \phi + \frac{1}{2}\phi^{(2)} + \frac{1}{6}\phi^{(3)}$

Connecting Poisson gauge to GLC gauge (2)

- Now our goal is finding the coordinates transformation at the desired perturbative order
- In particular, we have to expand consistently all the quantities up to the desired order:

$$
\begin{aligned}
g_{PG}^{\mu\nu}\left(y^{\mu}\right) &= g_{PG}^{\mu\nu}\left(y^{(0)}(x)\right) + \left(\frac{\partial g_{PG}^{\mu}\nu}{\partial y^{\rho}}\right)_{y=y^{(0)}} \left(y^{\rho}\right)^{(1)}(x) + \dots \\
\Upsilon^{-1}(x) &= \left(\Upsilon^{-1}\right)^{(0)} + \left(\Upsilon^{-1}\right)^{(1)} + \dots \\
U^{a}(x) &= \left(U^{a}\right)^{(0)} + \left(U^{a}\right)^{(1)} + \dots \\
\gamma^{ab}(x) &= \left(\gamma^{ab}\right)^{(0)} + \left(\gamma^{ab}\right)^{(1)} + \dots
\end{aligned}
$$

Connecting Poisson gauge to GLC gauge - First order (1)

- Having in mind that the zero-th order transformation is given by:

$$\left(\Upsilon^{-1}\right)^{(0)} = a^{-1}, \qquad (U^a)^{(0)} = 0, \qquad \left(\gamma^{ab}\right)^{(0)} = a^{-2} r^{-2} \mathrm{diag}\left(1, \sin^{-2}\theta\right)$$

$$\eta^{(0)} = \int_{\tau_{in}}^{\tau} \frac{d\tau'}{a(\tau')}, \qquad \eta^{+(0)} = w, \qquad \theta^{a(0)} = \tilde{\theta}^a$$

  we perform an iterative approach for solving the equations of the coordinates transformation.

- In this way, first order deflection angles is given by solving:

$$\begin{aligned}
\partial_\tau \eta^{(1)} &= -\frac{\psi}{a} - \frac{1}{a}\partial_w \eta^{(1)} - H\eta^{(1)} \\
\partial_\tau \eta^{+(1)} &= -2\frac{\psi}{a} \\
\partial_\tau \theta^{a(1)} &= \frac{1}{a}\gamma_0^{ab}\partial_b \eta^{+(1)}
\end{aligned}$$

Connecting Poisson gauge to GLC gauge - First order (1)

- Of course, the coordinates transformation gives us all the first order quantities ($\eta^{(1)}$, $\eta^{+(1)}$, $\Upsilon^{(1)}$, $U^{a(1)}$). However we are interested in deflection angles:

$$\theta^{a(1)} = -2\int_{\tau}^{\tau_o} \frac{d\tau'}{a(\tau')}\gamma_0^{ab}(\tau', w, \tilde{\theta})\int_{\tau'}^{\tau_o} \frac{d\tau''}{a(\tau'')}\partial_b\psi(\tau', w, \tilde{\theta})$$

- By defining, as usual, the lensing potential

$$\mathcal{P} = -\frac{2}{\eta_o - \eta}\int_{\eta}^{\eta_o} d\eta' \frac{\eta' - \eta}{\eta_o - \eta'}\psi(\eta')$$

we can rewrite it as:

$$\theta^{a(1)} = \hat{\gamma}_0^{ab}\partial_b\mathcal{P}$$

with $\hat{\gamma}_0^{ab} = r^2\gamma_0^{ab}$

- According to what concerns lensing, this results is completely leading. Moreover, it can be view directly as a function of redshift and angles

Connecting Poisson gauge to GLC gauge - Second order (1)

- As already said, we can built an iterative approach for finding second order deflection angles, since we have the first order ones
- The expression for $\theta^{a(2)}$ is now given by

$$\begin{aligned}\partial_\tau \theta^{(2)a} =& \frac{1}{a}\gamma_0^{ab}\partial_b \eta^{(2)+} - \frac{1}{a}\gamma_0^{ac}\partial_b \eta^{(1)+}\partial_c \theta^{(1)b} + \frac{1}{a}\partial_b \eta^{(1)+}\theta^{(1)c}\partial_c \gamma_0^{ab} \\ &+ \frac{1}{a}\gamma_0^{ab}\left[+2\psi\partial_b \eta^{(1)} + \psi \partial_b \eta^{(1)+} - \partial_b \eta^{(1)+}\partial_w \eta^{(1)} \right. \\ &\left. - aH\,\eta^{(1)}\,\partial_b \eta^{(1)+} - 2\,\frac{\eta^{(1)+} - \eta^{(1)}}{\eta^{(0)+} - \eta^{(0)}}\,\partial_b \eta^{(1)+}\right]\end{aligned}$$

Connecting Poisson gauge to GLC gauge - Second order (2)

Hence, after some algebraic manipulation the full expression for the second order deflection angles is given by

$$
\begin{aligned}
\left[\theta_s^{a(2)}(z_s,\theta_o)\right]_{\rm Full} = {}& 4\,\partial_b\left[\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\gamma_0^{ac}\partial_c\int_{\eta'}^{\eta_o} d\eta''\psi(\eta'')\right]\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\gamma_0^{bd}\partial_d\int_{\eta'}^{\eta_o} d\eta''\psi(\eta'') \\
& + 2\left\{\frac{1}{\mathcal{H}_s}\left[v_{o\|} - v_{s\|} + \psi_o - \psi_s - 2\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\partial_{\eta'}\psi(\eta')\right]\right. \\
& \left. - \int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\psi(\eta')\right\}\frac{1}{(\eta_o-\eta_s)^2}\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\,\hat\gamma_0^{ab}\partial_b\psi(\eta') \\
& + 4\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\psi(\eta')\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\frac{1}{(\eta_o-\eta')^3}\int_{\eta'}^{\eta_o} d\eta''\hat\gamma_0^{ab}\partial_b\psi(\eta'') \\
& - 4\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\,\psi(\eta')\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\gamma_0^{ab}\partial_b\left[\frac{\psi(\eta')-\psi_o}{2} + \int_{\eta'}^{\eta_o} d\eta''\,\partial_{\eta''}\psi(\eta'')\right] \\
& + 2\int_{\bar\eta_s^{(0)}}^{\eta_o} d\eta'\left[\gamma_0^{ac}\zeta_c(\eta') + \psi(\eta')\xi^a(\eta') + \lambda^a(\eta')\right]
\end{aligned}
$$

Connecting Poisson gauge to GLC gauge - Second order (3)

- Differently from the first order ones, second order corrections also contain some non leading terms for lensing
- By focusing on leading terms (three angular derivatives of the bardeen potential, at the second order), we have

$$
\begin{aligned}
(\theta_s^{a(2)})_{\mathrm{Leading}} &= -\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o} d\eta' \frac{\eta'-\eta_s}{\eta_o-\eta'}\hat{\gamma}_0^{ab}\hat{\gamma}_0^{cd}\partial_d\mathcal{P}(\eta')\partial_c\partial_b\psi(\eta') \\
&= -\frac{2}{\eta_o-\eta_s}\int_{\eta_s}^{\eta_o} d\eta' \frac{\eta'-\eta_s}{\eta_o-\eta'}\hat{\gamma}_0^{ab}\theta^{c(1)}(\eta')\partial_c\partial_b\psi(\eta')
\end{aligned}
$$

Connecting Poisson gauge to GLC gauge - Third order

- Following the same procedure, we get also the third order deflection angles

$$(\theta_s^{a(3)})_{\rm Leading} = -\frac{2}{\eta_o - \eta_s}\int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ab} \Big[(\theta^{c(2)})_{\rm Leading}(\eta')\partial_c\partial_b\psi(\eta') + \frac{1}{2}\theta^{c(1)}(\eta')\theta^{d(1)}(\eta')\partial_c\partial_d\partial_b\psi(\eta') \Big]$$

Amplification matrix and deflection angles (1)

- From the amplification matrix, we can try to build a general iterative approach for having the deflection angles up to each desired order

$$\mathcal{A}^a_b = \frac{\partial \theta^a}{\partial \theta^b_o} \equiv \delta^a_b - \Psi^a_b$$

- From the previous definition, we get that

$$(\Psi^a_b)^{(n)} = -\frac{\partial \theta^{a\,(n)}}{\partial \tilde{\theta}^b}$$

- Hence, according to the solutions that we found for angles, we obtain

$$(\Psi^a_b)^{(1)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \, \frac{\eta' - \eta_s}{\eta_o - \eta'} \, \hat{\gamma}_0^{ac} \partial_c \partial_b \psi(\eta', \eta_o - \eta', \theta^a_o)$$

$$(\Psi^a_b)^{(2)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \, \frac{\eta' - \eta_s}{\eta_o - \eta'} \, \hat{\gamma}_0^{ac} \left[\partial_c \partial_b \partial_d \psi(\eta') \theta^{d(1)} - \partial_c \partial_d \psi(\eta') \Psi^{d(1)}_b \right]$$

$$(\Psi^a_b)^{(3)} = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \, \frac{\eta' - \eta_s}{\eta_o - \eta'} \, \hat{\gamma}_0^{ac} \left[\partial_c \partial_b \partial_d \psi(\eta') \theta^{d(2)} + \frac{1}{2} \partial_c \partial_b \partial_d \partial_e \psi(\eta') \theta^{d(1)} \theta \right.$$
$$\left. - \partial_c \partial_d \partial_e \psi(\eta') \theta^{e(1)} \Psi^{d(1)}_b - \partial_c \partial_d \psi(\eta') \Psi^{d(2)}_b \right]$$

Amplification matrix and deflection angles (2)

- Those expressions seems to behave perfectly for solving the so called lens equation

$$\Psi^a_b = \frac{2}{\eta_o - \eta_s} \int_{\eta_s}^{\eta_o} d\eta' \frac{\eta' - \eta_s}{\eta_o - \eta'} \hat{\gamma}_0^{ac} \partial_c \partial_d \psi(\eta', \eta_o - \eta', \theta^a) \left[\delta^d_b - \Psi^d_b \right]$$

- In order to have the full agreement between our results and the lens equation, we have to expand the angles appearing in ψ

Summary and Conclusion

- This new approach allows to get directly the expression of the desired physical quantities in terms of the observer angles and redshift, i.e. other physical observables
- The same evaluation has been applied for deriving the angular/luminosity distance up to second order in perturbation theory. Result agrees with other evaluation given in literature, within the framework of the GLC gauge
  (Ben-Dayan, Marozzi, Nugier, Veneziano, JCAP 1211 (2012) 045)
  (Ben-Dayan, Gasperini, Marozzi, Nugier, Veneziano, JCAP 1306 (2013) 002)
  (F., Gasperini, Marozzi, Veneziano, JCAP 1311 (2013) 019)
- This evaluation directly allows to get the (full) deflection angles up to the desired order

Future applications

- The knowledge of the third order deflection angles allows us to evaluate the second order corrections to the power spectra to the CMB (Di Dio, Durrer, F., Marozzi, work in progress)
- Having a well defined amplification matrix via the Jacobi map in order to consider even all the non leading lensing terms (see F., Nugier, JCAP 1502 (2015) 02, 002 for a preliminary work in this sense)