COSMO-15

Correlation Functions in Stochastic Inflation

Based on VV & A Starobinsky, arXiv:1506.04732, accepted in EPJC

Vincent Vennin

ICG Portsmouth

University of Portsmouth

Outline

- The Stochastic Inflation Formalism
  & Correlation Functions
- The δN-stochastic formalism
- Results: Can we Observe Stochastic Effects?
- Conclusions & Prospects

Stochastic Inflation

Starobinsky, 1984, 1986, see also 1982
Rey, 1987
Goncharov, Linde & Mukhanov, 1987
Nakao, Nambu & Sasaki, 1988

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How do we calculate correlation functions of cosmological perturbations in stochastic inflation?

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Sasaki, Stewart, 1996
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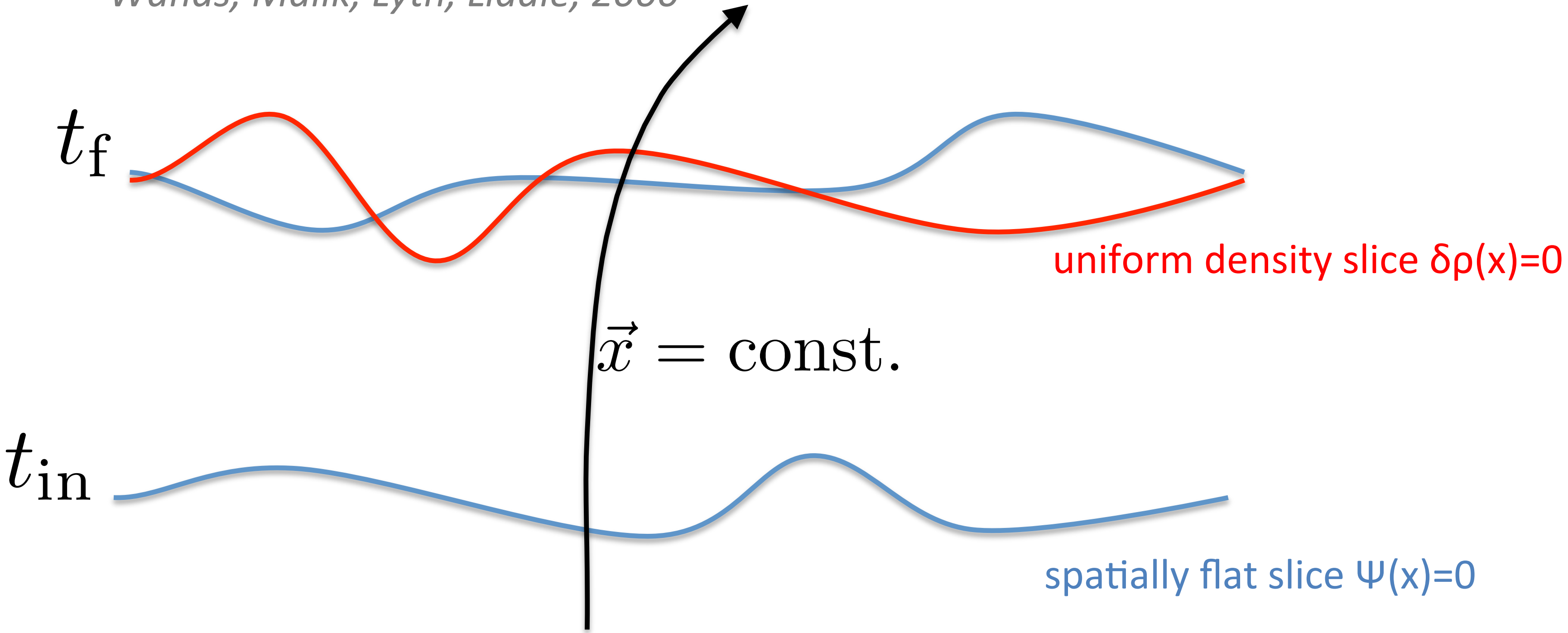

On large scales, the **curvature perturbation** on the uniform density surface is equal to the **perturbation in the number of e-folds** between the uniform density surface and the initial flat slice

$$\zeta(t,x) = N\,(t,x) - N_0(t) \equiv \delta N$$

The stochastic-δ*N* formalism

Enqvist, Nurmi, Podolsky, Rigopoulos, 2008
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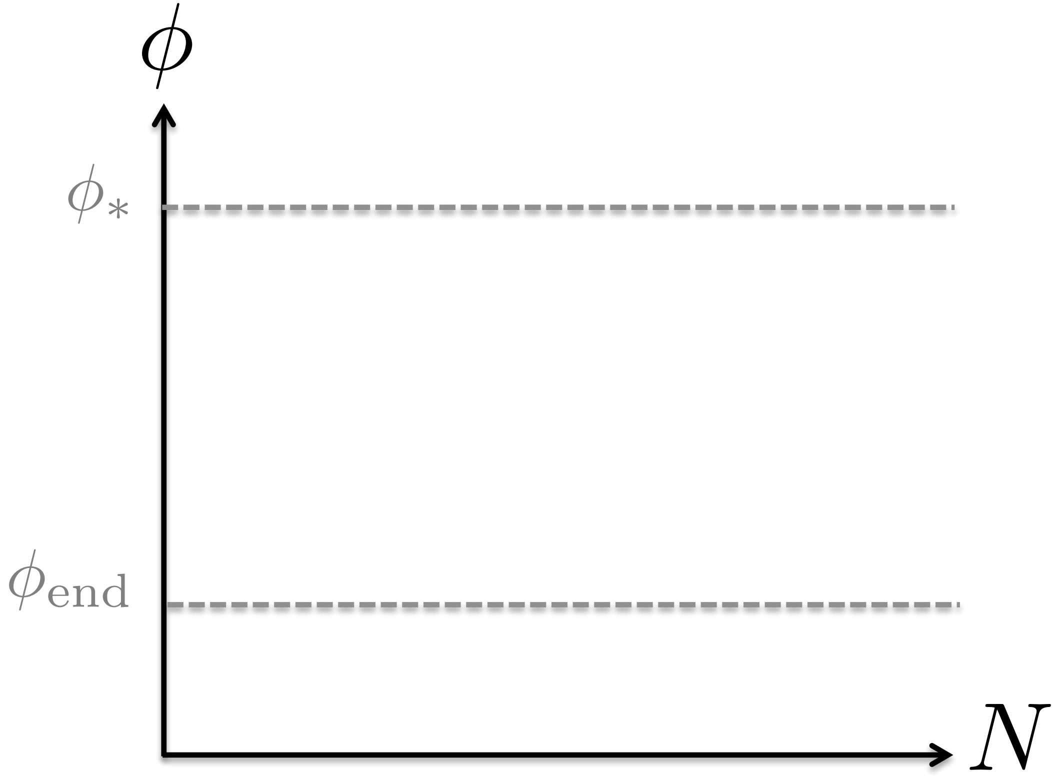

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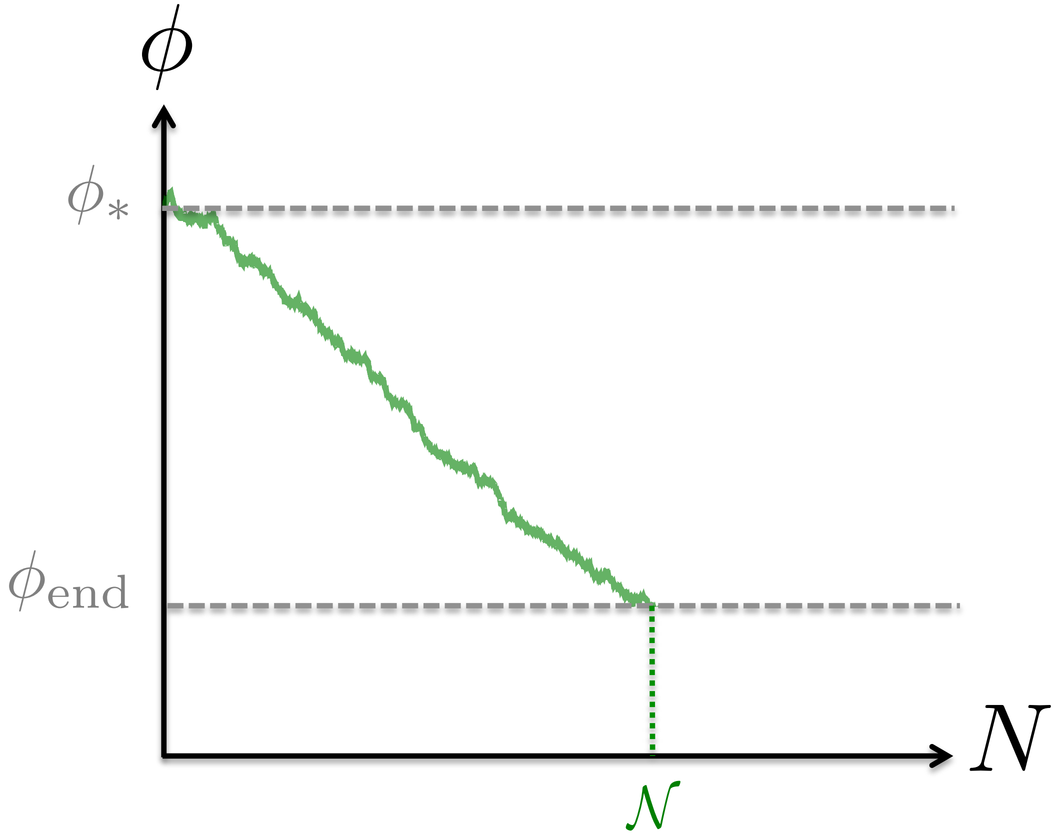

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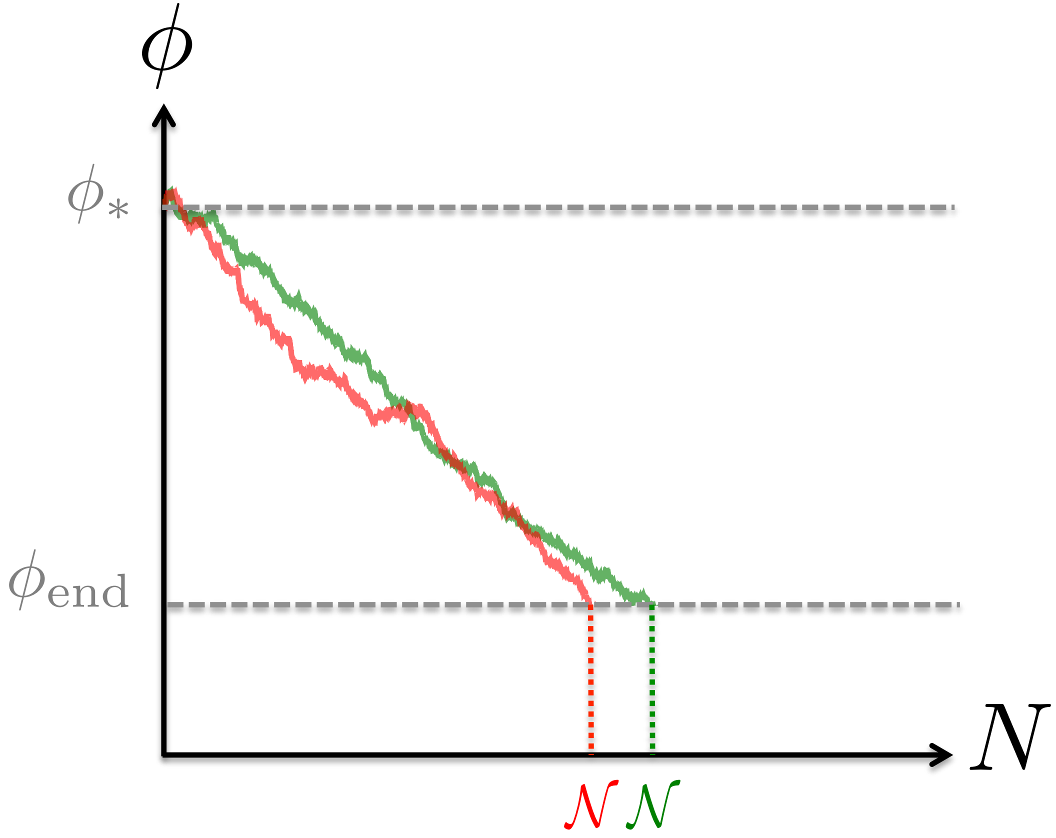

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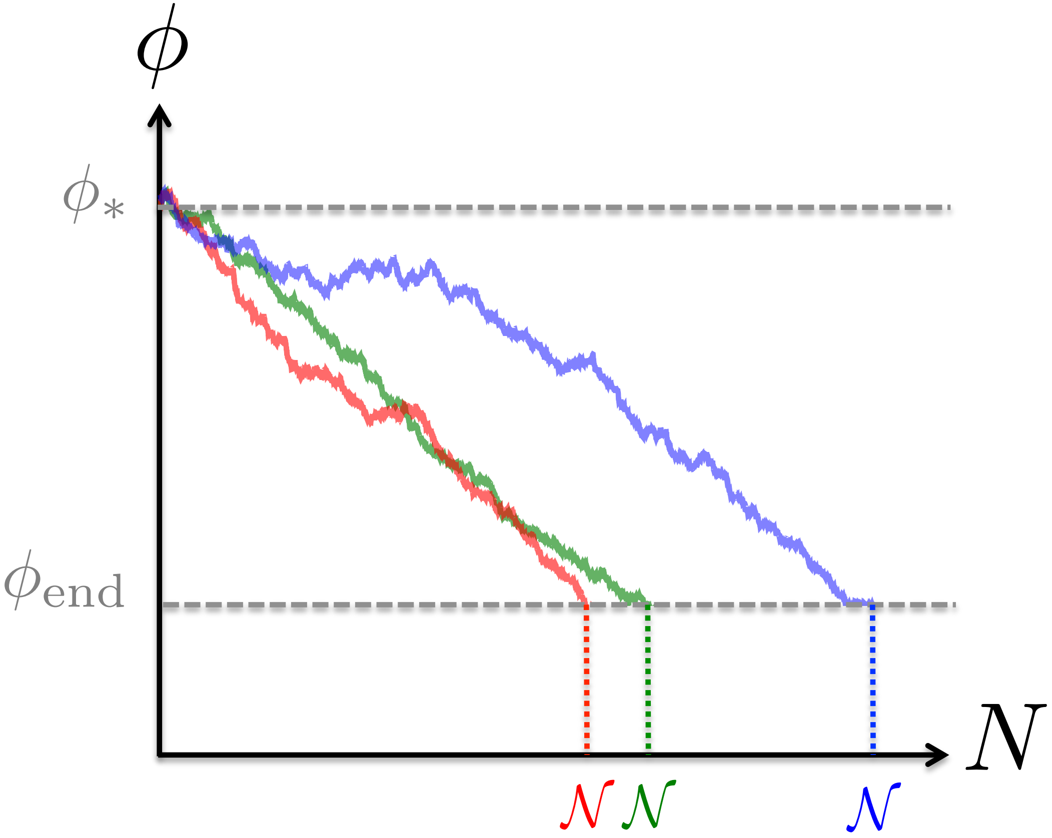

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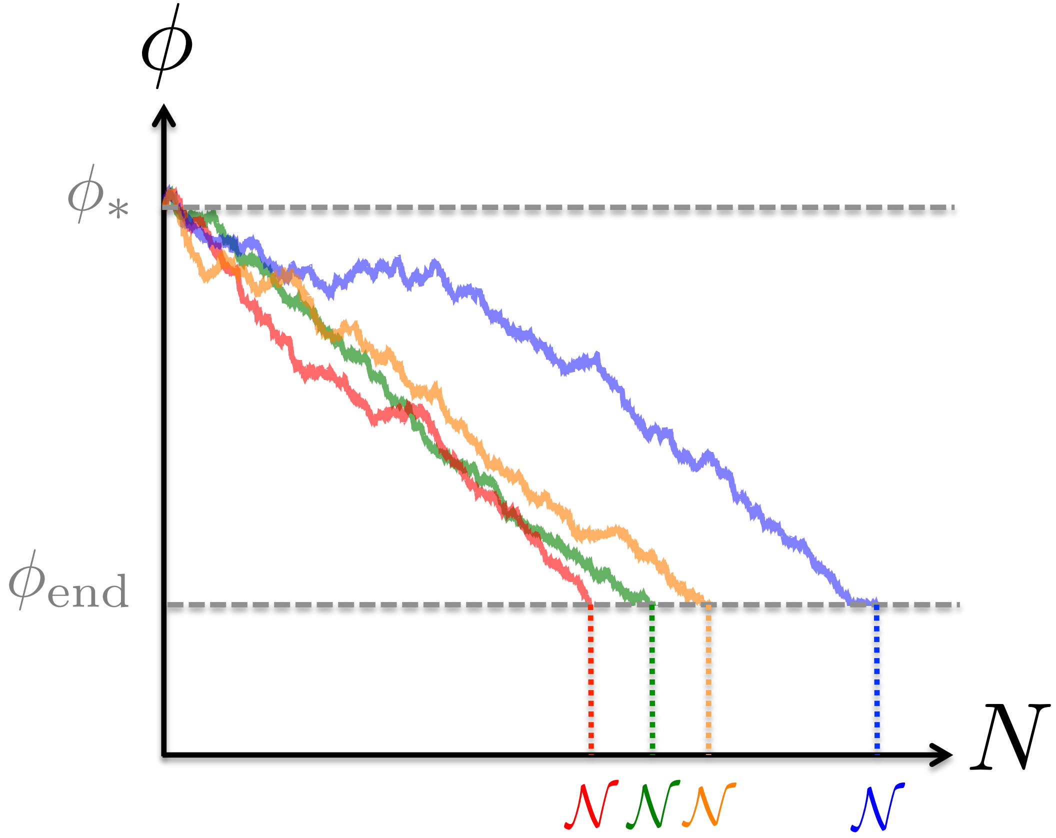

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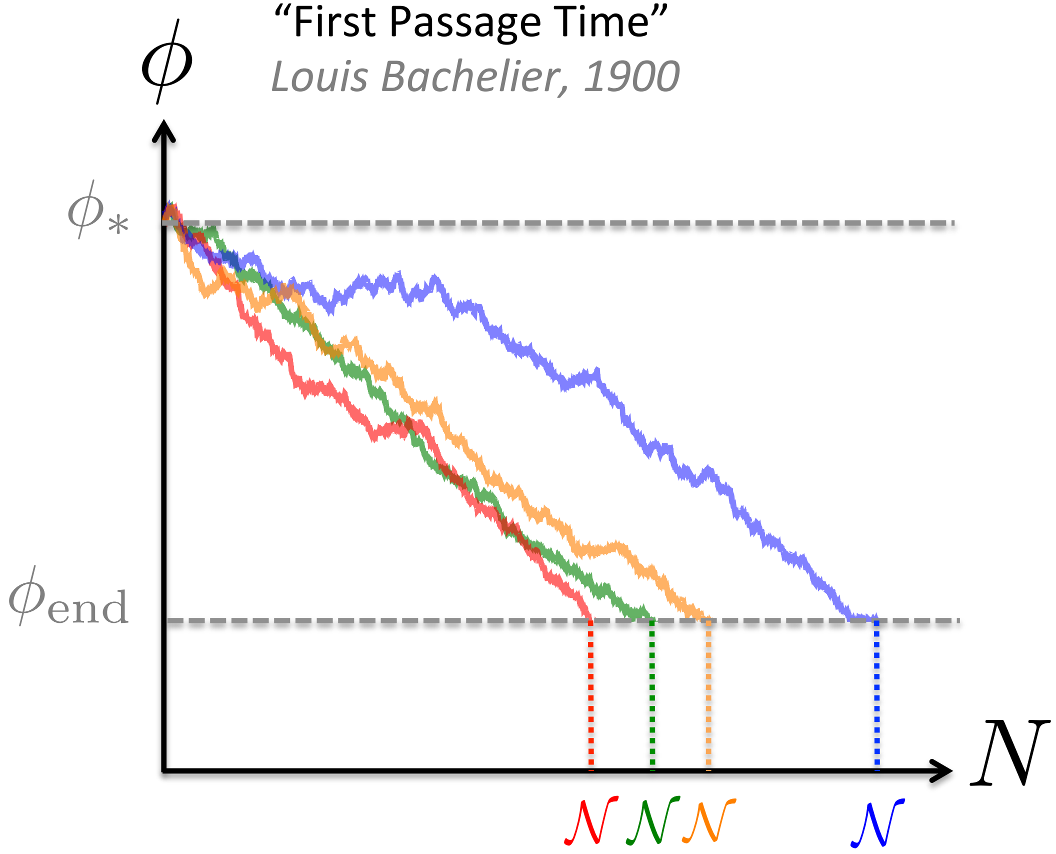

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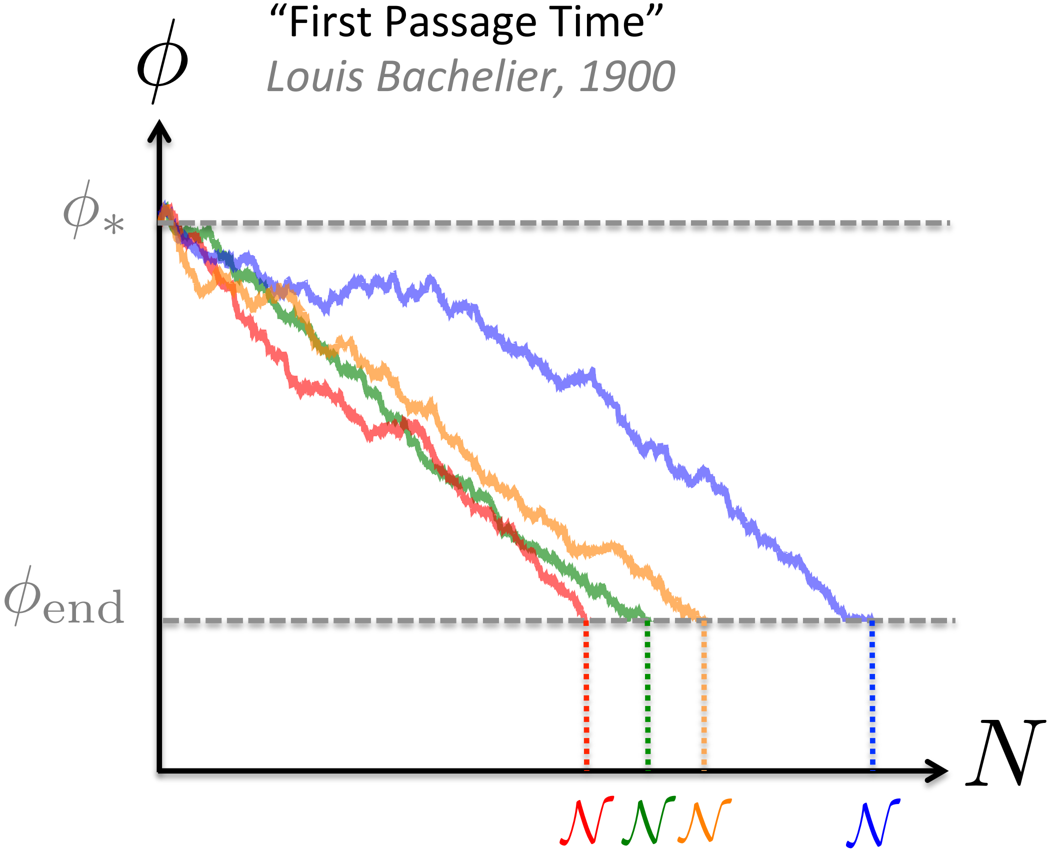

Scalar Power Spectrum

$$\boxed{\begin{aligned}\mathcal{P}_\zeta\left(\phi_*\right) = 2\left\{\int_{\phi_*}^{\bar{\phi}}\frac{\mathrm{d}x}{M_{\mathrm{Pl}}}\frac{1}{v\left(x\right)}\exp\left[\frac{1}{v\left(x\right)}-\frac{1}{v\left(\phi_*\right)}\right]\right\}^{-1}\times \\ \int_{\phi_*}^{\bar{\phi}}\frac{\mathrm{d}x}{M_{\mathrm{Pl}}}\left\{\int_{x}^{\bar{\phi}}\frac{\mathrm{d}y}{M_{\mathrm{Pl}}}\frac{1}{v\left(y\right)}\exp\left[\frac{1}{v\left(y\right)}-\frac{1}{v\left(x\right)}\right]\right\}^2\exp\left[\frac{1}{v\left(x\right)}-\frac{1}{v\left(\phi_*\right)}\right]\end{aligned}}$$

$$v = V/(24\pi^2 M_{\mathrm{Pl}}^4)$$

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Saddle Point Approximation

$$\left|2v - \frac{v''v^2}{v'^2}\right| \ll 1$$

$$\mathcal{P}_\zeta\left(\phi_*\right) \simeq \frac{2}{M_\mathrm{Pl}^2} \frac{v^3\left(\phi_*\right)}{v'^2\left(\phi_*\right)} \left[1 + 5v\left(\phi_*\right) - 4\frac{v^2\left(\phi_*\right) v''\left(\phi_*\right)}{v'^2\left(\phi_*\right)} + \cdots\right]$$

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Classical result

First order correction

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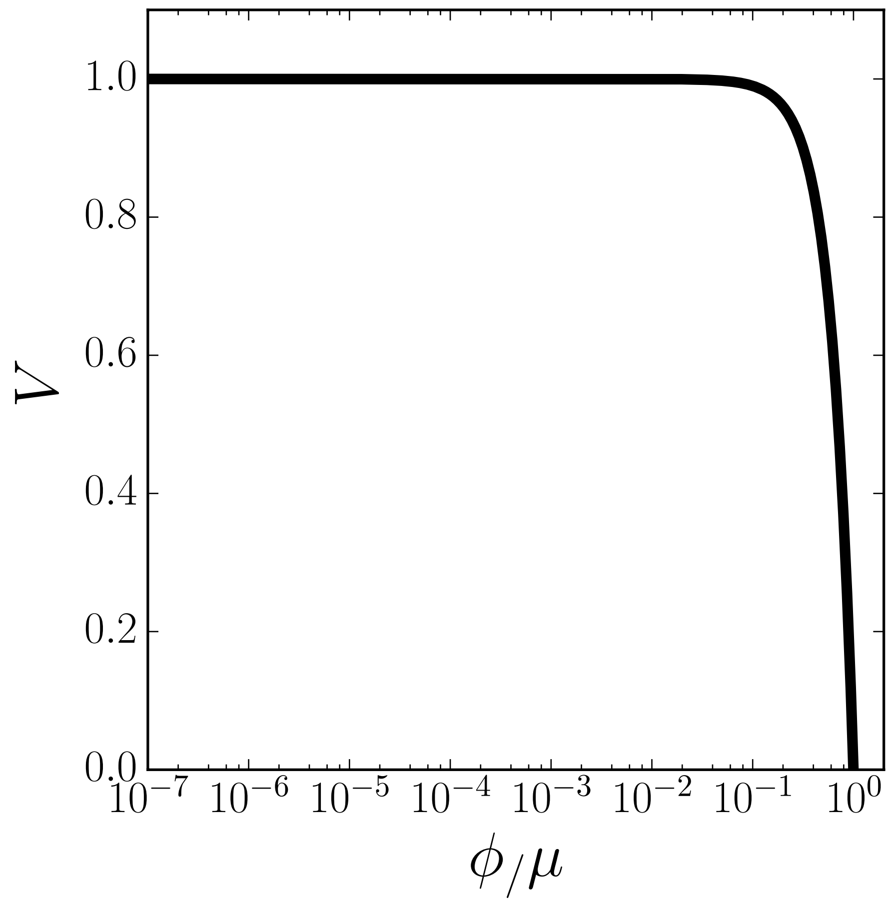
Example #1: Hilltop Inflation

$$V = M^4 \left[1 - \left(\frac{\phi}{\mu}\right)^2\right]$$



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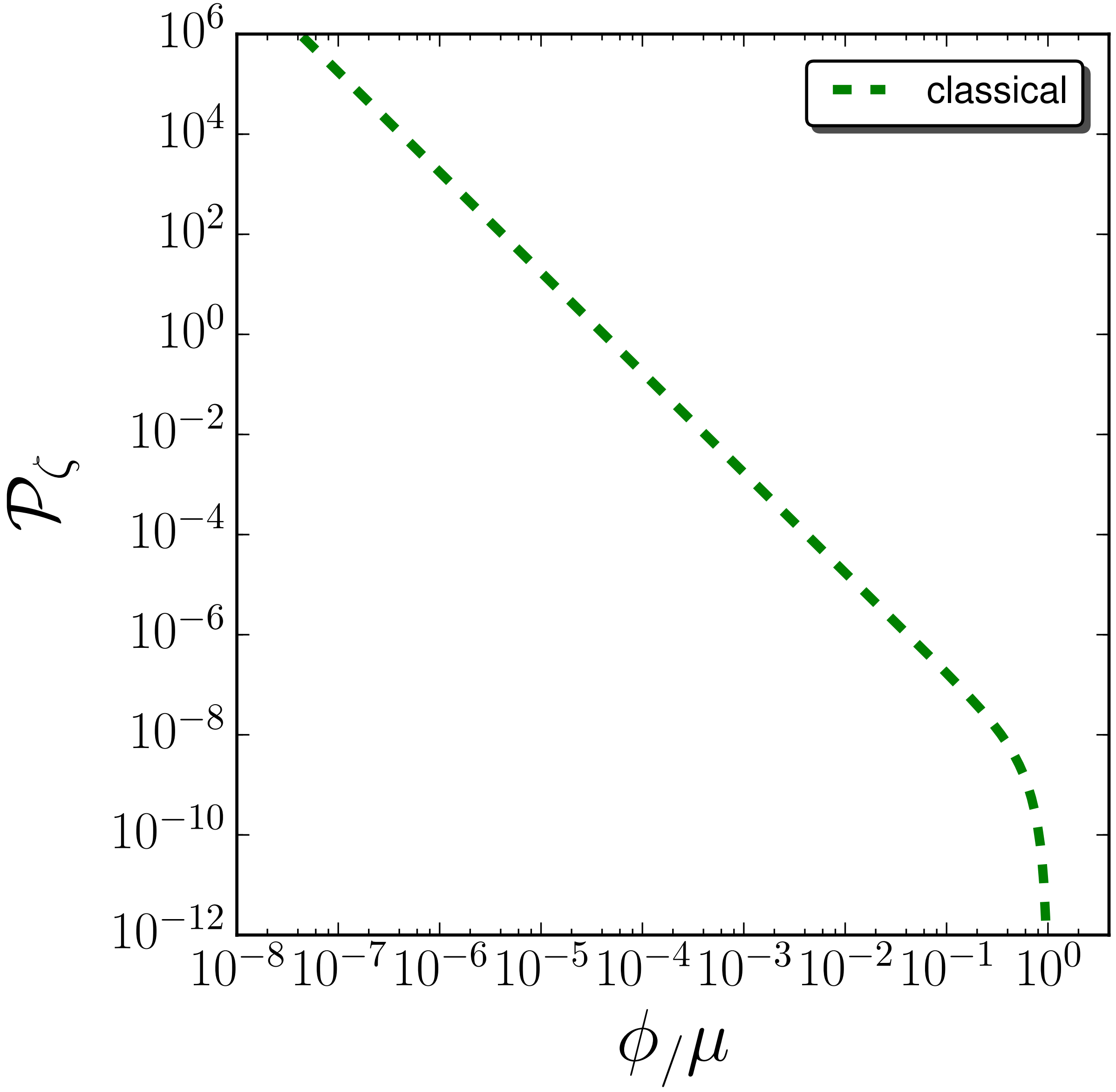

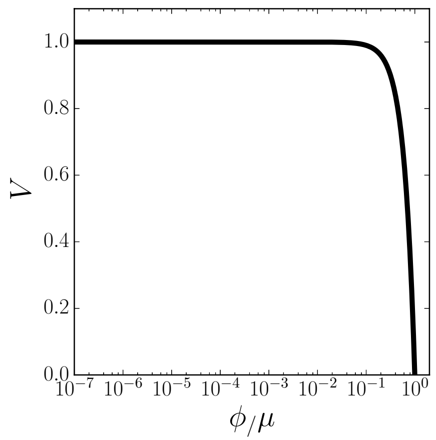

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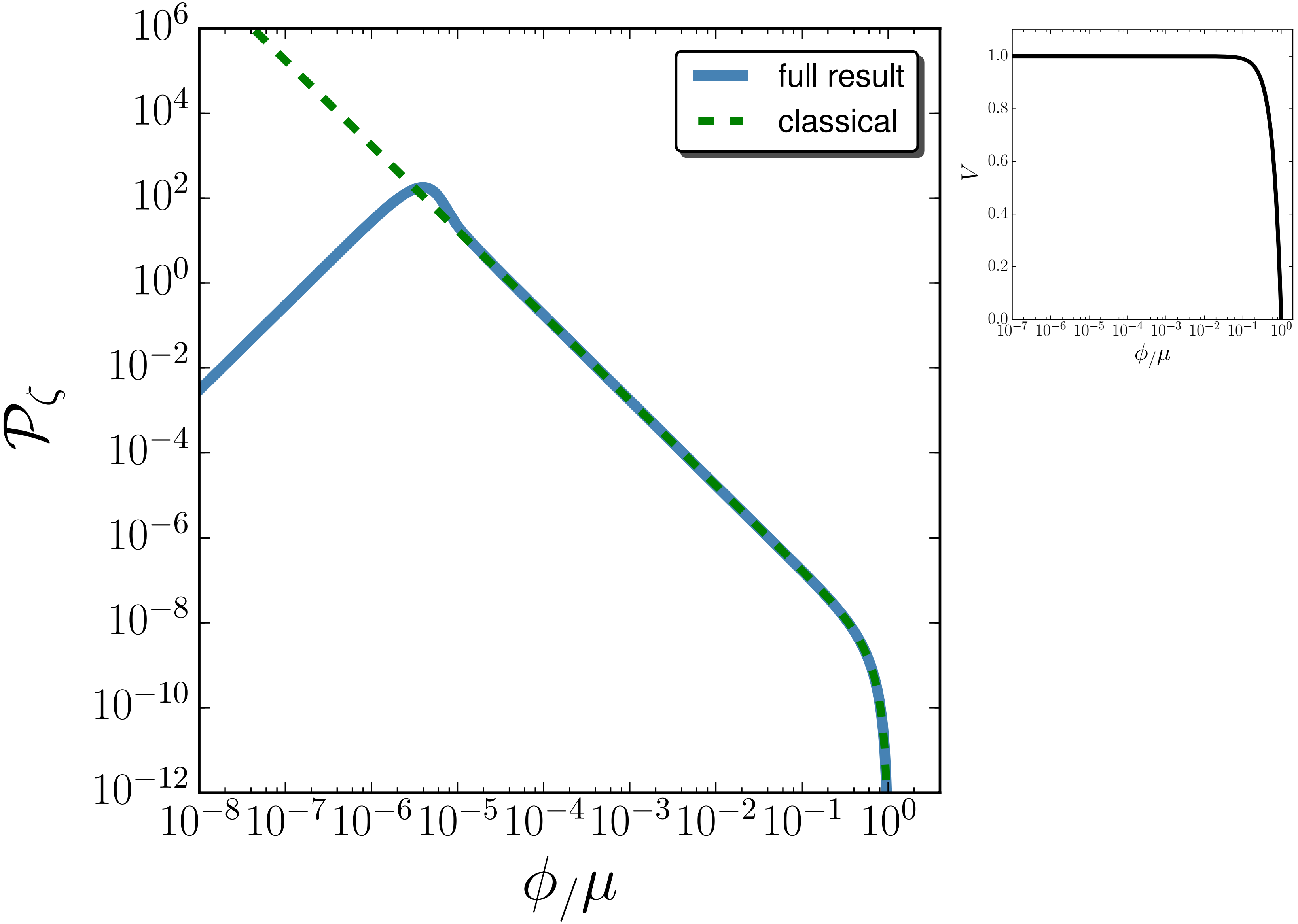

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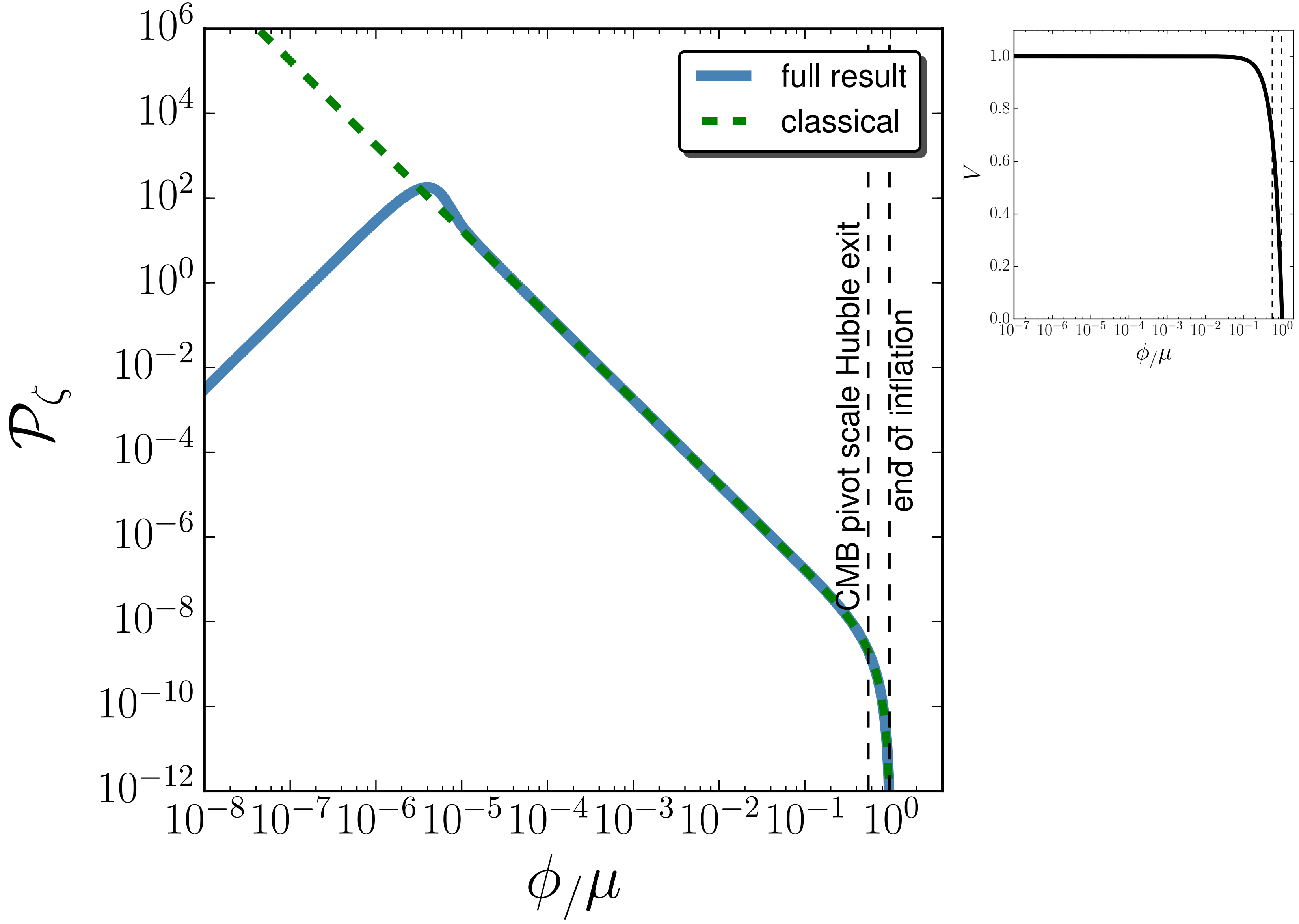

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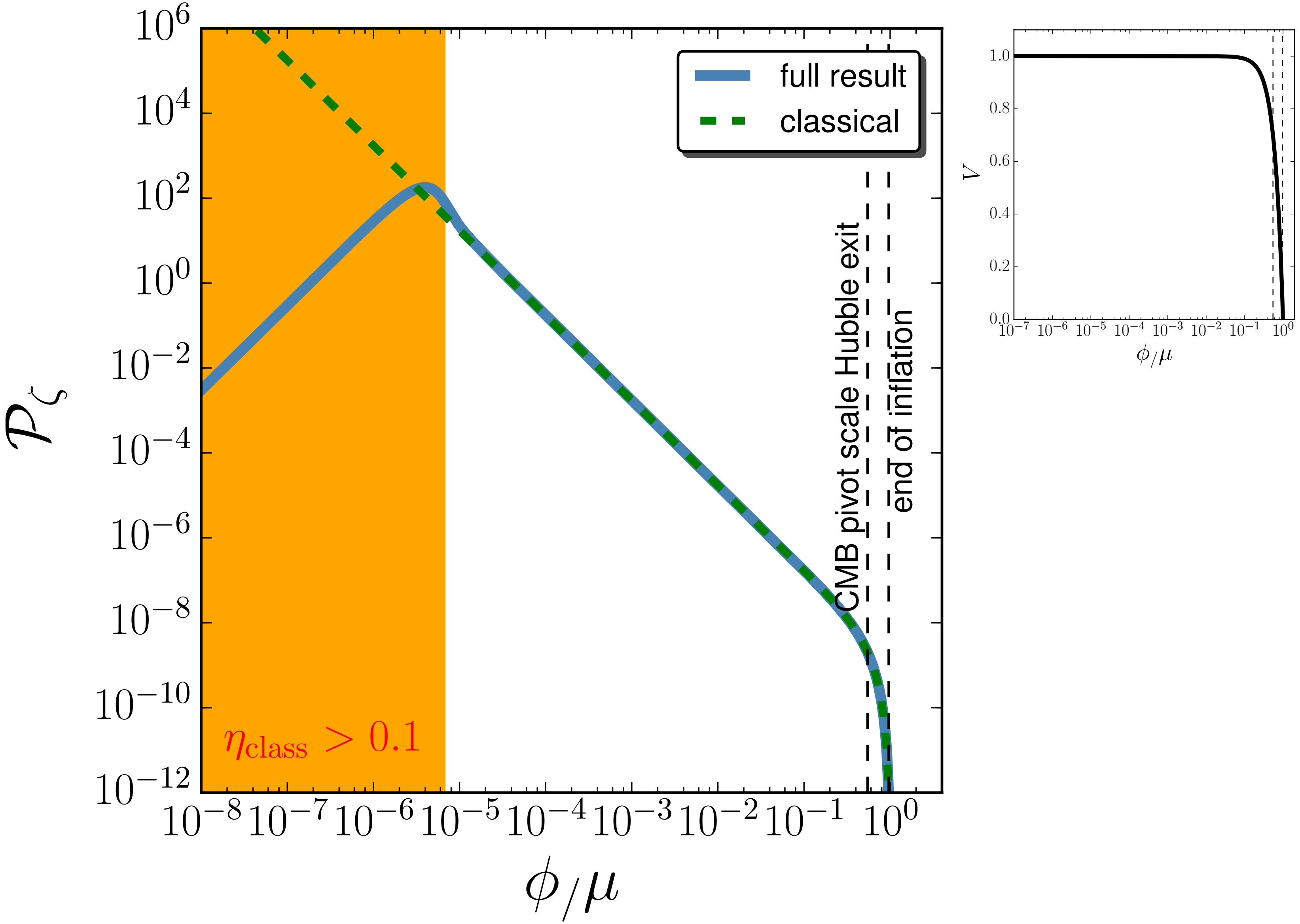
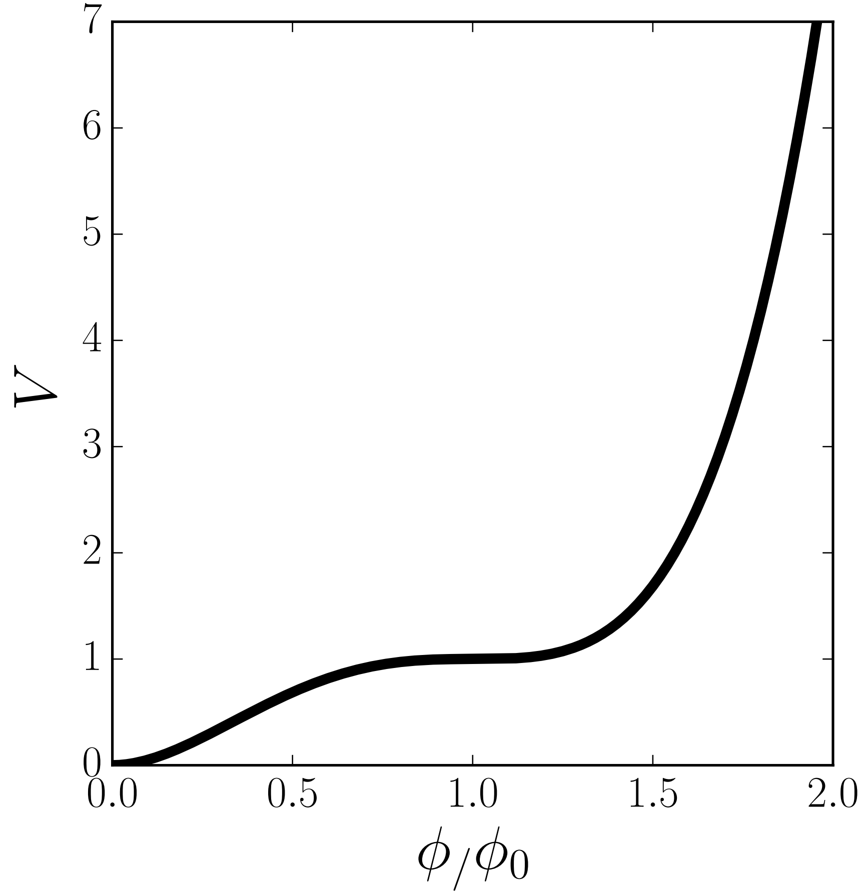


Example #2: Inflection Point Inflation

$$V = M^4 \left[\left(\frac{\phi}{\phi_0} \right)^2 - \frac{4}{3} \left(\frac{\phi}{\phi_0} \right)^3 + \frac{1}{2} \left(\frac{\phi}{\phi_0} \right)^4 \right]$$



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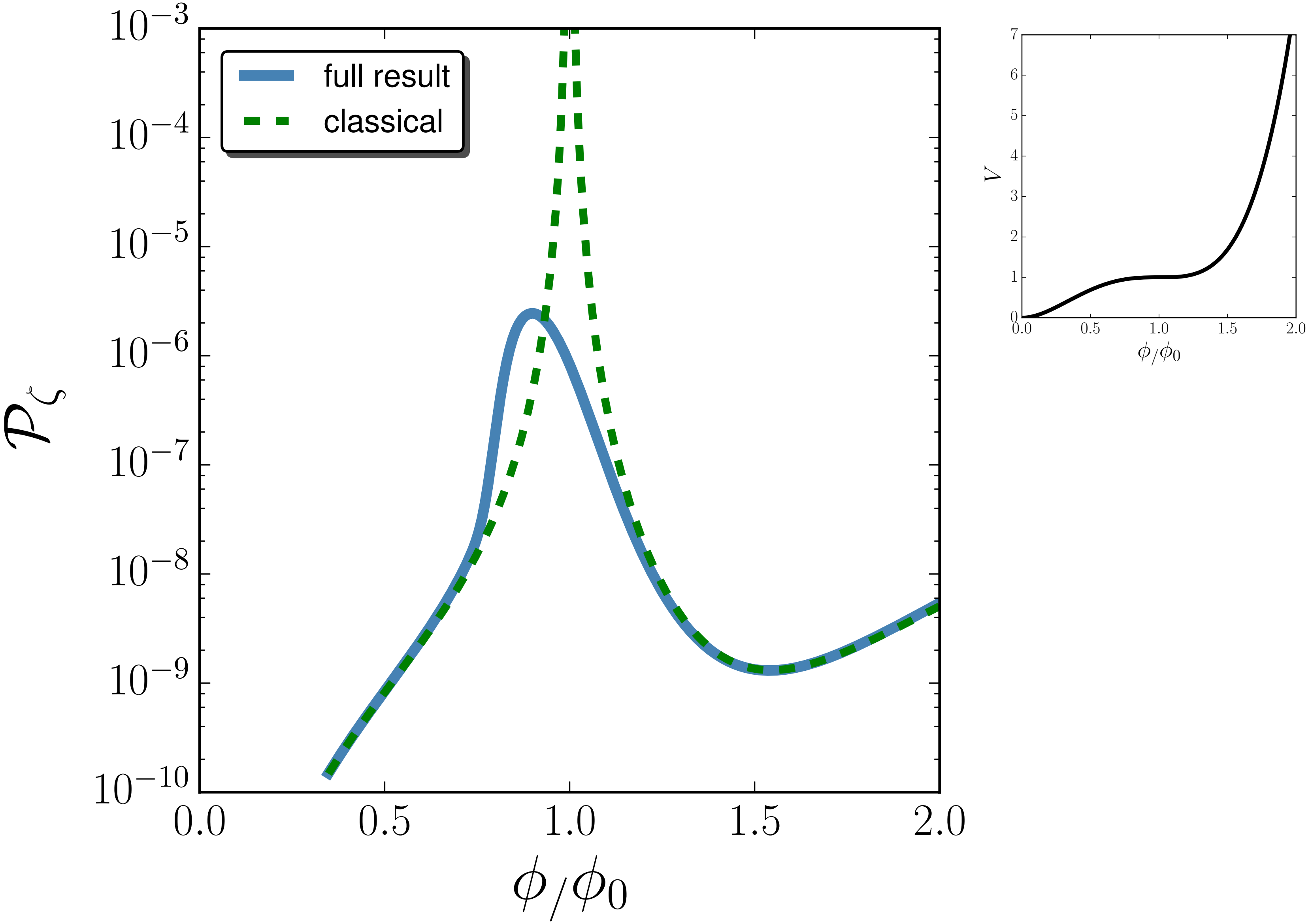

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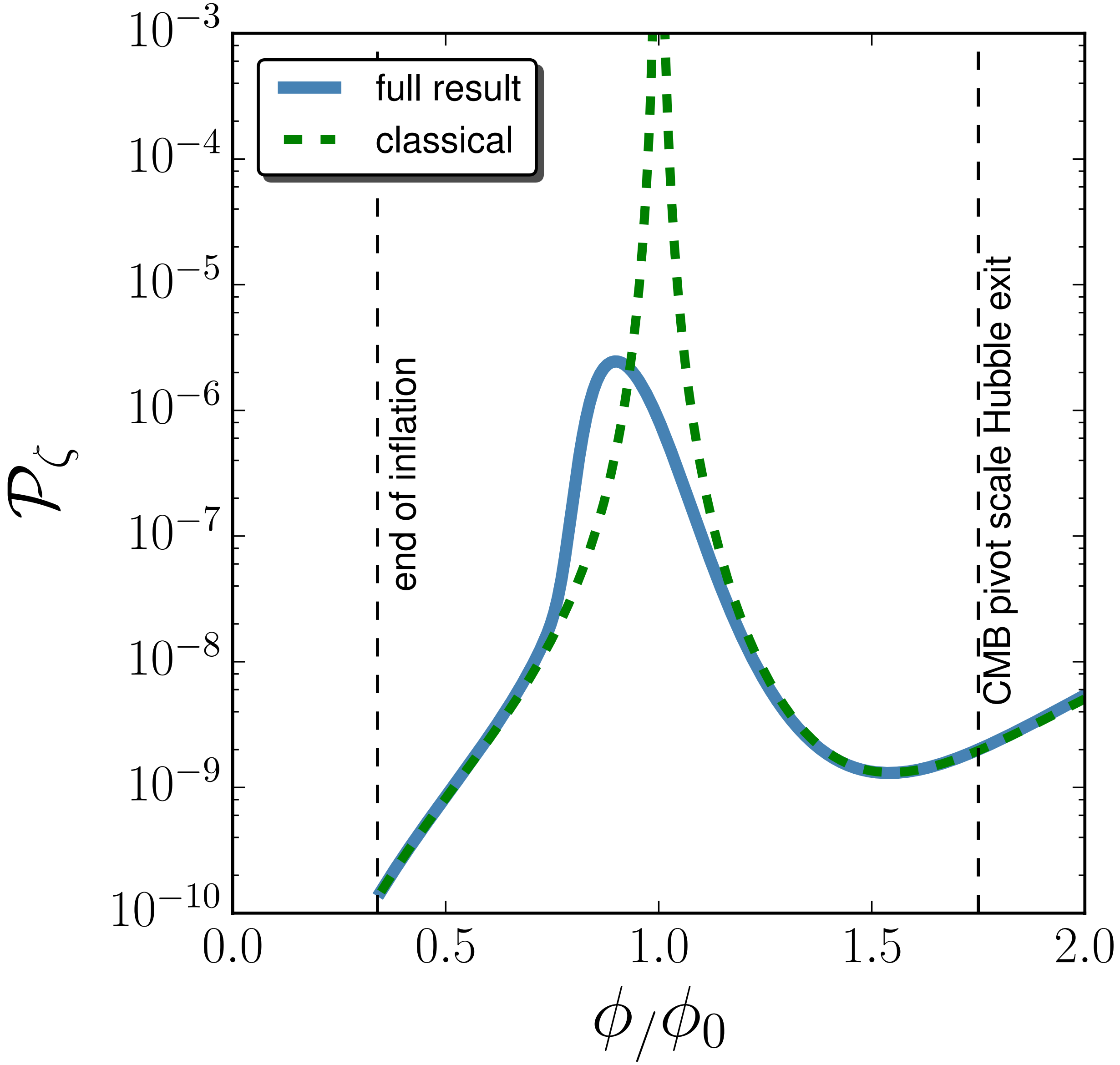

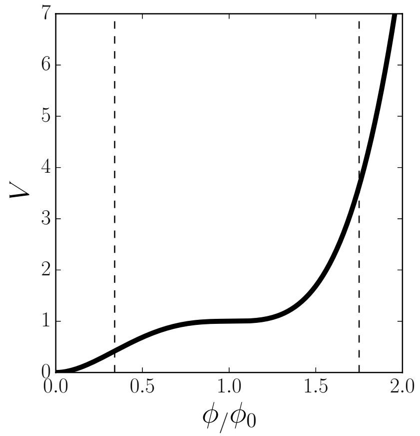

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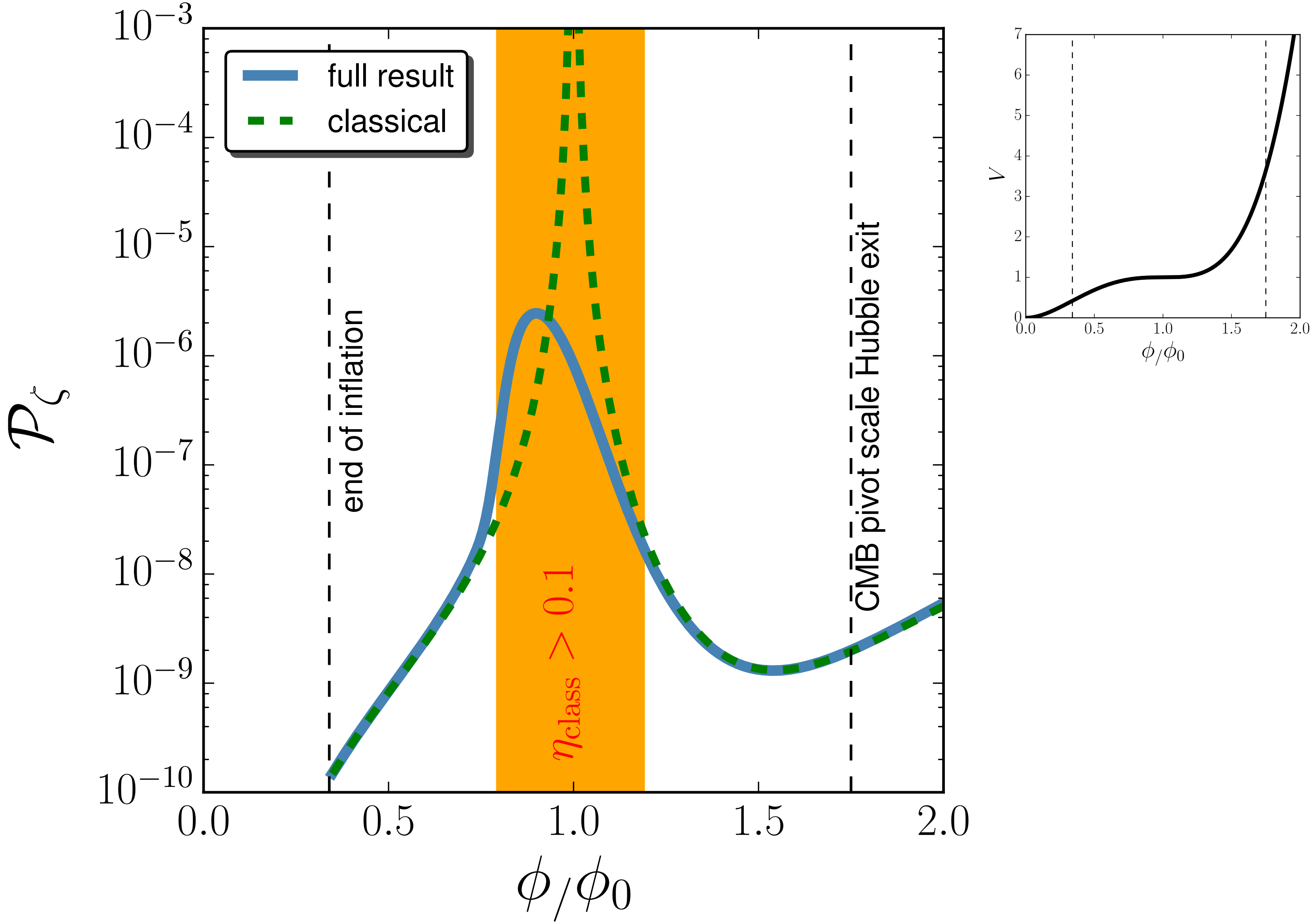

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- **Multi-Field** setups? **Non-canonical** Fields?

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- Making use of the ***δN* formalism**, we have derived all the **n-point correlation functions** of scalar perturbations in **stochastic inflation**

$$f_{\rm NL} = \frac{5}{24} M_{\rm Pl}^2 \left[6\frac{v'^2}{v^2} - 4\frac{v''}{v} + v\left(11\frac{v'^2}{v^2} - 158\frac{v''}{v} - 9\frac{v'''}{v'} + 118\frac{v''^2}{v'^2} \right) + \cdots \right]$$

- The **classical** results arise as a **saddle-point** limit

- A new **classicality criterion** has been derived: $\left| 2v - \frac{v''v^2}{v'^2} \right| \ll 1$

$\hookrightarrow$ **quantum gravity** effect … but **not only**!

- Can we **see the stochastic effects**? $2v - \frac{v''v^2}{v'^2} = \mathcal{P}_\zeta \left(\epsilon_1 + \frac{\epsilon_2}{4} \right)$

- **Primordial Black Holes** Physics?

- **Multi-Field** setups? **Non-canonical** Fields?

- What about **tensor perturbations**?

Thank you for your attention!

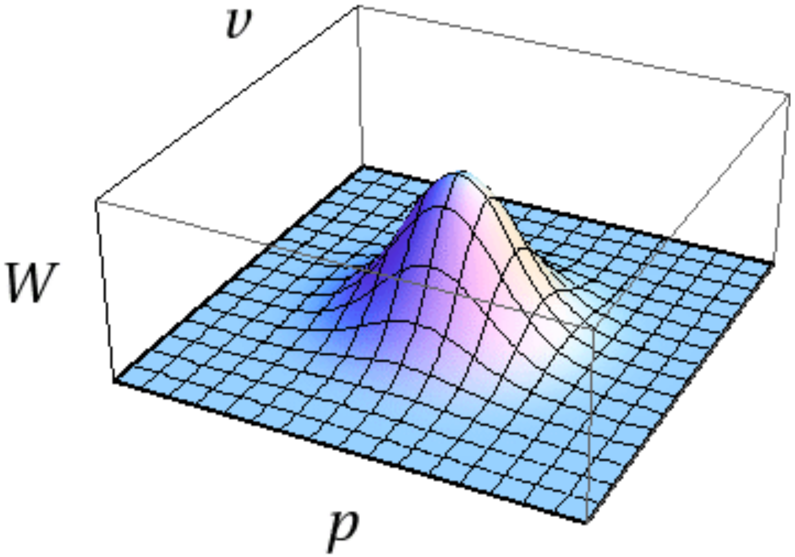
Back Up Slides

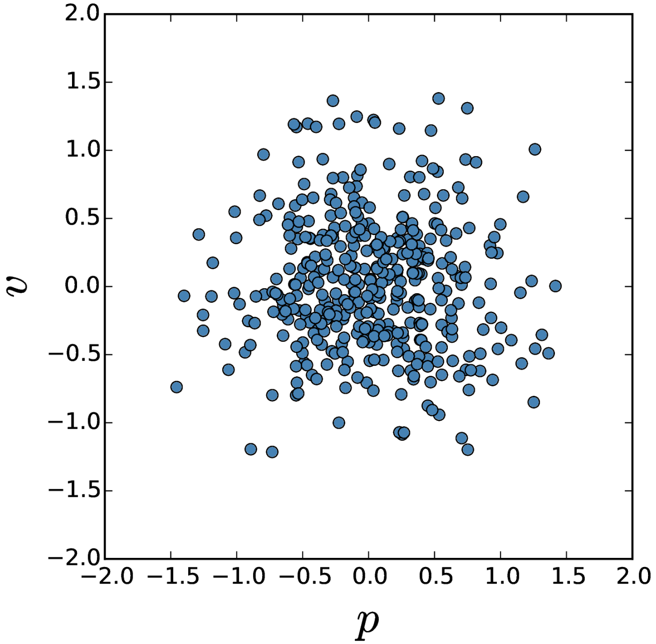
Quantum State of Cosmological Perturbations

Wigner Function
$$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\ \mathrm{e}^{-ip_{\boldsymbol{k}}x}\ \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

- Evolution Equation
$$\frac{\partial}{\partial t} W(v,p,t) = -\{W(v,p,t)\ ,\ H(v,p,t)\}_{\mathrm{Poisson\ \ Bracket}}$$

for quadratic Hamiltonian





Quantum State of Cosmological Perturbations

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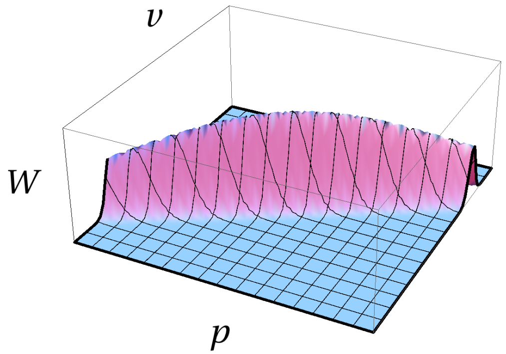
- Quantum Mean Value and Stochastic Average



$$\left\langle \hat{\mathcal{O}}(\hat{v}, \hat{p}) \right\rangle_{\text{quant}} \simeq \int W(v, p)\, \mathcal{O}(v, p)\, \mathrm{d}v\, \mathrm{d}p$$

in the high squeezing limit

Quantum State of Cosmological Perturbations

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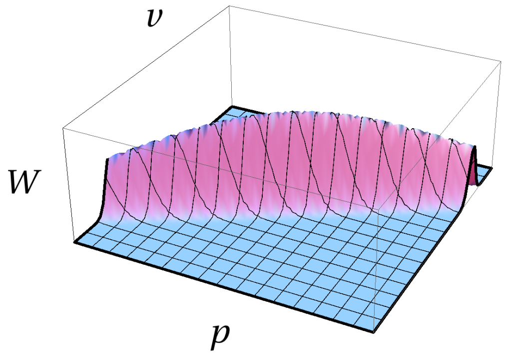

$$\left\langle \hat{\mathcal{O}}\left(\hat{v}, \hat{p}\right) \right\rangle_{\text{quant}} \simeq \int W\left(v, p\right) \mathcal{O}\left(v, p\right) \mathrm{d}v\, \mathrm{d}p$$

in the high squeezing limit

Example: $\langle vp \rangle \xrightarrow[\text{squeezed}]{} e^{\Delta N_*} + \frac{i}{2}\hbar$

Quantum State of Cosmological Perturbations

Wigner Function $$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\ \mathrm{e}^{-ip_{\boldsymbol{k}}x}\ \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

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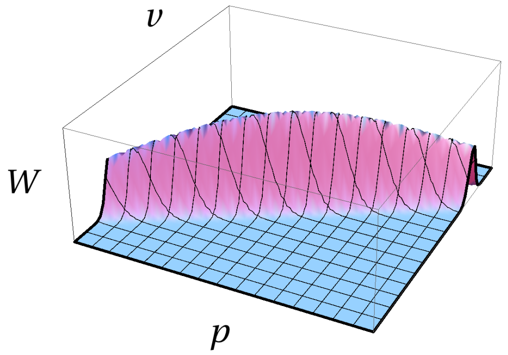

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in the high squeezing limit

Stochastic distribution
of classical processes

Stochastic Formalism

The physical scales probed in the CMB are **super-Hubble** at the end of inflation

$$\hat{\phi}\left(x\right)=\hat{\phi}_{\mathrm{cg}}+\int\frac{\mathrm{d}k}{\left(2\pi\right)^{3/2}}W\left(\frac{k}{\sigma aH}\right)\left[\phi_k\left(t\right)e^{-ikx}\hat{a}_k+\mathrm{h.c.}\right]$$

Upshot: derive a (stochastic and classical) **effective theory** for the coarse-grained part of the field, integrating out the small wavelength modes.

At the level of the action, this can be done using the **Schwinger-Keldysh formalism**

Morikawa, 1990
Hu & Sinha, 1995
Matarrese, Musso & Riotto, 2003

Stochastic Formalism

Heuristically, this can be done at the level of the **equation of motion**

Starobinsky, 1984, 1986, see also 1982
Rey, 1987
Goncharov, Linde & Mukhanov, 1987
Nakao, Nambu & Sasaki, 1988

Let us insert the decomposition

$$\hat{\phi}\left(x\right) = \hat{\phi}_{\mathrm{cg}} + \int \frac{\mathrm{d}k}{\left(2\pi\right)^{3/2}} W\left(\frac{k}{\sigma a H}\right)\left[\phi_k\left(t\right)e^{-ikx}\hat{a}_k + \mathrm{h.c.}\right]$$

In the Klein-Gordon equation of motion

$$\ddot{\hat{\phi}} + 3H\dot{\hat{\phi}} + V'\left(\hat{\phi}\right) = 0$$ and expand in $\phi - \phi_{\mathrm{cg}}$

Stochastic Formalism

At leading order in slow roll:

$$\dot{\hat{\phi}}_{\mathrm{cg}} + \frac{V'\left(\hat{\phi}_{\mathrm{cg}}\right)}{3H^2} = \hat{\xi}_1$$

with

$$\hat{\xi}_1 = -\int \frac{\mathrm{d}k}{(2\pi)^{3/2}} \frac{\partial}{\partial t} \left[W\left(\frac{k}{\sigma a H}\right)\right] \left[\phi_k\left(t\right) e^{-ikx} \hat{a}_k + \mathrm{h.c.}\right]$$

Modes smaller than the coarse-graining scale are constantly escaping the Hubble radius and **source the coarse-grained sector**.

Stochastic Formalism

Large Squeezing Approximation: $\hat{\xi}_1 \rightarrow \xi_1$

quantum operator

stochastic variable

ξ_1 is a Gaussian stochastic variable with two-point correlation

$$\langle \xi_1(x,t)\xi_1(x',t')\rangle \equiv \left\langle \hat{\xi}_1(x,t)\hat{\xi}_1(x',t')\right\rangle_{\mathrm{qu}}$$

$$= \frac{\sin\left(\sigma a H|x-x'|\right)}{\sigma a H|x-x'|}\frac{\sigma^3 H^5}{2\pi^2 a^3}\left|\phi_k\right|^2_{k=\sigma a H}\delta\left(t-t'\right)$$

→ 1 if x and x’ are in the same Hubble patch
→ 0 if x and x’ are in different Hubble patches

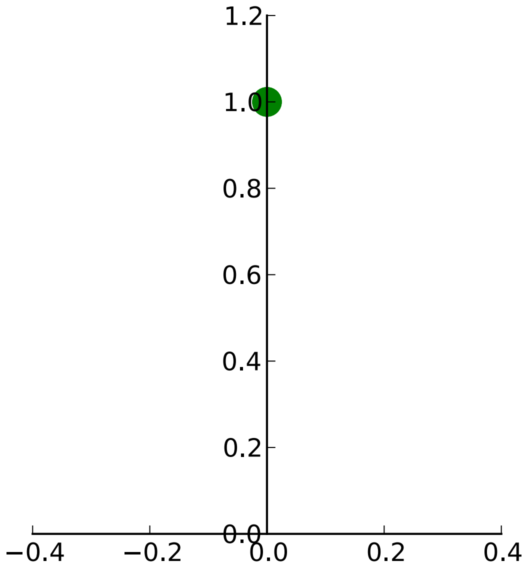
$(H/2\pi)^2$
in de Sitter

for a step
window function
(Markovian)

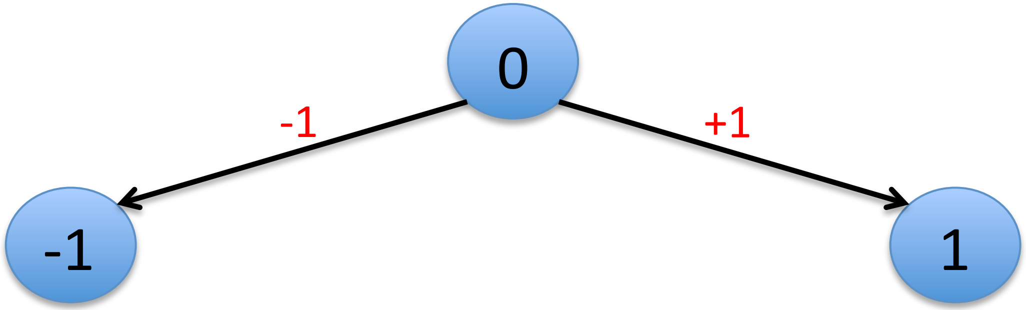
$$\frac{\partial \phi_{\mathrm{cg}}}{\partial N} + \frac{V'}{3H^2} = \frac{H}{2\pi}\xi \qquad \text{with} \qquad \langle \xi(N)\xi(N')\rangle = \delta(N-N')$$

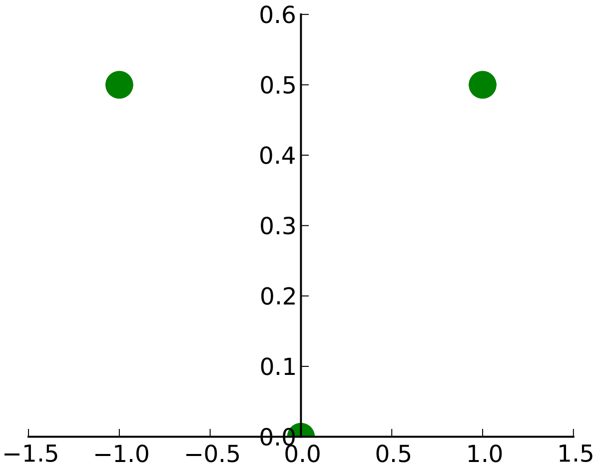
Separate Universe Picture

0



Separate Universe Picture





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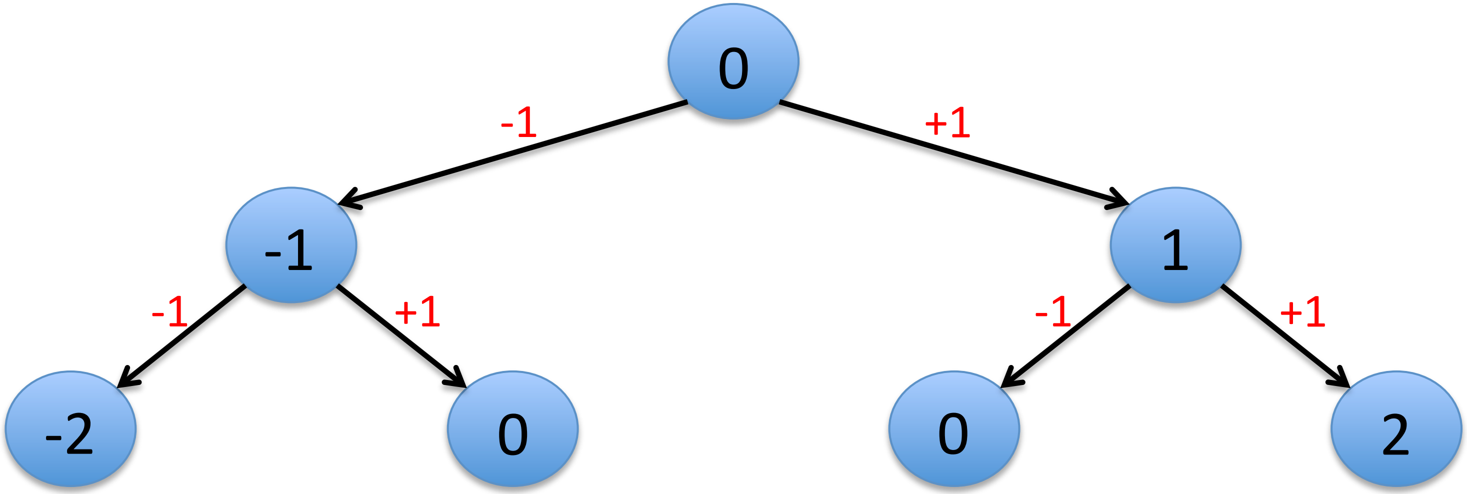

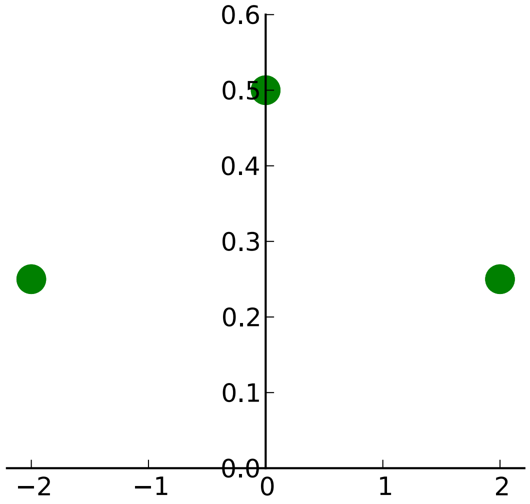

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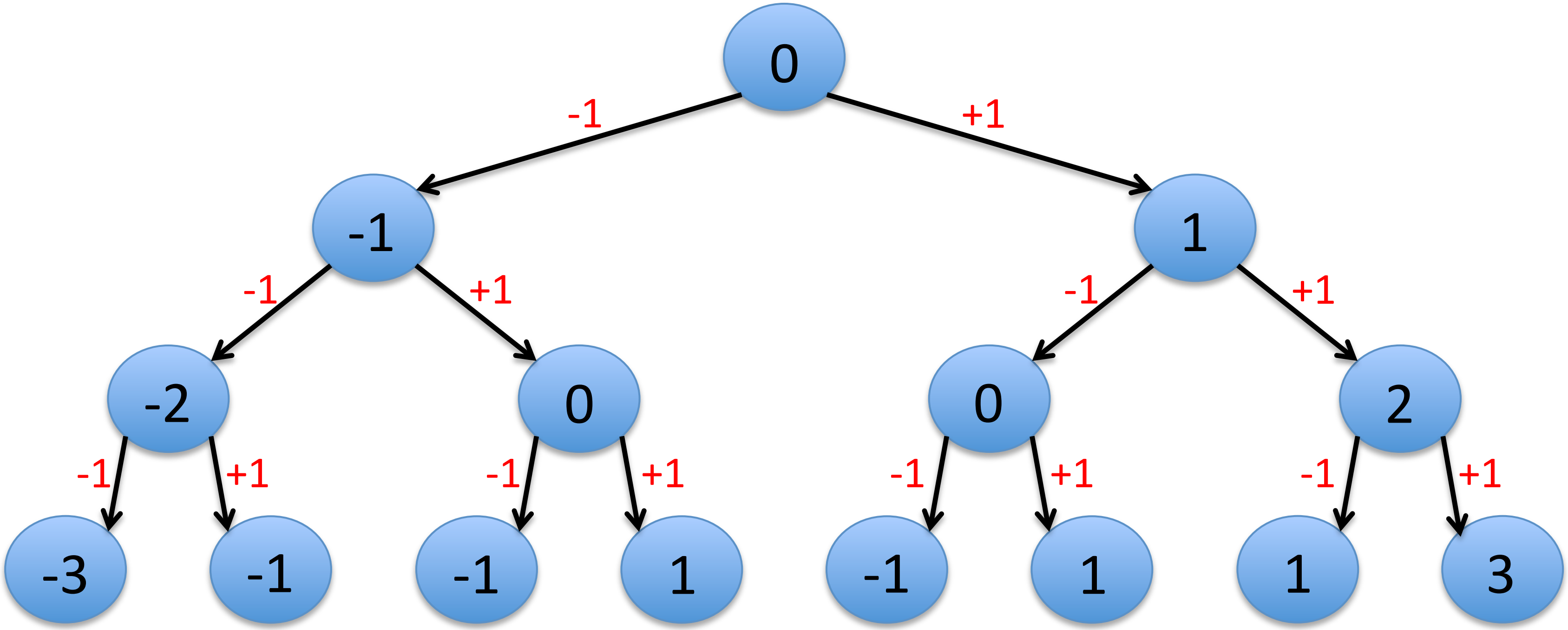

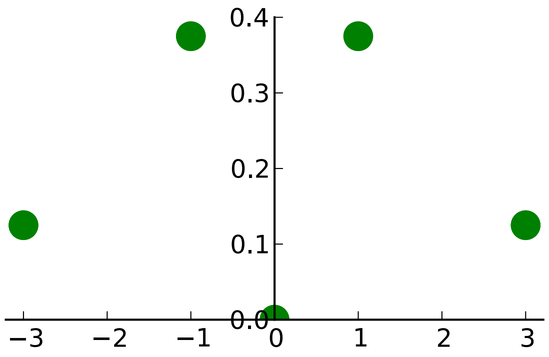

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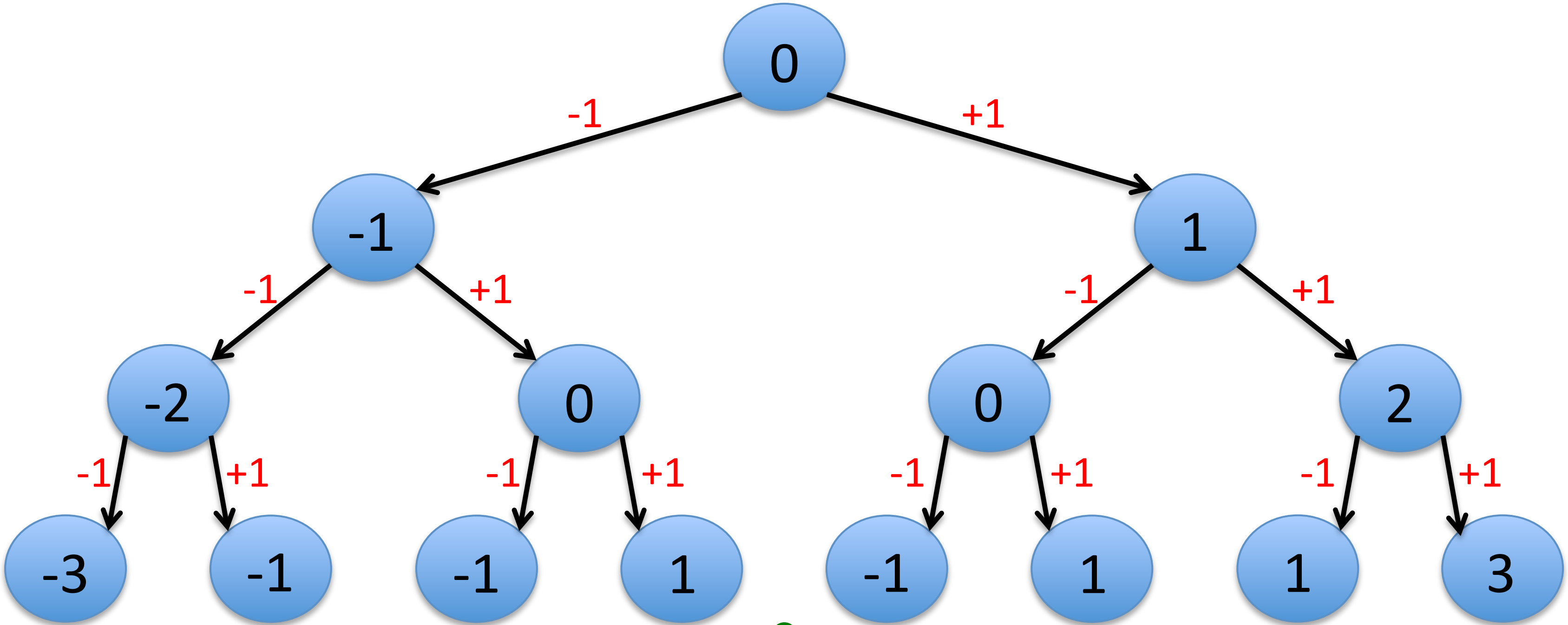
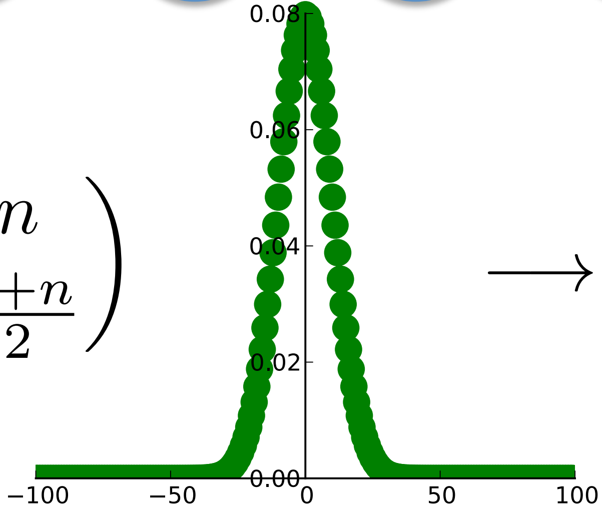




$$p_n(k) = 2^{-n}\binom{n}{\frac{k+n}{2}} \longrightarrow \sqrt{\frac{2}{n\pi}}\exp\left(-\frac{k^2}{2n}\right)$$

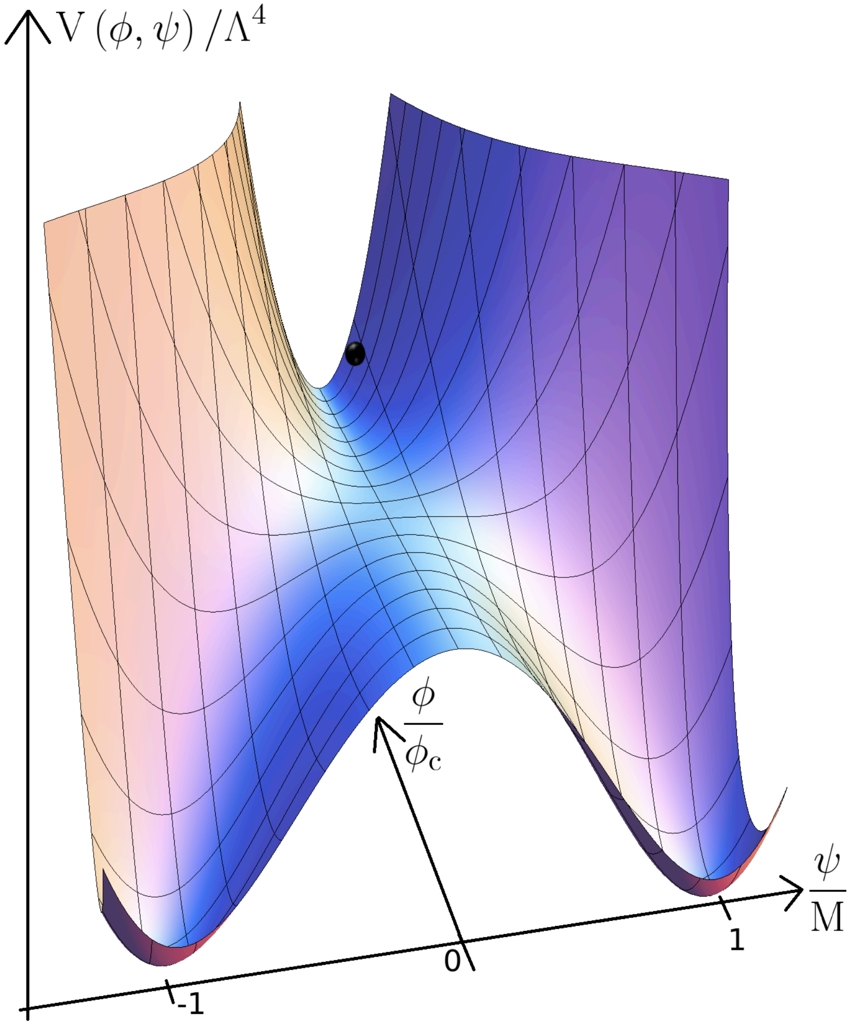
Stochastic Inflation

Hybrid Inflation

$$V\left(\phi,\chi\right)=\Lambda^4\left[\left(1-\frac{\psi^2}{M^2}\right)^2+\frac{\phi^2}{\mu^2}+2\frac{\phi^2\psi^2}{\phi_\mathrm{c}^2M^2}\right]$$

Linde, 1994

Copeland, Liddle, Lyth, Stewart, Wands, 1994



$$\frac{\mathrm{d}\phi}{\mathrm{d}N}=-\frac{2\Lambda^4\phi}{3H^2\mu^2}\left(1+\frac{2\psi^2\mu^2}{\phi_\mathrm{c}^2M^2}\right)$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}N}=-\frac{4\Lambda^4}{3H^2M^2}\psi\left(\frac{\phi^2-\phi_\mathrm{c}^2}{\phi_\mathrm{c}^2}+\frac{\psi^2}{M^2}\right)$$

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$$+\frac{H}{2\pi}\xi_\phi\left(N\right),$$

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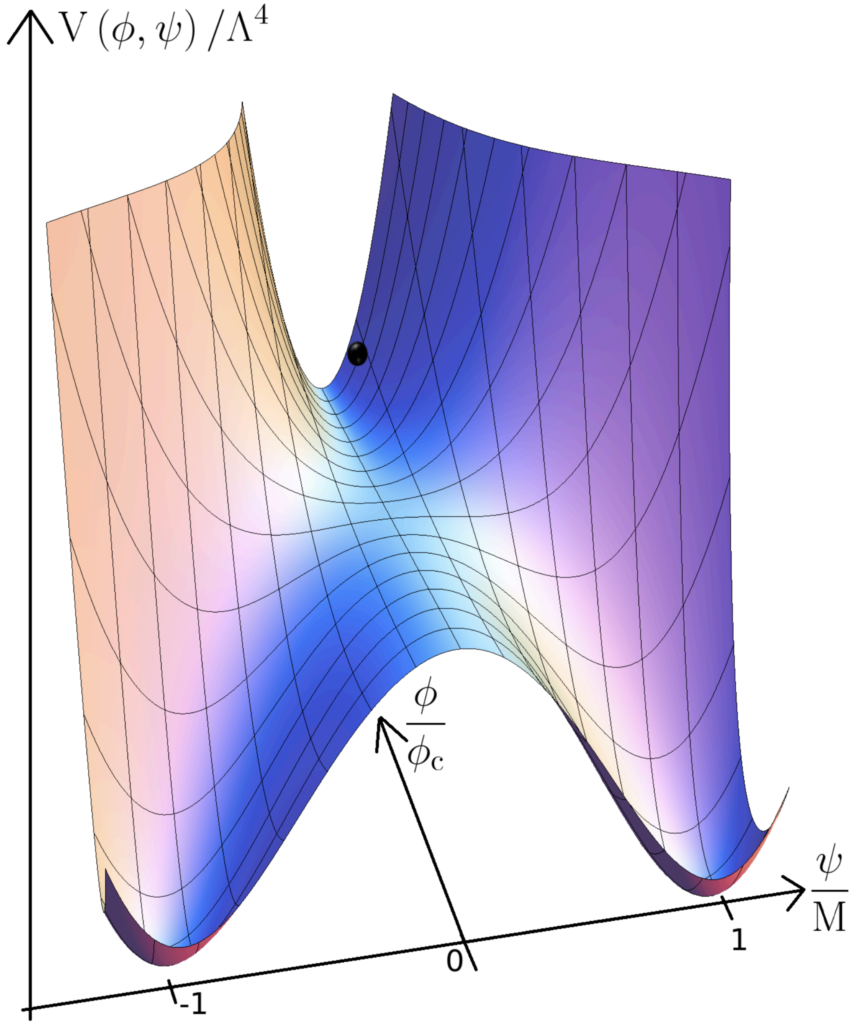

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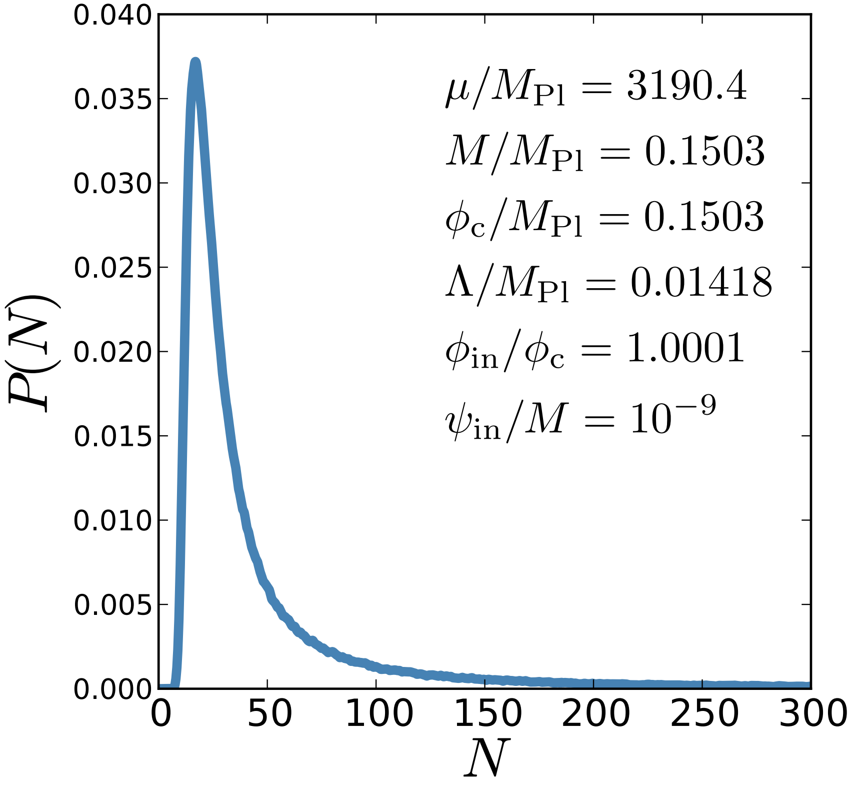
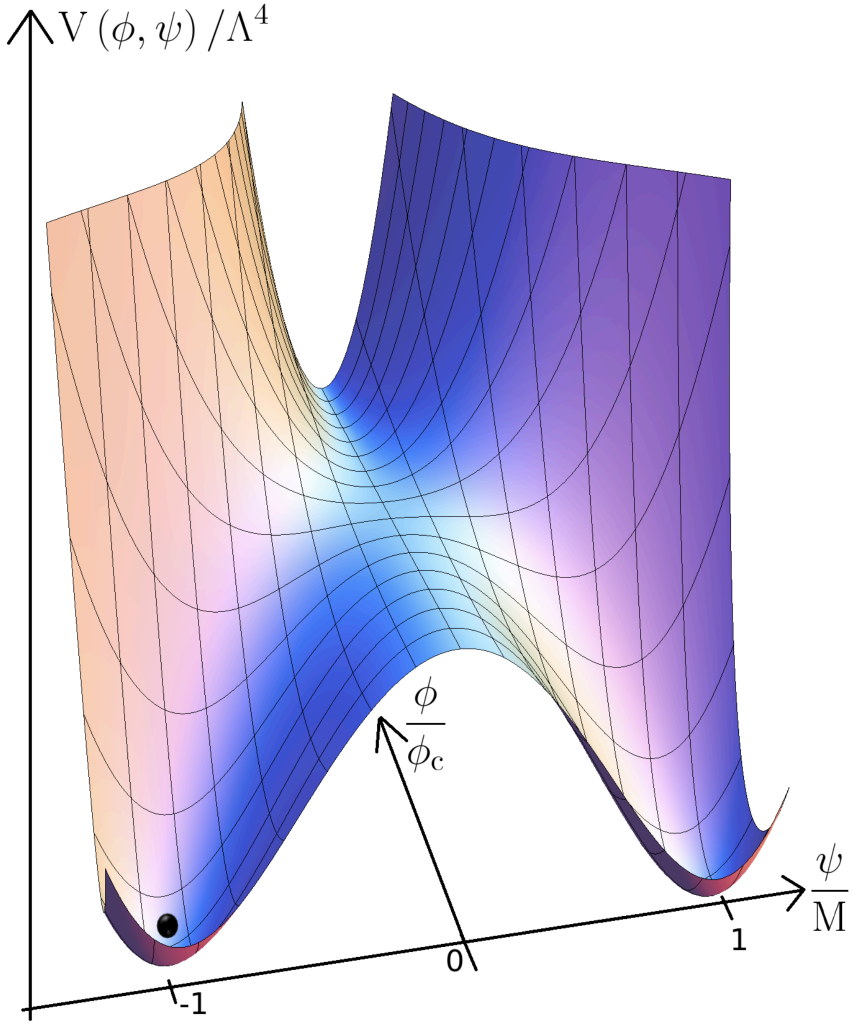


J.Martin & VV, 2011



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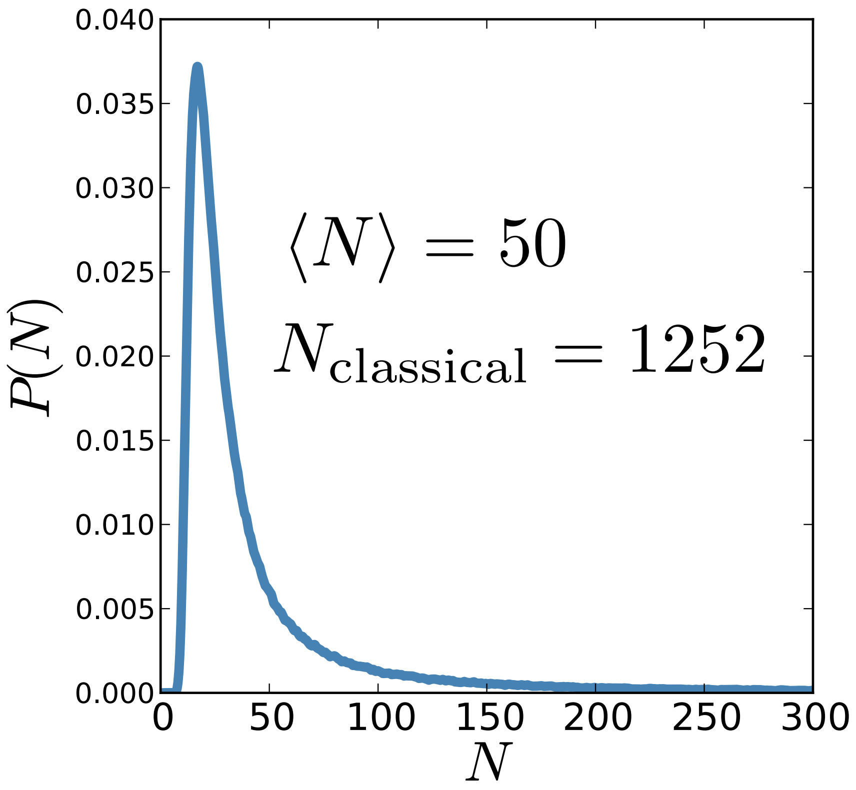

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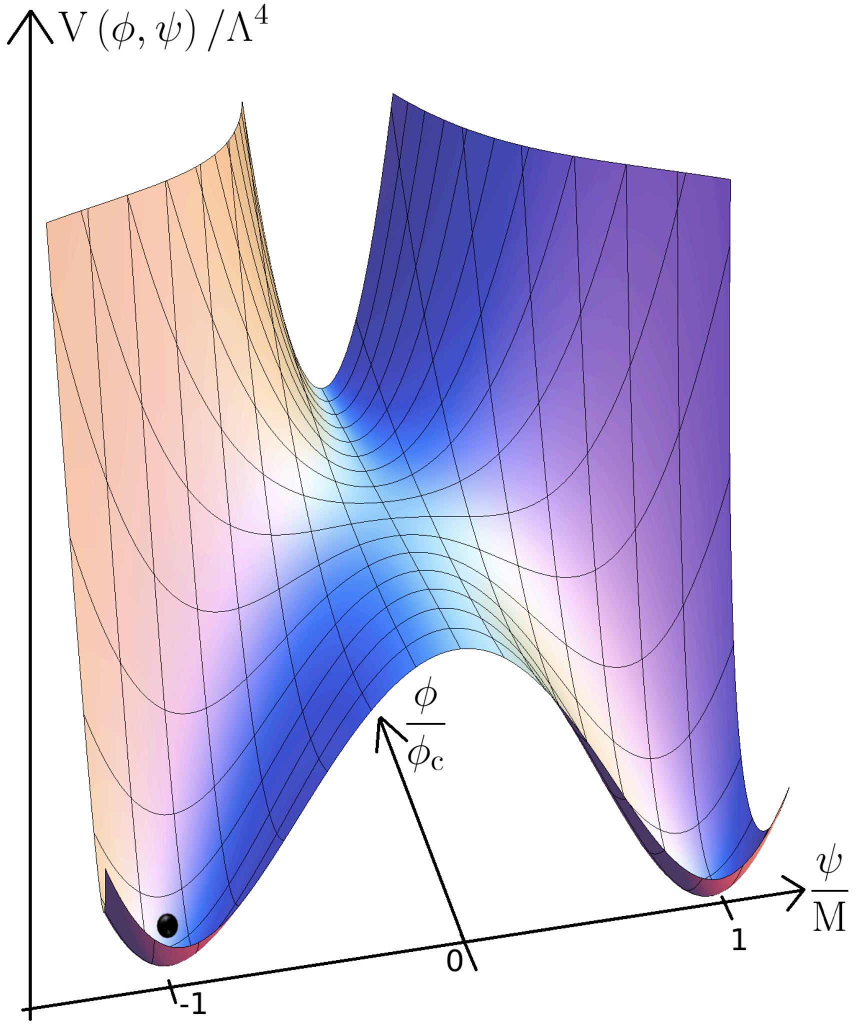

Correlation Functions in Stochastic Inflation

Test Fields

- Scalar field on inflationary background: Starobinsky, Yokoyama, 1994
  Finelli, Marozzi, Starobinsky, Vacca, Venturi, 2008 & 2010
  Garbrecht, Rigopoulos and Zhu, 2013
- Purely Gravitational Systems: Tsamis, Woodard, 2005
- Scalar electrodynamics: Prokopec, Tsamis, Woodard, 2007 &2008

↳ Standard QFT results recovered for <ϕ^2>

Perturbative Expansion

Martin, Musso, 2005
Kunze, 2006
Finelli, Marozzi, Starobinsky, Vacca, Venturi, 2008

$$\phi = \phi_{\rm cl} + \delta\phi$$

$$\langle \delta\phi^2 \rangle = \int^{\sigma a H} \mathcal{P}_{\delta\phi}(k) \mathrm{d}\log k$$

↳ Standard result recovered at leading order for $\mathcal{P}_\zeta$

Replica Field Theory Kuhnel and Schwarz, 2008 (test scalar field in de-Sitter)

Stochastic- δN formalism

Enqvist, Nurmi, Podolsky, Rigopoulos, 2008
Fujita, Kawasaki, Tada, Takesako, 2013 & 2014

The stochastic-δ*N* formalism

- Location of the observational window: $k \longrightarrow \phi_*(k)$
- Number of e-folds: $\mathcal{N}\left(\phi_*\right) \longrightarrow \delta\mathcal{N}^2 \equiv \left\langle \mathcal{N}^2 \right\rangle - \left\langle \mathcal{N} \right\rangle^2$
- Integrated Power: $\delta\mathcal{N}^2\left(k\right) = \int_k^{k_{\mathrm{end}}} \mathcal{P}_{\delta N}\left(k\right)\frac{\mathrm{d}k}{k}$

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$$= \int_{\ln k_\mathrm{end} - \langle\mathcal{N}\rangle(1-\epsilon_{1*}+\cdots)}^{\ln k_\mathrm{end}} \mathcal{P}_{\delta N}\mathrm{d}N$$

- Scalar Power Spectrum: $\mathcal{P}_\zeta\left(k\right) = \mathcal{P}_{\delta\mathcal{N}}\left(k\right) = \frac{\mathrm{d}\delta\mathcal{N}^2}{\mathrm{d}\left\langle\mathcal{N}\right\rangle}$

$$= \frac{\mathrm{d}\delta\mathcal{N}^2/\mathrm{d}\phi_*}{\mathrm{d}\left\langle\mathcal{N}\right\rangle/\mathrm{d}\phi_*}$$

Requires to compute $\left\langle\mathcal{N}\right\rangle\left(\phi_*\right)$ **and** $\delta\mathcal{N}^2\left(\phi_*\right)$

First Passage Time

$$\frac{\partial\phi}{\partial N}+\frac{V'}{3H^2}=\frac{H}{2\pi}\xi \longleftrightarrow \frac{\partial}{\partial N}P\left(\phi,N\right)=\frac{\partial}{\partial\phi}\left[\frac{V'}{3H^2}P\left(\phi,N\right)\right]+\frac{\partial^2}{\partial\phi^2}\left[\frac{H^2}{8\pi^2}P\left(\phi,N\right)\right]$$

$$=-\mathcal{L}_{\mathrm{FP}}\cdot P\left(\phi,N\right)$$

Langevin equation

Fokker-Planck equation

First Passage Time: Louis Bachelier, 1900

$$\mathcal{L}_{\mathrm{FP}}^{\dagger}\cdot\left\langle\mathcal{N}\right\rangle\left(\phi_*\right)=1$$

$$\left\langle\mathcal{N}\right\rangle'' v-\left\langle\mathcal{N}\right\rangle'\frac{v'}{v}=-1 \quad \text{where} \quad v=V/(24\pi^2 M_{\mathrm{Pl}}^4)$$

$$\left\langle\mathcal{N}\right\rangle=\int_{\phi_{\mathrm{end}}}^{\phi_*}\frac{\mathrm{d}x}{M_{\mathrm{Pl}}}\int_x^{\bar{\phi}}\frac{\mathrm{d}y}{M_{\mathrm{Pl}}}\frac{1}{v\left(y\right)}\exp\left[\frac{1}{v\left(y\right)}-\frac{1}{v\left(x\right)}\right]$$

How do we recover the **classical result?**

First Passage Time

Saddle Point Approximation

$$\langle \mathcal{N} \rangle = \int_{\phi_{\mathrm{end}}}^{\phi_*} \frac{\mathrm{d}x}{M_{\mathrm{Pl}}} \int_x^{\bar{\phi}} \frac{\mathrm{d}y}{M_{\mathrm{Pl}}} \frac{1}{v\left(y\right)} \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v\left(x\right)}\right]$$

$$\left|2v - \frac{v''v^2}{v'^2}\right| \ll 1$$

$$\langle \mathcal{N} \rangle \simeq \int_{\phi_{\mathrm{end}}}^{\phi_*} \frac{\mathrm{d}x}{M_{\mathrm{Pl}}^2} \frac{v(x)}{v'(x)} \left[1 + v\left(x\right) - \frac{v''\left(x\right) v^2\left(x\right)}{v'^2\left(x\right)} + \cdots\right]$$

Classical result

First order correction

First Passage Time

Higher Moments

$$\mathcal{L}_{\mathrm{FP}}^{\dagger} \cdot \left\langle \mathcal{N}^{p} \right\rangle \left(\phi_*\right) = p \left\langle \mathcal{N}^{p-1} \right\rangle \left(\phi_*\right)$$

$$\begin{aligned}
\delta\mathcal{N}^2 &= 2\int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \langle \mathcal{N} \rangle'^{2}\left(y\right) \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v\left(x\right)}\right] \\
\delta\mathcal{N}^3 &= 6\int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \left\langle \mathcal{N} \right\rangle'(y) \delta\mathcal{N}^{2\prime}(y) \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v\left(x\right)}\right] \\
\cdots \quad &= \quad \cdots \\
\delta\mathcal{N}^p\left(\phi_*\right) &= \int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \left[2p\left\langle \mathcal{N} \right\rangle'\left(y\right) \delta\mathcal{N}^{p-1\prime}\left(y\right) + p\left(p-1\right)\left\langle \mathcal{N} \right\rangle'^{2}\left(y\right) \delta\mathcal{N}^{p-2}\left(y\right)\right] \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v\left(x\right)}\right]
\end{aligned}$$

Analytical expression for **all moments !**

The δ*N* formalism

Usual Calculation:

$$\zeta\left(t,x\right)=\delta N\simeq\frac{\partial N}{\partial\phi}\delta\phi$$

$$\frac{1}{\sqrt{2\epsilon_1}M_{\mathrm{Pl}}}$$

classical trajectory

$$\frac{H}{2\pi}\xi$$

Power Spectrum:

$$\mathcal{P}_\zeta=\frac{1}{2M_{\mathrm{Pl}}^2\epsilon_1\left(k\right)}\left[\frac{H(k)}{2\pi}\right]^2$$

(standard result)