A convergent perturbation theory for Newtonian cosmological structure formation

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COSMO15

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Based on:
- V. Zheligovsky & U. Frisch, **J. Fluid Mech. 749 (2014) 404**
- CR, B. Villone & U. Frisch, **Mon. Not. Roy. Astron. 452 (2015) 1421**
- CR, A. Sobolevskiĭ, U. Frisch, work in progress

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Outline

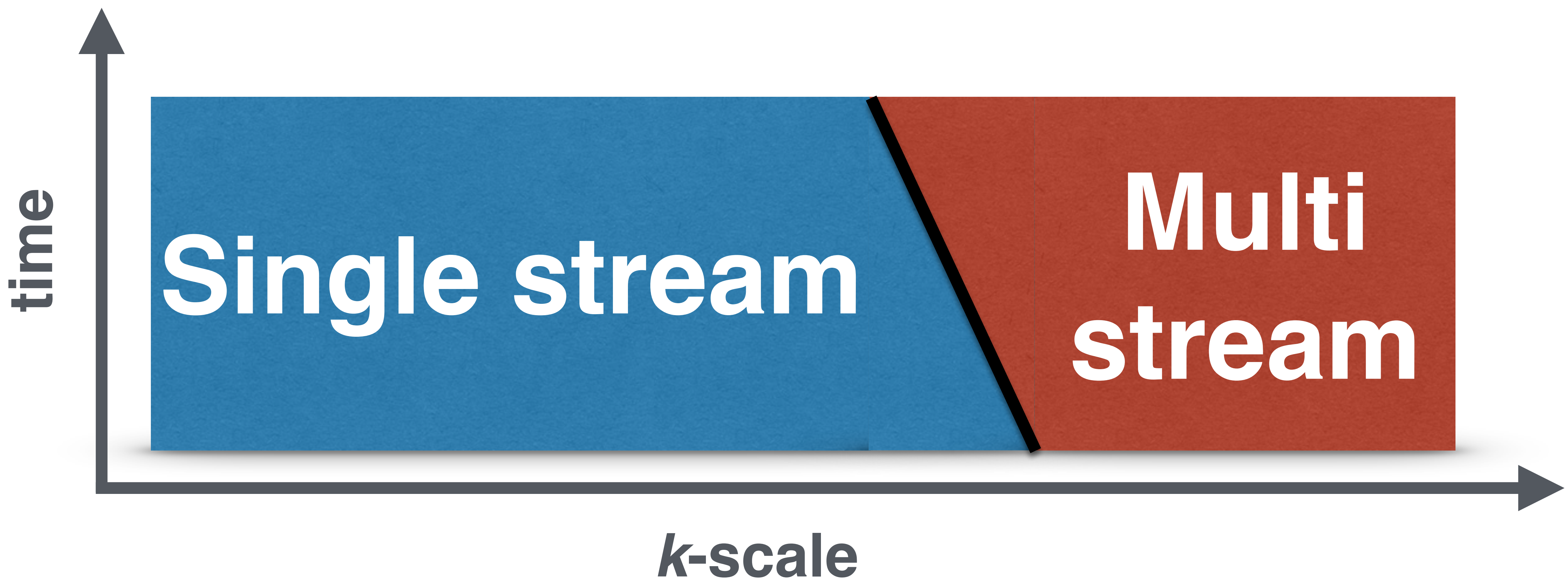
- Starting point: Dark matter evolution in $\Lambda\mathrm{CDM}$ is governed by **Euler-Poisson equations** until first shell-crossing
- how to obtain perturbative solutions to arbitrary **high accuracy?**
- revisit perturbation expansion ⟶ it's a **time-Taylor series!** valid for some **time range**, then re-expand, continue evolution
- Taylor series **representation** of fully **non-linear** particle trajectories
- practical realisation e.g. with a semi-Lagrangian method

Do we need so high accuracy?

- **cosmological reconstruction**

  to trace the whole past dynamical history of the LSS

- Analytical evidence of **shell crossing: (when?)**



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Quick recap of EoMs

The Eulerian description

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Euler-Poisson equations:

$$\partial_a \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla^{\mathrm{E}})\boldsymbol{v} = -\frac{3}{2a}\left(\boldsymbol{v} + \nabla^{\mathrm{E}}\varphi_{\mathrm{g}}\right)$$

$$\partial_a \delta + \nabla^{\mathrm{E}}\cdot[(1+\delta)\boldsymbol{v}] = 0$$

$$\nabla^2_{\mathrm{E}}\varphi_{\mathrm{g}} = \frac{\delta}{a}$$

here: EdS

cosmic scale factor (= modified time variable) a, satisfies Friedmann equations

density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$, and φ_{g} (rescaled) cosmological potential

The Lagrangian description

transform to Lagrangian space: $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q},a) = \boldsymbol{q} + \boldsymbol{\xi}(\boldsymbol{q},a)$

$\boldsymbol{v} = \partial_a^{\rm L} \boldsymbol{x}(\boldsymbol{q},a)$

Jacobian $J = \det\left(\nabla_i^{\rm L} x_j\right)$

$$\partial_{aa}^{\rm L} \boldsymbol{x} = -\frac{3}{2a}\left(\partial_a^{\rm L}\boldsymbol{x} + \nabla_x \varphi_{\rm g}\right)$$

$$(1+\delta)J = 1$$

$$\nabla_x^2 \varphi_{\rm g} = \frac{\delta}{a}$$

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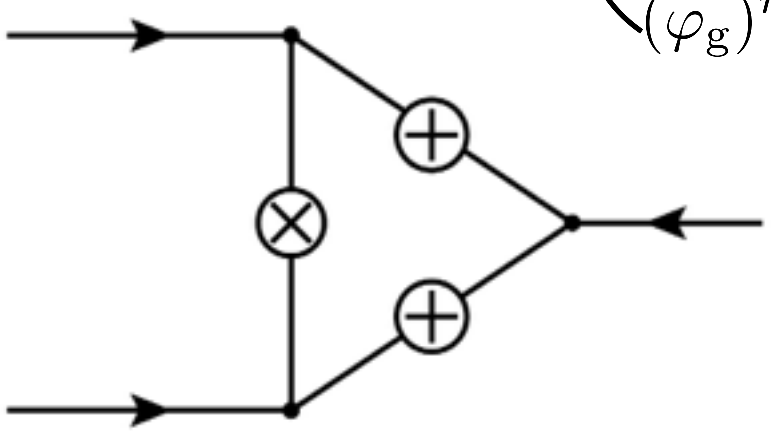
Approximative techniques in literature

- Eulerian or Lagrangian perturbation theory: **assume** that density contrast, peculiar velocity field, etc. are **small**

→ approximate fields without (really) specifying the expansion parameter, e.g.: $\delta(\boldsymbol{x},a) = \sum_{n=1}^{\infty} \delta^{(n)}(\boldsymbol{x})\, a^n$

$(\varphi_{\rm g})^n \sim \left(10^{-5}\right)^n$

→ loop expansion to get **statistical** information of density, etc.



- Many extensions exist: renormalisation, resummation, EFTofLSS, …

Limitations of such techniques

- No control of **shell-crossing**, i.e., no "idea" when exactly the **fluid description breaks down** in 3D!

- perturbation theory in its **weakest** formulation: only **approximative**, because it is unknown if the conventional perturbation series is converging

To overcome limitations,
need rethinking of perturbation theory
(be prepared that results might appear suddenly unknown!!)

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From approximative to **exact** solutions

One possibility to obtain solutions to **arbitrary high accuracy**:

1. choose appropriate time and expansion parameter (e.g., *a* or *D*)

2. derive explicit all-order recursion relations

3. prove that the respective series is converging

   (with it comes a validity regime)

4. use e.g. a semi-Lagrangian approach to obtain

   fully non-linear solutions of Euler-Poisson eqs.

   (until shell-crossing)

Exact solutions for the Lagrangian displacement

[Zheligovsky&Frisch'14]

- **expansion** & time parameter is e.g. **cosmic scale factor**

(or e.g. $D(a) = a - (2/11)\Lambda a^4 + \cdots$)

$$\boldsymbol{\xi}(\boldsymbol{q}, a) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\, a^n$$

Taylor series!! here around *a*=0

- (simple!!) all-order recursion relations

- series expansion is an **exact** solution for (here EdS)

$$0 \leq a \leq T^{\rm EdS} = \frac{0.0204}{||\boldsymbol{\nabla}^{\rm L} \boldsymbol{v}^{\rm (init)}||}$$

← norm of initial velocity gradients

- displace particles from $a = 0$ until $T^{\rm EdS}$ (as the **first** step!)

[CR, Villone&Frisch'15] → generalisation to ΛCDM and beyond

Recursion relations for ΛCDM

$$\nabla^{\rm L}_\mu \xi^{(s)}_\nu = \nabla^{\rm L}_\mu v^{(\rm init)}_\nu \delta^s_1 - \Lambda \frac{s-3}{s+3/2} \mathcal{C}_{\mu\nu} \mu_1^{(s-3)} + \sum_{\substack{1\le j\le 3,\\ j\neq\nu}} \mathcal{C}_{\mu j} \left(\sum_{\substack{1\le k\le 3,\\ 0<n<s}} \frac{2n-s}{s} \left(\nabla^{\rm L}_\nu \xi^{(n)}_k\right) \nabla^{\rm L}_j \xi^{(s-n)}_k \right)$$

$$+ \mathcal{C}_{\mu\nu} \left(\sum_{0<n<s} \left\{ \frac{(3-s)/2 - n^2 - (s-n)^2}{(s+3/2)(s-1)} \mu_2^{(n,s-n)} + \Lambda \frac{4s - n^2 - (s-n)^2 - 6}{(s+3/2)(s-1)} \mu_2^{(n,s-n-3)} \right\} \right.$$

$$\left. + \sum_{n_1+n_2+n_3=s} \left\{ \frac{(3-s)/2 - n_1^2 - n_2^2 - n_3^2}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3)} + \Lambda \frac{4s - 9 - n_1^2 - n_2^2 - n_3^2}{(s+3/2)(s-1)} \mu_3^{(n_1,n_2,n_3-3)} \right\} \right)$$

$\mu, \nu, i, j = 1, 2, 3$

$\mathcal{C}_{ij} = \nabla^{-2} \nabla^{\rm L}_i \nabla^{\rm L}_j$

→ $\boldsymbol{\xi}^{(s)}(\boldsymbol{q})$

→ $\boldsymbol{\xi}(\boldsymbol{q}, a) = \sum_{s=1}^{\infty} \boldsymbol{\xi}^{(s)}(\boldsymbol{q})\, a^s$

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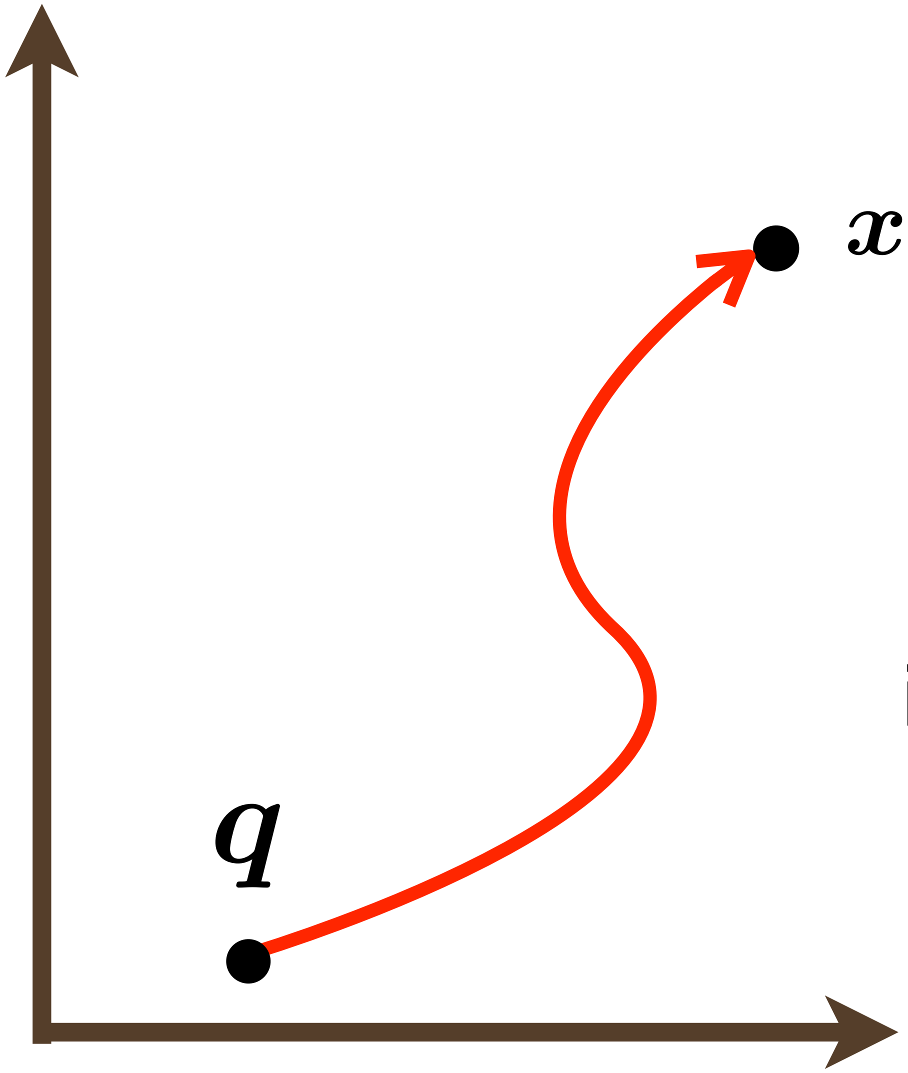


$$\boldsymbol{x}(\boldsymbol{q},a) = \boldsymbol{q} + \sum_{s=1}^{\infty} \boldsymbol{\xi}^{(s)}(\boldsymbol{q}, a=0)\, a^s$$

initial position of particle

Taylor coefficient at $a=0$

- pushing the initial time up to $a=0$ is a consequence of our treatment (-> CMB physics reduced to simple boundary conditions)
- **fully non-linear trajectories** from $0 \leq a \leq T^{\rm EdS} = \dfrac{0.0204}{||\boldsymbol{\nabla}^{\rm L}\boldsymbol{v}^{\rm (init)}||}$
- how to obtain trajectories beyond the radius of convergence?

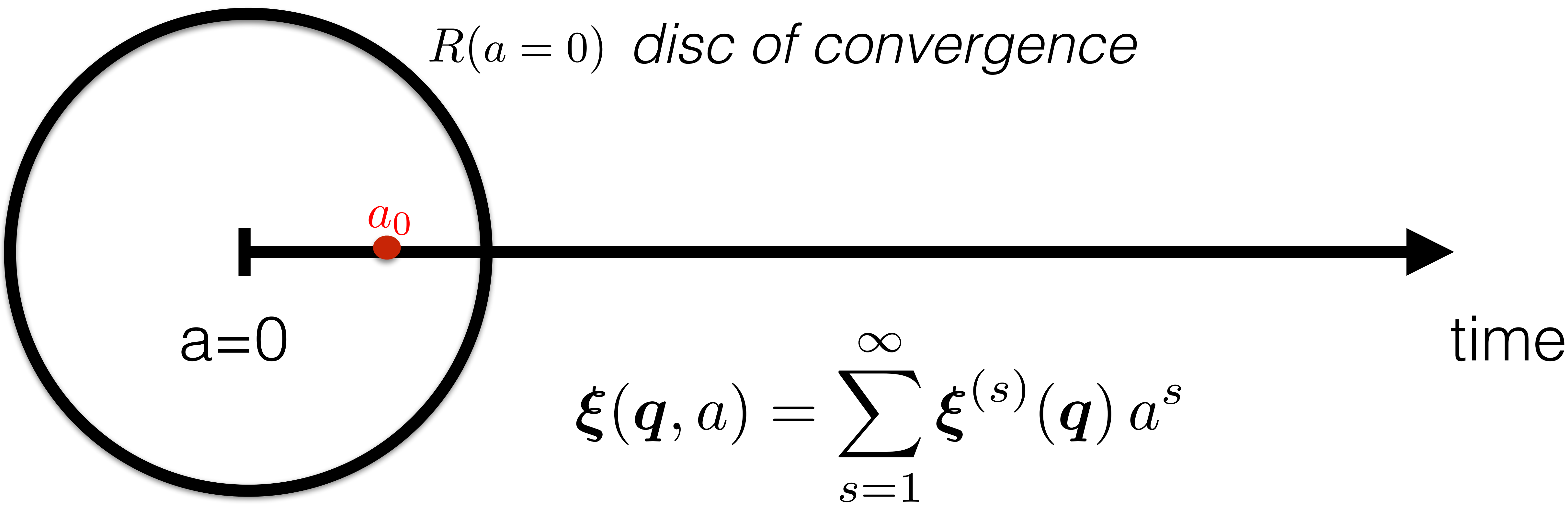
Analytic continuation:

[CR,Sobolevskiĭ&Frisch, *work in progress*]

a semi-Lagrangian approach for all-time solutions

choose $0 < a_0 < R$.

Numerically, the larger a_0, the more Taylor coefficients are required



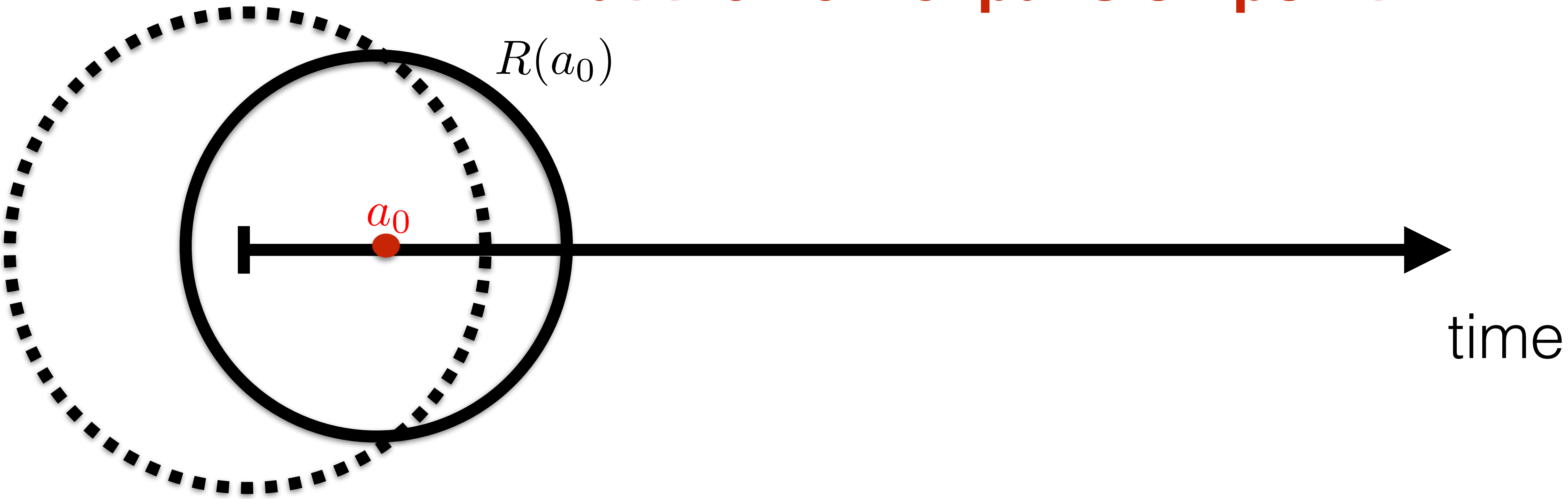
$$\boldsymbol{\xi}(\boldsymbol{q}, a) = \sum_{s=1}^{\infty} \boldsymbol{\xi}^{(s)}(\boldsymbol{q})\, a^s$$

Evolve until a_0. Then transform back to Eulerian coordinates to initiate a fresh start

Analytic continuation: a semi-Lagrangian approach for all-time solutions

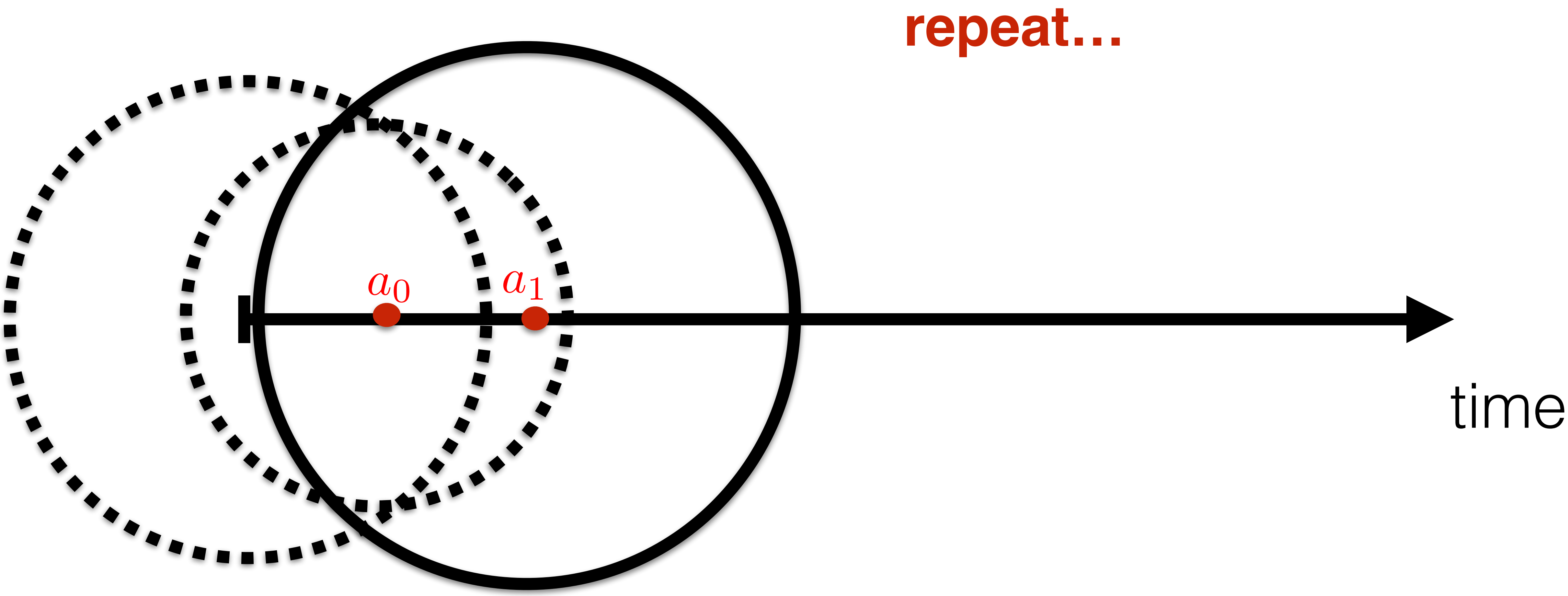
[CR,Sobolevskiĭ&Frisch, *work in progress*]

Next time step, now at a_0. Taylor coefficients obtained at the new expansion point



Analytic continuation: a semi-Lagrangian approach for all-time solutions

[CR,Sobolevskiĭ&Frisch, *work in progress*]



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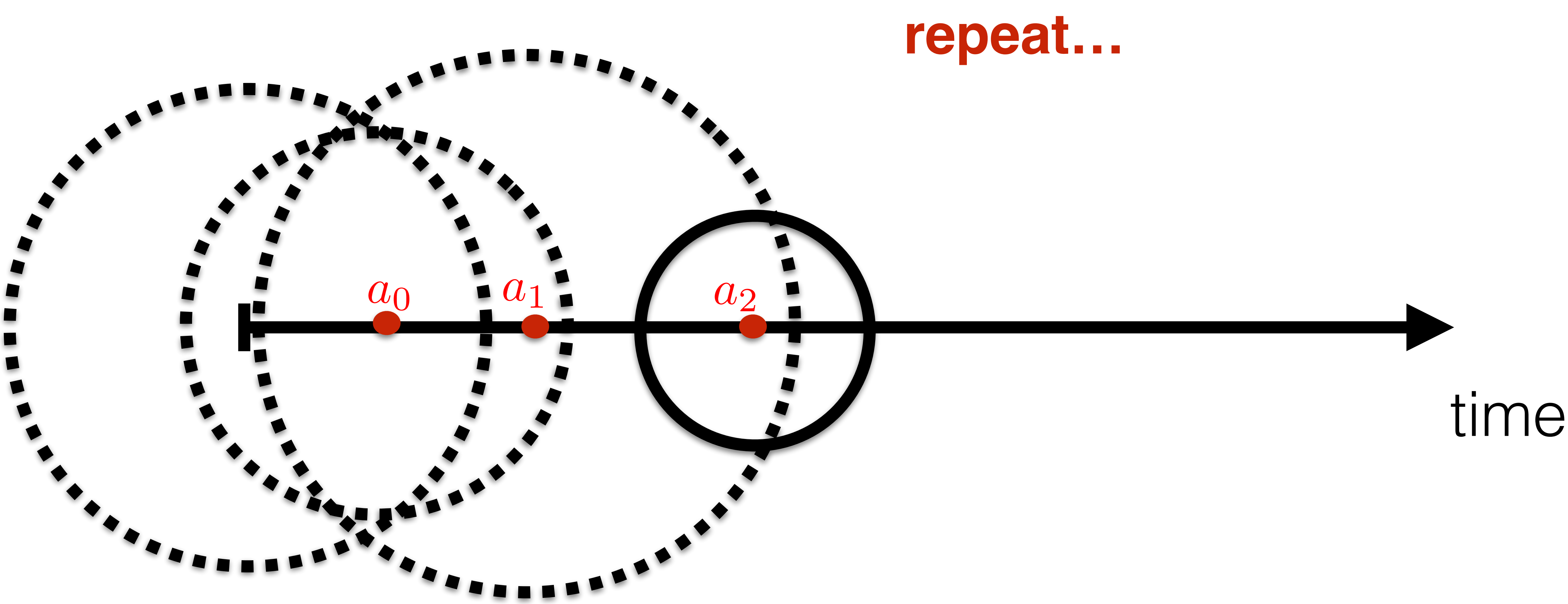
Analytic continuation: a semi-Lagrangian approach for all-time solutions

[CR,Sobolevskiĭ&Frisch, *work in progress*]



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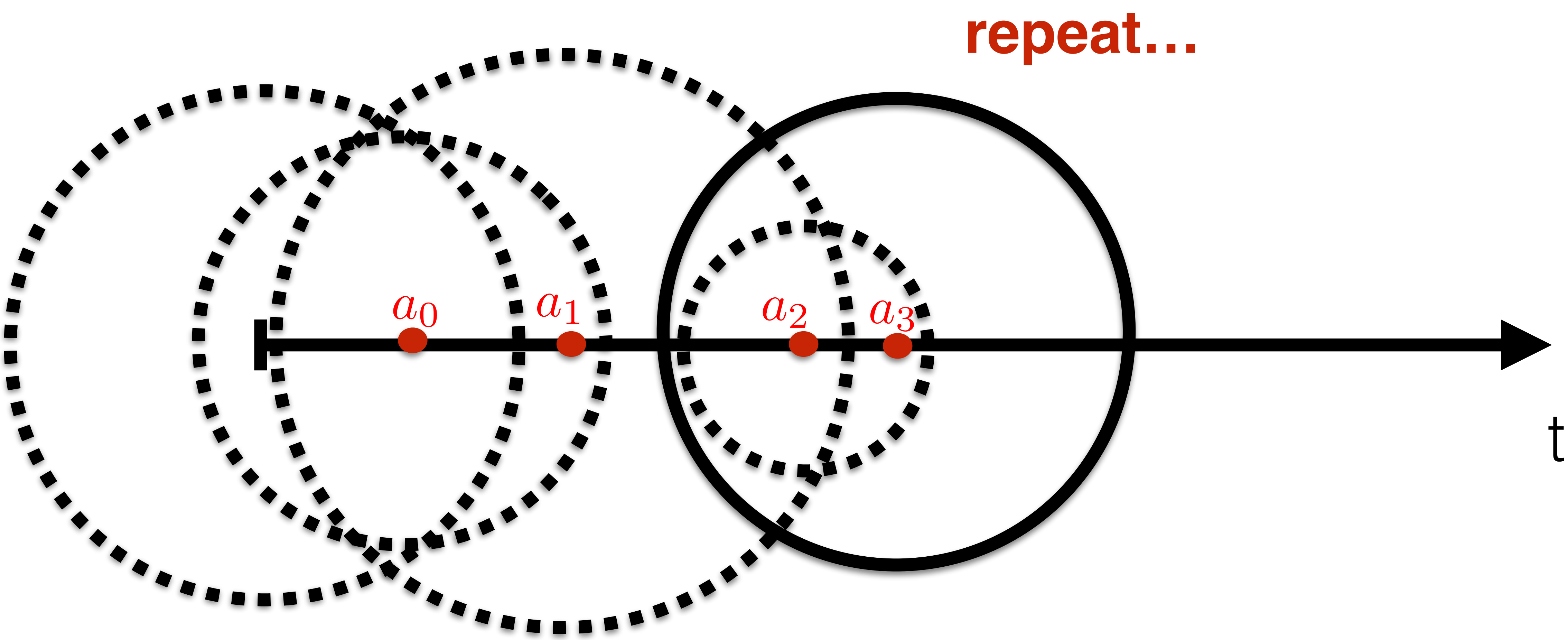
Analytic continuation: a semi-Lagrangian approach for all-time solutions

[CR,Sobolevskiĭ&Frisch, *work in progress*]



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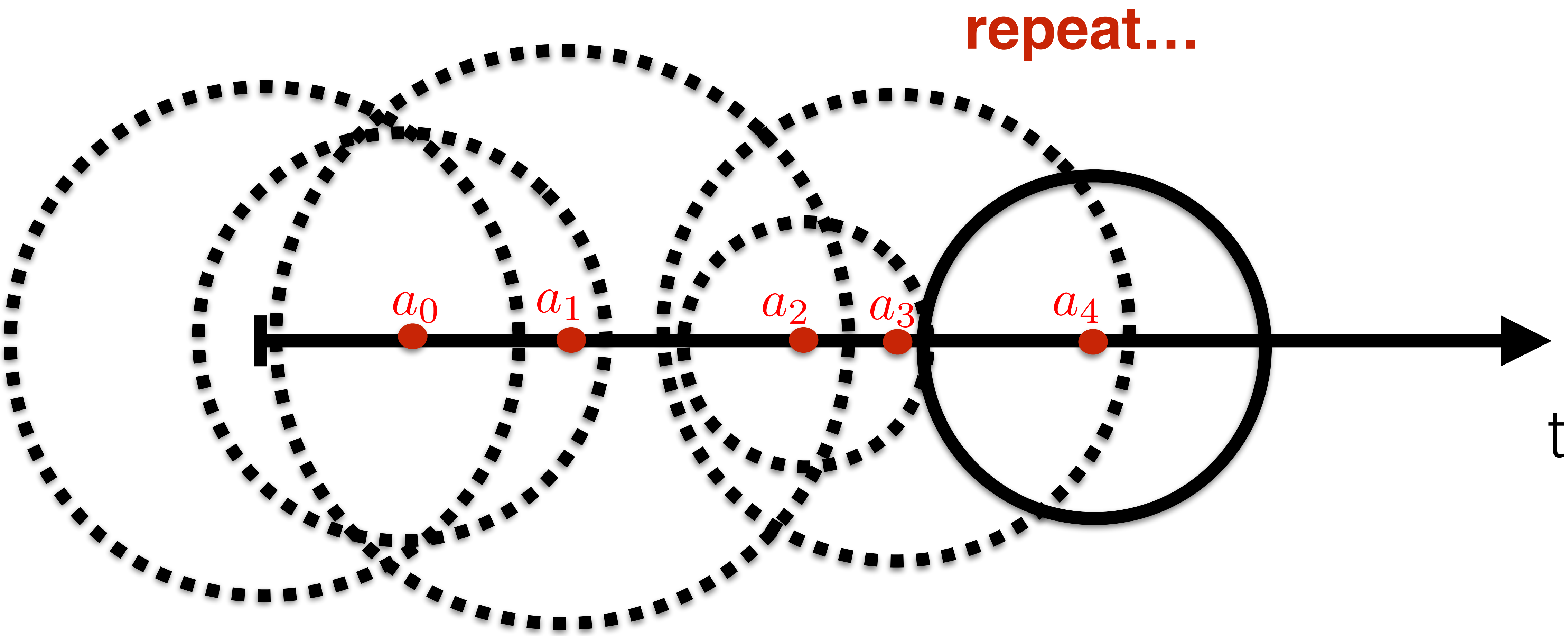
[CR,Sobolevskiĭ&Frisch, *work in progress*]

Analytic continuation: a semi-Lagrangian approach for all-time solutions



Until some finite time (or shell crossing!)

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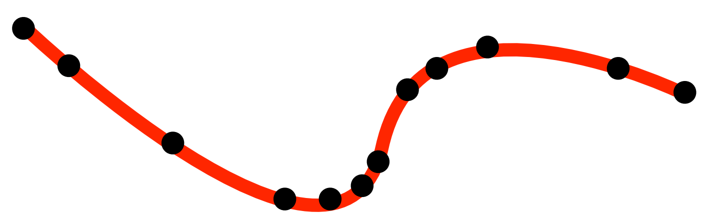
Conclusions

arXiv:1504.00032

- all-order recursion relations for the **trajectories** in $\Lambda\mathrm{CDM}$
- we have made use of an ***a*-time-Taylor expansion**
- we proved **convergence** of the series
  (from *a*=0 until some finite *a*-time)
- obtained analytic **bounds on shell-crossing**

Applications?



- e.g. efficient semi-Lagrangian method to obtain trajectories to **arbitrary high accuracy** (even higher than *N*-body!)

→ study shell crossing / birth of **multi-streams**

→ **deterministic** cosmological reconstruction