Aspects of topos theory in cosmology

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Motivations

Nowadays, formulation of physical theories entirely in terms of classical, boolean logic is regarded almost as a rule. Basically, such a logic is internal to a category of sets **Set**. Here we propose to weaken this principle and *locally* refer to a logic internal to the specific *topos*, as a mathematical tool suitable also for building cosmological models. In particular, we discuss the effects of this modification for a theory of distributions applied to general relativity and quantum field theory. The perspectives cover also the possibility to model the evolution of the Universe by a change of logic, hence also the underlying geometry.

Topos theory

A *topos* is a particular category that behaves very much like **Set** — it contains counterparts of $\{*\}$ and $\varnothing$, the powerset X^Y etc. The main difference is that internal logic of a topos is an *intuitionistic* one, i.e. no axiom of choice nor exluded middle $\phi \vee \neg\phi = 1$ at hand, only constructive reasoning allowed. We take a rather special, *Basel* topos $\mathcal{B}$ as a stage for further analysis.

The Basel topos $\mathcal{B}$

A Basel topos $\mathcal{B}$ is a topos of sheaves on a particular site of C^∞-rings, constructed in detail in [1]. What favours $\mathcal{B}$ among other topoi is that it contains both nilpotent and invertible infinitesimals, together with infinite (smooth, s-) natural numbers N. The former makes $\mathcal{B}$ a natural environment for *synthetic differential geometry* (SDG), while the latter equips $\mathcal{B}$ with the machinery of *non-standard analysis* (NSA). It can be argued that s-natural numbers N in $\mathcal{B}$ are more “natural” to $\mathcal{B}$ than ordinary $\mathbb{N}$, e.g. smooth interval $[0,1]$ is s-compact while not compact in $\mathcal{B}$. Since infinite numbers are inherently built in N, this could be a premise to regard $\mathcal{B}$ as a suitable tool to handle various divergent phenomena and interpret directly divergent expressions in $\mathcal{B}$. However, it is even better to look at a concrete theory, i.e. theory of distributions. It can be shown that in $\mathcal{B}$ every distribution T is *internally* T_f, i.e. it is represented by a smooth function. Recall that classical distributions *can not* be multiplied in general (e.g. there is no δ^2). In $\mathcal{B}$, as in NSA, this is no longer the case. Let us look at some applications.

Applications to astrophysics and cosmology

Remark: the following applications were originally derived by use of *Colombeau algebras* — the other way to speak about multiplication of distributions rigorously. However, Colombeau algebras and NSA are in strict correspondence, hence the former is also available in $\mathcal{B}$.

- the Reissner–Nordström (charged) black hole in the ultrarelativistic limit generates a formal problem of vanishing field tensor together with non-zero energy density [4]:
  $$F_{ik} \approx 0 \text{ (i.e. field tensor is infinitesimal), while } T_{00} \sim F_{ik}^2 \approx \delta$$
  This fact could not be explained on the ground of classical distribution theory; instead, one has to refer to non-linear distribution theory. This is a rather natural conclusion when interpreted in $\mathcal{B}$;
- with these methods, numerous results have been derived concerning the Schwarzschild metric [5]: a distributional character of the Ricci tensor and curvature scalar
  $$R_i^i \sim \delta,\ R \approx \pi m \delta$$
  Also, by non-linear distributional analysis it can be showed (without leaving Schwarzschild coordinates) that horizon singularity is a of pure coordinate type.

Renormalization in $\mathcal{B}$

- Let us take as an example renormalization of a coupling constant in the scalar ϕ^4 theory. From the equation of motion $\left(\Box + m^2\right) G(x) = \delta(x)$ one concludes $G(x) \sim \frac{1}{x^2}$. Therefore distribution involved in the 2-point 1-loop integral is $G^2(x) \sim \frac{1}{x^4}$. Such a distribution can not be defined on whole space of test functions $\mathcal{D}\left(\mathbb{R}^4\right)$, since unless $\phi(0) = 0$, we have $G^2(\phi) = \int \mathrm{d}^4 x \frac{\phi(x)}{x^4} \to \infty$. We proceed with Epstein–Glaser renormalization as follows: first we restrict $\mathcal{D}\left(\mathbb{R}^4\right)$ to $\mathcal{D}_0\left(\mathbb{R}^4\right) = \{\phi | \phi(0) = 0\}$. The unique extension of G^2 to whole $\mathcal{D}\left(\mathbb{R}^4\right)$ is of the form $G^2_{ren}(\phi) = \int \mathrm{d}x \frac{\phi(x) - \phi(0)}{x} + c\delta(\phi)$, which equals the Cauchy principal value of G^2. It can be shown [2] that in $\mathcal{B}$ there is a non-standard smooth function f^* such that $G^2_{ren}(\phi) = \int \mathrm{d}x f^*(x)\phi(x)$. Thus a problem of regularization of distributions is automatically reformulated in $\mathcal{B}$.
- In the end, there is one more interesting remark: in $\mathcal{B}$ the finite Casimir effect can be obtained without imposing a cut-off on integrals representing contributions of EM field. Recall that classically [4] the finite result is obtained by regularization of a difference in energy with a cut-off function $f(k/k_m)$:
  $$\Delta E = \hbar c L_x L_y \int \frac{\mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^2} \left(\sum_{n=0}^{\infty} - \int_0^{\infty} \frac{\mathrm{d}n}{\pi} \right) \omega_n \times f(k/k_m) = -\hbar c \frac{\pi^2}{720 a^3}$$
  However, it was shown in [5] that E–M formula can be fully transferred to non-standard domain (the upper limits replaced by non-standard real numbers). The conclusion is that no regularization, hence no physical argument about cut-off, is needed to obtain the same, finite result, as long as the effect is interpreted in $\mathcal{B}$.

Literature

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