# Newtonian perturbations from the Schrödinger-Poisson equations


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based on *Banik, AJC, Sikivie & Todarello,* *arXiv:1504.05968 (PRD, 2015)*

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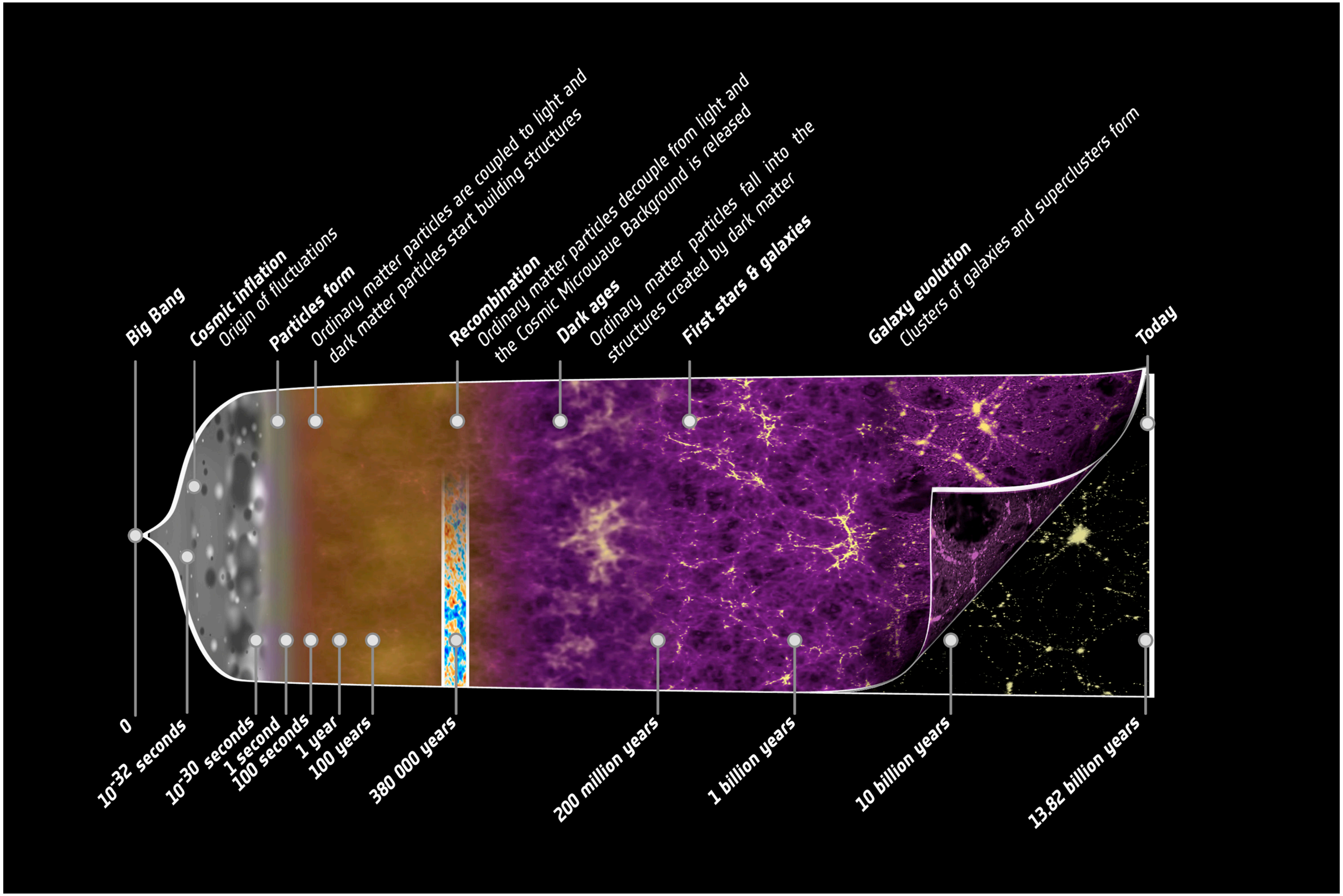
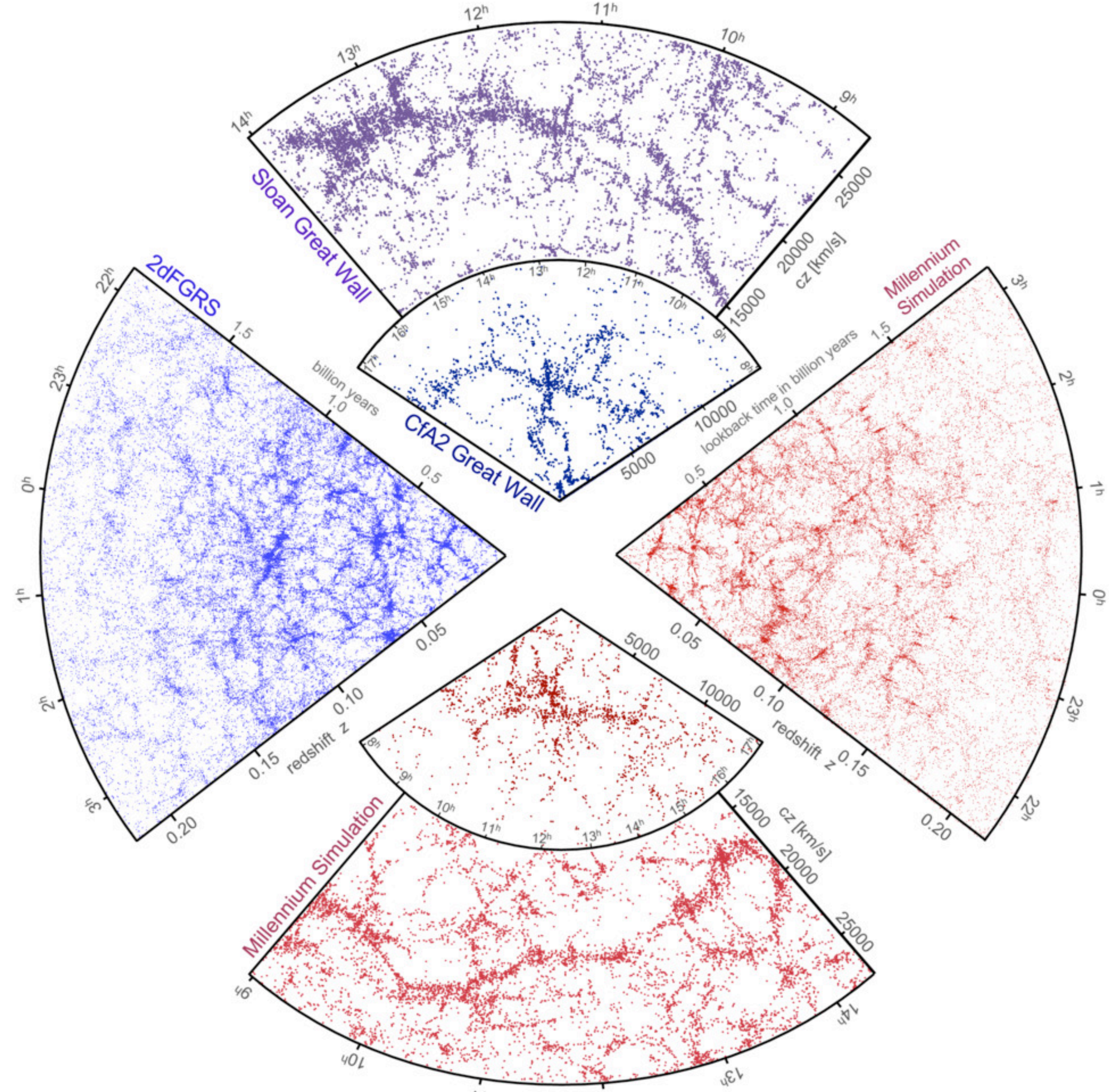
# Motivation



Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
0
10-32 seconds
10-30 seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years


Sloan Great Wall
2dFGRS
CfA2 Great Wall
Millennium Simulation
Millennium Simulation
billion years
lookback time in billion years
redshift z
redshift z
cz [km/s]
cz [km/s]

# What is the dark matter?

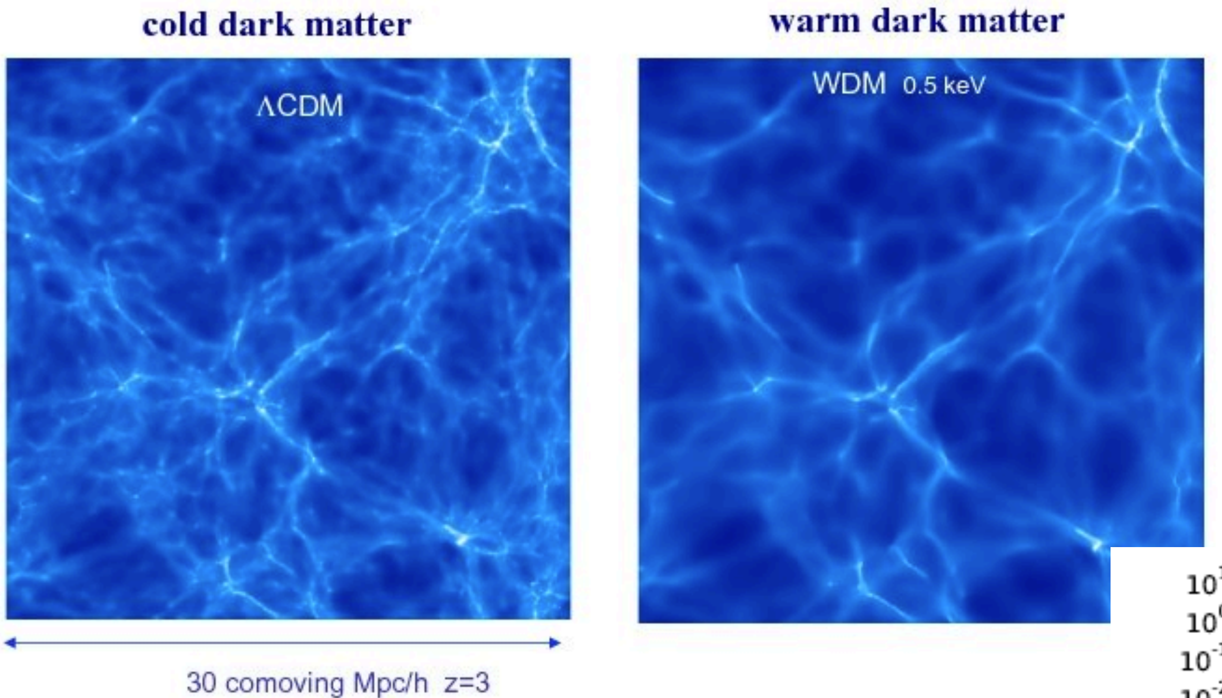
Properties:

- Dark (does not interact with light)
- Massive (interacts gravitationally)
- Weakly interacting (no interactions detected)

Three classes of candidate:

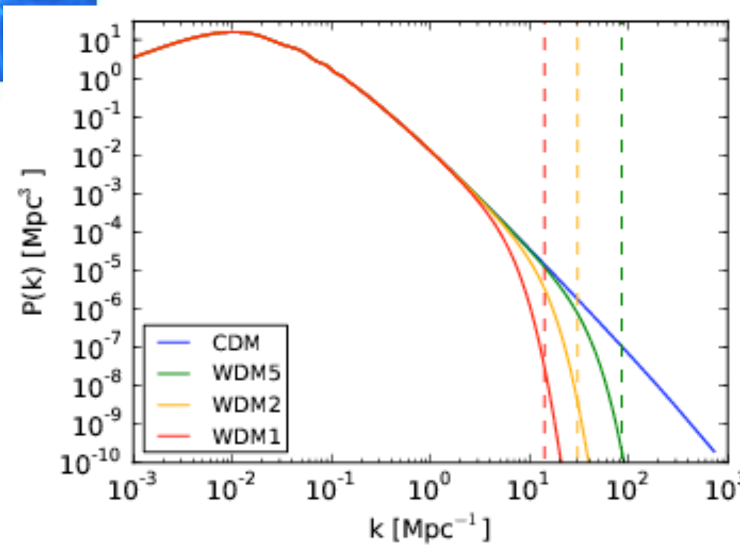
- Weakly Interacting Massive Particle (WIMP)
- Axions and Axion-like particles
- Sterile neutrinos

# Cold vs. Warm Dark Matter




Courtesy: kicc.cam.ac.uk

Warm dark matter forms less structure on smaller scales




arXiv:1308.1088

# Cold dark matter

What do we mean by CDM?

- Cold – slow moving (primordial velocity dispersion very small ~0)
- Pressureless fluid (collisionless)

Since structure formation takes place on subhorizon scales -> Newtonian physics!

- WIMPS ✔
- Axions ?

# Schrödinger-Poisson equations

- Schrödinger equation

$$i\partial_t \psi(\vec{r},t) = \Big( -\frac{1}{2m}\nabla^2 + m\Phi(\vec{r},t)\Big)\psi(\vec{r},t)$$

- Poisson equation

$$\nabla^2 \Phi = 4\pi G m n(\vec{r},t)$$

particle number density: $n(\vec{r},t) = N\psi(\vec{r},t)^*\psi(\vec{r},t)$

- Fluid density satisfies continuity eqn:

$$\partial_t n + \vec{\nabla} \cdot \vec{j} = 0$$

with $\vec{j} = \frac{N}{2m}(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$

- Write $\psi(\vec{r}, t) = \sqrt{n(\vec{r}, t)} \mathrm{e}^{i\beta(\vec{r}, t)}$

so fluid velocity, $\vec{v} = \frac{1}{m} \vec{\nabla} \beta$ obeys the Euler equation

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\Phi} + \vec{\nabla} \left[ \frac{1}{2m^2} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right]$$

# Homogeneous background

Aim: model expanding, homogeneous universe

- Wavefunction: $\psi_0(\vec{r},t) = \sqrt{n_0(t)}\mathrm{e}^{i\frac{1}{2}mH(t)r^2}$
- Velocity $\vec{v} = H\vec{r}$
- Schrödinger eqn satisfied if

$$\partial_t n_0 + 3Hn_0 = 0$$

$$\Phi_0 = -\frac{1}{2}(\partial_t H + H^2)r^2$$

# Homogeneous equations

This reproduces the homogeneous background equations:

- Friedmann $H^2 = \frac{8\pi G}{3} m n_0$
- continuity $\partial_t n_0 + 3 H n_0 = 0$
- acceleration $\partial_t H + H^2 = -\frac{4\pi G}{3} m n_0$

# Inhomogeneities

- To model structure formation, consider perturbations about homogeneous solution
- Much of structure formation takes place in the *linear regime*
- Expand wavefunction

$$\psi(\vec{r}, t) = \psi_0(\vec{r}, t) + \psi_1(\vec{r}, t)$$

# Linear Schrödinger-Poisson eqns

- Schrödinger equation

$$i\partial_t \psi_1 = -\frac{1}{2m}\nabla^2\psi_1 + m(\Phi_0\psi_1 + \Phi_1\psi_0)$$

- Poisson equation

$$\nabla^2\Phi_1 = 4\pi G m(\psi_0^*\psi_1 + \psi_0\psi_1^*)$$

- Switch to Fourier space, e.g.

$$\psi(\vec{r},t) = \psi_0(\vec{r},t)\int d^3k\,\psi_1(\vec{k},t)\mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}$$

- Expand the ansatz wavefunction

$$\psi(\vec{r},t) = \sqrt{n_0(t) + n_1(\vec{r},t)}\mathrm{e}^{i\left(\beta_0(\vec{r},t)+\beta_1(\vec{r},t)\right)}$$

- Gives evolution equation for $\delta \equiv n_1(\vec{r},t)/n_0(t)$

$$\partial_t^2\delta + \frac{4}{3t}\delta - 4\pi G\rho_0\delta + \frac{k^4}{4m^2a^4(t)}\delta = 0$$

- and the velocity perturbation

$$\vec{v}(\vec{k},t) = \frac{ia(t)\vec{k}}{\vec{k}\cdot\vec{k}}\partial_t\delta(\vec{k},t)$$

# Solutions

- The additional term in the evolution equation modifies the Jeans scale of the system

$$k_J^4 = 16\pi G m^2 a^4(t)\rho_0(t)$$

- Assuming k>>$k_J$

$$\delta(\vec{k},t) = A(\vec{k})\left(\frac{t}{t_0}\right)^{2/3} + B(\vec{k})\left(\frac{t_0}{t}\right)$$

# Limitations of fluid treatment

- Consider the example wavefunction

$$\psi(\vec{r},t) = A(\mathrm{e}^{i\vec{k}\cdot\vec{r}} + \mathrm{e}^{-i\vec{k}\cdot\vec{r}})\mathrm{e}^{-i\omega t}$$

  this describes two flows, with:

  - densities $n_1 = n_2 = N|A|^2$
  - velocities $\vec{v}_1 = \vec{k}/m \quad \vec{v}_2 = -\vec{k}/m$

$$\psi(\vec{r},t) = A(\mathrm{e}^{i\vec{k}\cdot\vec{r}} + \mathrm{e}^{-i\vec{k}\cdot\vec{r}})\mathrm{e}^{-i\omega t}$$

- However, the fluid description describes one flow with

$$n(\vec{r}) = 4|A|^2\cos^2(\vec{k}\cdot\vec{r})$$

- and $\vec{v}(\vec{r},t) = 0$

- These are *mathematically* identical, but physically different

# Limitations of *classical* treatment

- Axion fluid is highly degenerate; well described by classical field?
- In the Newtonian limit, classical field obeys

$$i\partial_t\phi = -\frac{c^2}{2\omega_0}\nabla^2\phi + \frac{\lambda}{8\omega_0}|\phi|^2\phi + \frac{\omega_0}{c^2}\Psi\phi$$

- Expand quantum axion field in a box; Hamiltonian for $\Phi(\vec{r},t)$ which obeys this & Poisson

$$H = \sum_{\vec{n}} \hbar\omega_{\vec{n}}\, a^\dagger_{\vec{n}} a_{\vec{n}} + \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3,\vec{n}_4} \frac{1}{4}\,\hbar\Lambda^{\vec{n}_3,\vec{n}_4}_{\vec{n}_1,\vec{n}_2}\, a^\dagger_{\vec{n}_1} a^\dagger_{\vec{n}_2} a_{\vec{n}_3} a_{\vec{n}_4}$$

*Erken, Sikivie, Tam & Yang (2012)*

- Using Heisenberg eom, for $\mathcal{N}_{\vec{n}}(t) = a^{\dagger}_{\vec{n}}(t)\ a_{\vec{n}}(t)$

$$\dot{\mathcal{N}}_i = \sum_{k,i,j=1} \frac{1}{2}|\Lambda^{kl}_{ij}|^2 \left[\mathcal{N}_i\mathcal{N}_j(\mathcal{N}_l+1)(\mathcal{N}_k+1) - \mathcal{N}_l\mathcal{N}_k(\mathcal{N}_i+1)(\mathcal{N}_j+1)\right] 2\pi\delta(\omega_i+\omega_j-\omega_k-\omega_l)$$

- Similarly, for classical case

$$\dot{N}_i = \sum_{k,i,j=1} \frac{1}{2}|\Lambda^{kl}_{ij}|^2 \left[N_iN_jN_l + N_iN_jN_k - N_lN_kN_i - N_kN_lN_j\right] 2\pi\delta(\omega_i+\omega_j-\omega_k-\omega_l)$$

- Clear difference between the cases:

$$i + j \to k + l$$

where quanta in i,j move to k,l cannot happen in classical, but does happen in quantum case

# Thermalisation

- After a sufficiently long period of time, each system will reach equilibrium distribution

- Classical and quantum results are different:
  - Classical $N_i \epsilon_i = k_B T$
  - Quantum (Bose-Einstein distribution)

$$\mathcal{N}_i = \frac{1}{e^{\frac{\epsilon_i}{k_B T}} - 1}$$

# Summary

- For axions (or other scalar field dark matter), quantum is important - is this complete?
- The Schrödinger-Poisson equations reproduces linear perturbation theory
- Axions form Bose Einstein Condensate, and in doing so not described by classical field theory
- Future work: are there observational consequences of treating axion (properly) as a quantum field