General Relativity: its creation, classical tests and new effects in rotating systems.

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Content

- The beginning of General Relativity (Pais 1982; Sauer 1999, 2015; Straumann 2015; and others).
- Present status of classical (Einsteinian) tests of GR.
- New effects in rotating hydrodynamic systems.

The beginning of General Relativity

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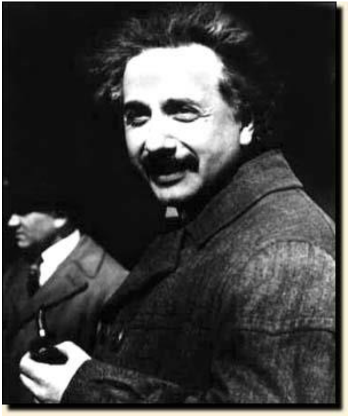

The beginning of General Relativity

- Einstein's review of special relativity theory 1907. Local flatness. Freely falling bodies do not feel gravity — acceleration can eliminate gravity (equivalence principle).
- Intensive phase started in 1912. In summer Einstein realized, that he needs (Pais 1982, Straumann 2015)
- the metric ("metric potentials")!!!
- Marcel Grossmann (the dean at ETH at that time) had succeeded in securing position for Einstein, effective from 1st October 1912 (Sauer 2015)).

The beginning of General Relativity

- In Einstein's words *" The problem of gravitation was thus reduced to a purely mathematical one.* **Do differential equations exist for the g_{ik}, which are invariant under non-linear coordinate transformations?** *Dfferential equations of this kind and only of this kind were to be considered as field equations of the gravitational field. The law of motion of material points was then given by the equation of the geodesic line. With this problem in mind I visited my old friend Grossmann who in the meantime had become professor of mathematics at the Swiss polytechnic. He at once caught fire, although as a mathematician he had a somewhat skeptical stance towards physics.* (Sauer 2015)

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- "Grossmann, Du musst mir helfen, sonst werd' ich verrückt!" (Grossmann, you must help me, or else I will go crazy!) — recollection (from that period) of Einstein's colleague from student's times at ETH Louis Kollros in 1956.

Zürich 1912: Einstein-Grossmann

- In the subsequent collaboration with Einstein, Grossman was responsible for the mathematical part of the collaboration.
- Grossmann and Einstein: "Zurich notebook" and " Outline (Entwurf) of a general theory of relativity and a theory of gravitation." Leipzig, Berlin: Teubner, 1913. (also: in the Zeitschrift für Mathematik und Physik, 30 January 1914). The "Entwurf" is separated into two parts, one (physical) written by Einstrein and the other (mathematical) by Grossmann.
- Grossmann introduced the name "tensor" and (probably) the Ricci curvature tensor (Sauer 2015).

Zürich 1912: Einstein-Grossmann

- These attempts culminated in gravitational equations

  $$R_{\mu\nu} = 8\pi T_{\mu\nu}.$$

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- Einstein wrote linearized equations ($h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$, $h = h^{\mu}_{\mu}$)

  $$\eta^{\mu\nu}\partial_\mu\partial_\nu\left(h_{\alpha\beta} - \frac{\eta_{\alpha\beta}}{2}h\right) = 8\pi T_{\alpha\beta}$$

  that are correct — we know now (Straumann 2015) — and follow from the correct equations

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- Einstein realized that the energy momentum tensor in the standard formulation does not satisfy the conservation laws. These linearized equations allow for the conservation of $T_{\mu\nu}$!

Zürich 1912: Einstein-Grossmann

- The original model of matter was dust, $T_{\mu\nu} = U_\mu U_\nu \rho$, thus $T_\mu^\mu \neq 0$. For that reason, Einstein and Grossmann failed to get from their incomplete theory the Newton-Poisson potential equation in the limit $c \to \infty$.
- That led to the 2-year withdrawal of Einstein from the Riemannian-based geometry approach.
- Einstein contemplated resignation from full covariance and formulated the hole argument in the appendix of the January 1914 article in the Zeitschrift für Mathematik und Physik. Its complete resolution — the Darmois-Israel junction conditions — has been formulated decades later (Stachel LRR 2014).

Einstein-Grossmann: resume

- Sauer (2015):
- Grossmann's *collaboration with Einstein in Zurich ... resulted in the Einstein-Grossmann theory of 1913.* **This theory is a precursor version of the final theory of general relativity with all the ingredients of that theory except for the correct gravitational field equations.**
- *Although in later recollections, Einstein credited Grossmann mainly for showing him the relevant literature,* **we must assume that Grossmann actually helped clarify the very mathematical status of the objects that were entering the center stage of their theoretical efforts.**

Einstein-Grossmann: resume

- Straumann (2011-2015):
- Zürich Notebook .... research notes are an extremely illuminating source for understanding Einstein' s main physical arguments and conceptual difficulties that delayed his discovery of general relativity by about three years. Together with the "Entwurf theory" in collaboration with Marcel Grossmann, these notes also show that **the final theory was missed late in 1912 within a hair's breadth.** The Einstein-Grossmann theory, published almost exactly hundred years ago, contains, however, virtually all essential elements of Einstein's definite gravitation theory.

Einstein-Grossmann: resume.

- Tilman Sauer (2015) quotes Marcel Grossmann's ("A New Worldview" in the Neue Schweizer Zeitung, 1920):
- *Laymen have an entirely misleading conception of the essence of mathematical and generally scientific research. Also in this field of human intellect, something new is only being created by intuition, by creative imagination. The great mathematicians and physicists are not 'good calculators,' in this respect they are outplayed by your average able accountant; nor is someone who plays the piano with virtuosity a great musician! Original achievements in all fields of human knowledge and capability are artistic achievements and follow their own laws. To a person who witnessed Einstein's first laborious attempts in the years 1912 and 1913, as the composer of these lines did, they must appear like the ascent of an inaccessible mountain in the dark of the night, without path or trail, without foothold or direction. Experience and deduction provided only few and insecure handholds. All the higher we have to value this intellectual deed.*

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- October 1915: Einstein resumes the "Riemannian path".
- Einstein's seminar of 4th November 2015: A restricted form of equations is shown to yield the Newton-Poisson equation

  $$\Delta\Psi = 4\pi\rho$$

  in the Newtonian limit.

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- Einstein actually derives the contracted Bianchi identity, assuming the energy-momentum conservation!

The birth of Einstein equations: Hilbert.

- 1915. Known facts (Pais 1982; Sauer 1999). Hilbert tried, after the Einstein's summer (1915) visit to Göttingen (17th July, Hilbert to Schwarzschild: "The lectures of the latter were memorable"), to merge the Mie's theory and the Einstein's approach to gravitation based on "metric potentials".

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- In another letter to Seidelmann: *Hilbert's team was working on extremely difficult calculations for Einstein and "none of us understand what they are for"*.

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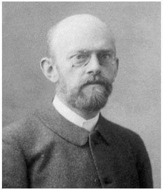

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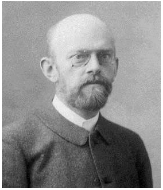

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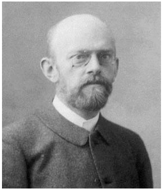

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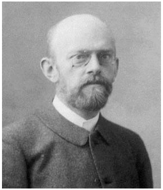

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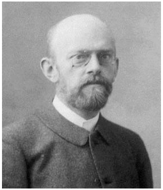

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- More on Hilbert's programme concerning gravitation: Sauer (1999).
- Hugo Steinhaus (a student of Hilbert, PhD in Goettingen in 1912; later discoverer of Stefan Banach and professor at the Jan Kazimierz University in Lvov, one of prominent members of the Polish school of mathematics) on the role of David Hilbert: in Wspomnienia i zapiski
- **His "Foundations of Geometry"... removed any doubts concerning non-euclidean geometry and opened way to the understanding of the theory of relativity."** **— HS probably confused "Foundations of Geometry" (1899) with "Foundations of Physics" (1916)**

The birth of Einstein equations: 20. 11. or 25. 11?

- Corry, Renn, and Stachel (1997): analyze printer's proofs of Hilbert's 1915 paper (submitted on 20th November) on the general theory of relativity and claim that Hilbert amended his published (in March 1916) version with the correct form of the gravitational field equations only after having seen Einstein's final paper (published on December 2nd 1915).
- There was an emotional discussion afterwards, best reported by Walter Isaacson in *Albert Einstein. His life and Universe* (2007).
- Sauer (2015) ("The relativity of discovery: Hilbert's first note on the Foundations of Physics") states that the main point — the action — was not changed in Hilbert's proofs, albeit the analysis was difficult, because a part (about 10 lines) of the page 7/8 is missing.

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- The first (documented) correct form of the gravitational action (and thus of the **Einstein equations**) had been written by Hilbert. But recall that **the final theory was missed late in 1912 within a hair's breadth.**

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- Sauer (1999): *...the independence of Einstein's discovery was never a point in dispute between Einstein and Hilbert. Nor was the independence of Hilbert's derivation of the field equations disputed by Einstein. Hilbert claimed priority for the introduction of the Ricci scalar into the action principle and the derivation of the field equations from it, and Einstein publicly admitted that Hilbert (and Lorentz) had succeded in giving the equations of general relativity a particularly lucid form by deriving them from a single variational principle.*

Confirmation of GR — classical triad of experiments: periastron advance

- The anomalous perihelion advance of Mercury — 43 arcseconds per century — constitutes roughly 1% of its total perihelion advance.
- Einstein calculated the effect a few times (mostly erronously); the crucial, correct calculation was reported on November 18th 1915. Its explanation gave Einstein confidence in his theory, but that fact was not significant (Will 2015) in the early discussion on the validity of general relativity.
- Global fits of solar system orbital data verify the relativistic perihelion precession of Mercury to a few parts in 10^5. Helioseismology measurements show that the solar quadrupole moment is too small to have an effect.

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- Measurements of the effect in binary systems with pulsars give results in the strong field regime. An example: J0737-3039A,B (two pulsars), discovered in 2003, has short orbital period (0.1 days) and large periastron advance (16.88 degree’s per year).
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- Only in recent decades the measurements — with very-long-baseline radio interferometry (VLBI) and within the programme HIPPARCOS in optical spectrum — gave a high accuracy. The observed deflection angle agrees with the theoretical value

$$\frac{4GM}{c^2 d}.$$

up to 10^{-4}.

Confirmation of GR — classical triad of experiments: gravitational redshift

- Einstein 1907 - first derivation.
- (Will 2006) The first reliable experiment (with the accuracy circa 1%): — Pound-Rebka measured in 1960 the frequency shift of (gamma ray) photons from 57Fe as they ascended or descended the Jefferson Physical Laboratory tower at Harvard University.
- R. F. C. Vessot et al. (1980) *The results of a test of general relativity with use of a hydrogen-maser frequency standard in a spacecraft launched nearly vertically upward to 10 000 km are reported. The agreement of the observed relativistic frequency shift with prediction is at the* 70×10^{-6} *level.*
- In 1991, LoPresto et al. measured the solar shift in agreement with the prediction to about 2 percent.
- Shapiro effect (light travel delay)

Confirmation of GR — Global Positioning System

- Ashby (2003)
- The Global Positioning System (GPS) uses accurate, stable atomic clocks in satellites and on the ground to provide world-wide position and time determination. **These clocks have gravitational and motional frequency shifts which are so large that, without carefully accounting for numerous relativistic effects, the system would not work for longer than half an hour**. Relativistic principles and effects include the constancy of the speed of light, the equivalence principle, the Sagnac effect, time dilation, gravitational frequency shifts, and relativity of synchronization.

Hulse-Taylor pulsar and GW

- The binary pulsar B1913+16 by Joseph Taylor and Russell Hulse (1974) — the first opportunity to see the effects of strong relativistic internal gravitational fields on orbital dynamics, and the effects of gravitational radiation reaction.
- The period 59 ms in a close binary orbit with a companion (a dead pulsar?). The orbital period is about 7.75 hours, and the eccentricity is 0.617. Mean rate of periastron advance 4.226595(5) degres per year.
- The rate of decrease of orbital period is compatible with General Relativity (76.5 μs/year).
- From modelling: the rate of decrease of semimajor axis is 3.5 meters per year, and the calculated lifetime to final inspiral is 3×10^8 years.
- Challenge: direct detection of gravitational radiation.

Future strong gravity tests

- Accretion phenomena associated with compact objects such as black holes or neutron stars.
- Gravitational wave detectors to explore and test GR in the strong-field, highly-dynamical regime associated with the formation and dynamics of these objects.
- Big Bang and later phenomena (inflation?).

New effects in rotating stationary systems

- Assume the barotropic equation of state ($p = p(\rho)$), rotation around a fixed axis ($\vec{V} = \Omega_0 \partial_\phi$) and axial symmetry.
- Euler equations:

$$-\Omega_0^2 r = -\nabla_r (U_0 + h)$$

$$0 = -\nabla_z (U_0 + h)$$

- The Poisson equation:

$$\Delta U_0 = 4\pi\rho$$

- Observations:
  — one needs to prescribe the angular velocity (the rotation curve), in order to close the system of equations;
  — angular velocity depends on r, the distance from a rotation axis: $\Omega_0 = \Omega_0(r)$ (!). This constitutes the integrability condition!
- If $\Omega = w/r^{2/(1-\delta)}$ then integration of Euler eqs yields the Bernoulli equation:

$$h_0 + U_0 - \frac{\delta - 1}{2(1+\delta)})\Omega^2 r^2 = c_0$$

New effects in rotating stationary systems

- The line element

$$ds^2 = -e^{\frac{2\nu}{c^2}}(dx^0)^2 + r^2 e^{\frac{2\beta}{c^2}}\left(d\phi - \frac{A_\phi}{r^2 c^3}(r,z)\,dx^0\right)^2 + e^{\frac{2\alpha}{c^2}}\left(dr^2 + dz^2\right).$$

  $x^0 = ct$ - the time coordinate; r, z, ϕ are cylindrical coordinates. Two Killing vectors — timelike and azimuthal.

- Assumptions: barotropic equation of state, $p = p(\rho)$; the velocity u^α and the angular velocity Ω:

$$u^\alpha = (u^0, u^r, u^z, u^\phi) = u^t(c, 0, 0, \Omega),$$

  where

$$\Omega = \frac{u^\phi}{u^t}$$

- The angular momentum per unit mass j

$$j \equiv u_\phi u^t = \frac{V^2}{\left(\Omega - \frac{A_\phi}{r^2 c^2}\right)\left(1 - \frac{V^2}{c^2}\right)},$$

  Here

$$V^2 = r^2\left(\Omega - \frac{A_\phi}{r^2 c^2}\right)^2 e^{2(\beta-\nu)/c^2}.$$

New effects in rotating stationary systems

- Komatsu, Eriguchi and Hachisu (c. 1990): the GR Euler eqs are solvable (and reducible to the GR Bernoulli eq.), iff $j = j(\Omega)$.
- Before 2015 there had been known two types of relations (linear rotation law, angular momentum per unit mass), 2-parameter (Butterworth and Ipser 1969; Bardeen 1970)

  $$j = A(\Omega - C);$$

  and a 3-parameter generalization (Galeazzi, Yoshida, and Eriguchi 2012).
- These choices would not recover the Newtonian rotation curves $\Omega = w/r^{2/(1-\delta)}$.

New effects in rotating stationary systems

- A new rotation law (Mach and Malec 2015):

$$j(\Omega) \equiv \frac{w^{1-\delta}\Omega^{\delta}}{1 - \frac{\kappa}{c^2} w^{1-\delta}\Omega^{1+\delta} + \frac{\Psi}{c^2}} = \frac{V^2}{\left(\Omega - \frac{A_\phi}{r^2 c^2}\right)\left(1 - \frac{V^2}{c^2}\right)}$$

- There are 4 parameters: $w, \delta \neq -1, \Psi$ and κ,

$$\kappa = \frac{1-3\delta}{1+\delta}.$$

- The relativistic Bernoulli law

$$\left(1 + \frac{h}{c^2}\right) e^{\frac{\nu}{c^2}} \sqrt{1 - \frac{V^2}{c^2}} \times \left(1 - \frac{\kappa}{c^2} w^{1-\delta}\Omega^{1+\delta} + \frac{\Psi}{c^2}\right)^{\frac{-1}{(1+\delta)\kappa}} = C.$$

New effects in rotating stationary systems

- **Lemma** Assume the rotation law with

  $$\Psi = 4c_0,$$

  where c_0 is the Newtonian hydrodynamic energy per unit mass. Then:
- the Bernoulli and Einstein equations are satisfied in 0PN and 1PN orders of approximation;
- Parameters w and δ are given by the Newtonian rotation law:

  $$\Omega_0 = \frac{w}{r^{2/(1-\delta)}}.$$
- a test disk made of dust rotating in Keplerian motion satisfies exactly the Bernoulli equation in the Schwarzschild spacetime.

New effects in rotating stationary systems

- (Mach, EM 2015) Angular velocity with GR corrections:

$$\Omega = \Omega_0 + \frac{v_1^{\phi}}{c^2} = \frac{w}{\tilde{r}^{2/(1-\delta)}}$$

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- The last term $\frac{2\Omega_0}{c^2(1-\delta)} \left(U_0 + \Omega_0^2 r^2\right)$ measures selfgravity.

New effects in rotating stationary systems

- Special cases:
- For a disk of dust with negligible mass, keplerian rotation ($\delta = -1/3$) and

$$\Omega = \Omega_0 + \frac{v_1^\phi}{c^2} = \frac{w}{\tilde{r}^{3/2}} + \frac{3A_\phi}{4r^2c^2}$$

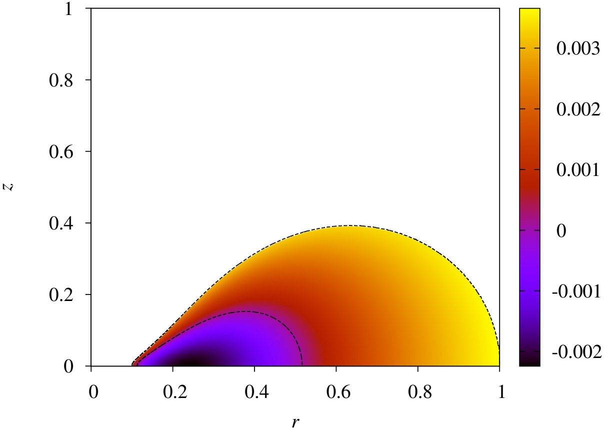
  — only dragging of frames.
- Rigid rotation ($\delta = -\infty$):

$$\Omega = \Omega_0$$

  — angular velocity is not effected.

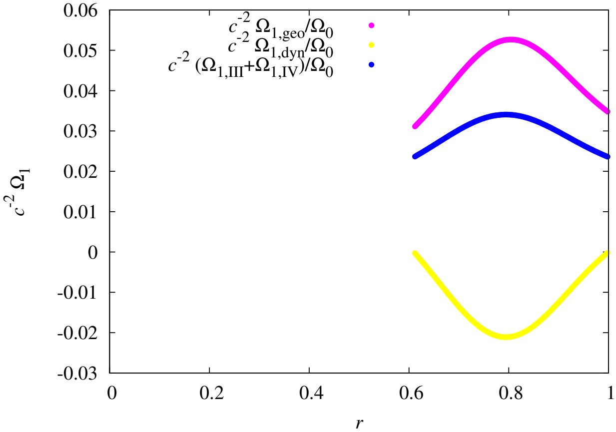
Numerics: anti-dragging.

- (Mach EM,Jaranowski and Piróg 2015) Dragging versus anti-dragging: a torus around the Schwarzschild black hole.



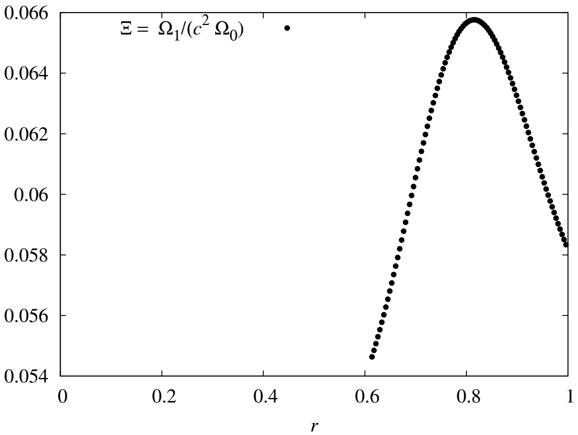
- The ratio $v_1^\phi/(c^2\Omega_0)$. Here 1 corresponds to $250R_S$. The inner boundary - $r_{\rm in} = 25R_S$, the outer boundary $r_{\rm out} = 250R_S$. The Newtonian mass $M_{\rm D} = 1.44 \times M_c$.

Numerics: comparing different effects.



- The ratio $v_1^{\phi}/(c^2\Omega_0)$. Here 1 corresponds to $50R_S$. The inner boundary - $r_{\mathrm{in}} = 30.5R_S$, the outer boundary $r_{\mathrm{out}} = 50R_S$. The Newtonian mass $M_{\mathrm{D}} = 1.8 \times M_c$.

Numerics: combined effects.



The ratio $v_1^\phi/(c^2\Omega_0)$ at the symmetry plane. Here 1 corresponds to $50R_S$. The inner boundary - $r_{\rm in} = 30.5R_S$, the outer boundary $r_{\rm out} = 50R_S$. The Newtonian mass $M_{\rm D} = 1.8 \times M_c$.

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