The two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier

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Introduction and Motivation: Generalised Mimetic Gravity

- Disformal transformations [J. D. Bekenstein, Phys.Rev. D 48, 3641 (1993)]
- "Mimetic" Dark Matter theory: Applied some special transformation on **Einstein action**: a scalar field mimics Dark Matter. $g_{\mu\nu} = [KE]\ell_{\mu\nu}$
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We shall apply full disformal transformation on a very general single field scalar-tensor theory of gravity $\longrightarrow$

1. Invariant properties under disformal transformation.
2. For a special subset, the scalar field can mimic almost any desired expansion history (**including Dark Energy and Dark Matter dominated universe**). We call it **generalized "mimetic" gravity**.
3. Application: cosmology in the **"mimetic" Horndeski model**

Disformal Transformation [3]

$$g_{\mu\nu} = A(\Psi, w)\ell_{\mu\nu}$$

[1] D. Bettoni & S. Liberati, [arXiv:1306.6724]

[2] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi [arXiv:1408.1952]

[3] J. Bekenstein "The Relation between physical and gravitational geometry", Phys. Rev. D48 (1993) 3641

Disformal Transformation [3]

$$g_{\mu\nu} = A(\Psi, w)\ell_{\mu\nu} + B(\Psi, w)\partial_\mu\Psi\partial_\nu\Psi,$$

$$\text{where} \quad w \equiv \ell^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi$$

A and B: **Arbitrary** disformal functions (depend on Ψ and w)

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A and B: **Arbitrary** disformal functions (depend on Ψ and w)

Pure Conformal

$A(\Psi, w) = A(\Psi, w), B = 0$

e.g.: $g_{\mu\nu} = \Psi^2\,\ell_{\mu\nu}$; where $A(\Psi, w) = \Psi^2$

Pure Disformal

$A(\Psi, w) = 1,$
$B(\Psi, w) = B(\Psi, w)$

Some applications were studied in many recent articles. [1],[2]

Chamseddine, Mukhanov and Vikman's Transformation

$A = w, B = 0$ (on Einstein-Hilbert action)

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2 J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi [arXiv:1408.1952]

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Theoretical Constraction:

Non-invertibility condition on a disformal transformation

Writing $\ell_{\mu\nu}$ in terms of $g_{\mu\nu}$:

- need to solve w in terms of $g_{\mu\nu}$.
- 'w' contains $\ell_{\mu\nu}$
- Assuming that Ψ is the same field as the scalar field in the scalar tensor action.
- Relating ten variables of $g_{\mu\nu}$ to eleven variables of $(\ell_{\mu\nu} + \Psi)$.

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- Assuming that Ψ is the same field as the scalar field in the scalar tensor action.
- Relating ten variables of $g_{\mu\nu}$ to eleven variables of $(\ell_{\mu\nu} + \Psi)$.
- Applying the inverse transformation theorem for a **fixed** given Ψ. Inverse function exists if $\frac{dG(\Psi,w)}{dw}|_{w=w_*} \neq 0$
  $G(\Psi, w) = \frac{w\left(1-B(\Psi,w)g^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi\right)}{A(\Psi,w)} \quad \left(= 1/b(\Psi) = g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi\right),$

Non-invertibility condition (Derived)

$$B(\Psi, w) = -\frac{A(\Psi, w)}{w} + b(\Psi); \quad b(\Psi) \equiv \text{int. const.}$$

EOM: Quick look

Considering **very general** scalar-tensor theory action,

$$S = \int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\mu_1}\ldots\partial_{\mu_{\rm p}} g_{\mu\nu}, \Psi, \partial_{\mu_1}\ldots\partial_{\mu_{\rm q}}\Psi] + S_{\rm m}[\phi_{\rm m}, g_{\mu\nu}]$$

Variation of the action in terms of fundamental fields, $g_{\mu\nu}$ and Ψ:

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}(E^{\mu\nu} + T^{\mu\nu})\delta g_{\mu\nu} + \int d^4x\ \Omega_\Psi \delta\Psi \tag{1}$$

Equation of motion are $\delta S/\delta g_{\mu\nu} = 0$ and $\delta S/\delta\Psi = 0$

$$(E^{\mu\nu} + T^{\mu\nu}) = 0; \quad \Omega_\Psi = 0$$

Definition:

$$\Omega_\Psi = \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta\Psi} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\Psi} + \sum_{h=1}^{q}(-1)^h \frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\left(\partial_{\lambda_1}\ldots\partial_{\lambda_h}\Psi\right)},$$

$$E^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}} + \sum_{h=1}^{p}(-1)^h\frac{d}{dx^{\lambda_1}}\cdots\frac{d}{dx^{\lambda_h}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\left(\partial_{\lambda_1}\ldots\partial_{\lambda_h} g_{\mu\nu}\right)}\right),$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}},\ \Omega_m = \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi_m},\ \text{where } S_m[g_{\mu\nu}] = \int d^4x\sqrt{-g}\mathcal{L}_m[g_{\mu\nu}, \phi_m].$$

Einstein EOM, $\delta S/\delta \ell_{\mu\nu} = 0$; KG EOM, $\delta S/\delta\Psi = 0$

$$A(E^{\mu\nu} + T^{\mu\nu}) = \left(\alpha_1 \frac{\partial A}{\partial w} + \alpha_2 \frac{\partial B}{\partial w}\right) (\ell^{\mu\rho}\partial_\rho\Psi)\,(\ell^{\nu\sigma}\partial_\sigma\Psi),$$

$$\frac{1}{\sqrt{-g}}\partial_\rho \left\{\sqrt{-g}\,\partial_\sigma\Psi\left[B(E^{\rho\sigma} + T^{\rho\sigma}) + \left(\alpha_1 \frac{\partial A}{\partial w} + \alpha_2 \frac{\partial B}{\partial w}\right)\ell^{\rho\sigma}\right]\right\} - \frac{\Omega_\Psi}{\sqrt{-g}} = \frac{1}{2}\left(\alpha_1 \frac{\partial A}{\partial \Psi} + \alpha_2 \frac{\partial B}{\partial \Psi}\right),$$

where we define a new set of variables of the system,

$$\alpha_1 \equiv (E^{\rho\sigma} + T^{\rho\sigma})\ell_{\rho\sigma} \quad \text{and} \quad \alpha_2 \equiv (E^{\rho\sigma} + T^{\rho\sigma})\partial_\rho\Psi\,\partial_\sigma\Psi\,,$$

Contracting metric EOM with respect to $\ell_{\mu\nu}$ and $\partial_\mu\Psi\,\partial_\nu\Psi$

$$\alpha_1\left(A - w\frac{\partial A}{\partial w}\right) - \alpha_2 w\frac{\partial B}{\partial w} = 0, \qquad \alpha_1\, w^2\frac{\partial A}{\partial w} - \alpha_2\left(A - w^2\frac{\partial B}{\partial w}\right) = 0.$$

Determinant of the system

$$\det(M) = w^2 A\frac{\partial}{\partial w}\left(B + \frac{A}{w}\right).$$

Solution: $\alpha_1 = \alpha_2 = 0$ for the **generic case** $(det(M) \neq 0)$.

Recovered EOM

$$A(E^{\mu\nu} + T^{\mu\nu}) = 0$$

$$\partial_\rho\left[\sqrt{-g}\,\partial_\sigma\Psi B(E^{\rho\sigma} + T^{\rho\sigma})\right] - \Omega_\Psi = 0 \quad \text{or} \quad \Omega_\Psi = 0$$

Comments

For the **Generic case**,

Invariant under disformal transformation (Classically)

All EOMs of **very general scalar-tensor theories** are the same before and after performing disformal transformation: **Physics is the same**

Valid for any **single field healthy theory**, Horndeski or Generalized Hordeski (G^3) (or beyond as proposed by X. Gao [arXiv: 1406.0822])

Less surprising result: well-behaved **invertible** change of variables.

Conformal transformation (as special case): The basic physics of the system are the same in Einstein and Jordan frame. **(Classically)**

Check: Recovering Deruelle and Rua's Result

$E^{\mu\nu} = -\frac{1}{\kappa} G^{\mu\nu}$ (on Einsten-Hilbert action)

Generalized Mimetic Gravity

The system to be indeterminate: Determinant of the system is zero, $det(M) = 0$,

$$B(\Psi, w) = -\frac{A(\Psi, w)}{w} + b(\Psi),$$ (Recall the non-invertibility condition: Same!!)

Generalized Mimetic Gravity EOM

$$E_{\mu\nu} + T_{\mu\nu} = (E + T)\, b\, \partial_\mu \Psi\, \partial_\nu \Psi,$$

$$\nabla_\rho \left[(E + T) b\, \partial^\rho \Psi\right] - \frac{\Omega_\Psi}{\sqrt{-g}} = \frac{1}{2}(E + T)\frac{1}{b}\frac{db}{d\Psi},$$

Completely **new** set of EOM "without any cost".

Non-invertibility or being **indeterminate** of the **full disformal transformation** yields the Mimetic Gravity.

Recovering earlier results from our Generalized Mimetic Gravity

For different models: Explicit form of mimetic EOM would be different. Hence the **Cosmological solutions** will be **different**.

Check: Recovering Deruelle and Rua's Result

$E^{\mu\nu} = -\frac{1}{\kappa} G^{\mu\nu},$ (on Einsten-Hilbert action)

Check: Recovering Chamseddine, Mukhanov and Vikman's Result

$A = w, B = 0,$ Hence $b = 1$ (on Einsten-Hilbert action)

All results can be recovered from our generalized form.

Mimetic gravity from a Lagrange multiplier

The “mimetic” or non-invertibility condition are equivalent to the normalization constraint,

$$b(\Psi)g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi = 1.$$

$$S_\lambda = \int d^4x\sqrt{-g}\mathcal{L} + S_m[g_{\mu\nu}, \phi_m] + \int d^4x\sqrt{-g}\lambda\Big(b(\Psi)g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - 1\Big),$$

Solve for λ in order to get the EOM.

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Solve for λ in order to get the EOM.

These **EOM are the same** as the earlier mimetic EOM derived by disformal transformations

The price to pay in this formulation

Introducing an additional scalar field, the Lagrange multiplier λ

However good news is that: one EOM is redundant.
The action does not contain any more higher-order derivatives than $\mathcal{L}$ in terms of $g_{\mu\nu}$, but may have in terms of $\ell_{\mu\nu}$.

Application in Cosmology:

Application - 1: A very simple example

$$\mathcal{L} = c_2 X + \frac{1}{2} R, \qquad S_m = 0,$$

$$\text{where,} \qquad X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \tag{2}$$

mimetic-EOM: $\quad b(\Psi)\dot{\Psi}^2 + 1 = 0, \quad 6H^2 + 4\dot{H} + c_2 \dot{\Psi}^2 = 0,$

Application - 1: A very simple example

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mimetic-EOM: $\quad b(\Psi)\dot{\Psi}^2 + 1 = 0, \quad 6H^2 + 4\dot{H} + c_2 \dot{\Psi}^2 = 0,$

Cosmological solution:

$$a(t) = t^{\frac{2}{3(1+\omega)}}, \quad \Psi(t) = \pm\sqrt{-\frac{\alpha}{c_2}} \log \frac{t}{t_0}, \quad b(\Psi) = -\frac{1}{\dot{\Psi}^2} = \frac{c_2}{\alpha} t^2 = \frac{c_2}{\alpha} t_0^2 e^{\pm 2\sqrt{-\frac{c_2}{\alpha}}\Psi},$$

t_0: an integration constant, $\alpha = -\frac{8\omega}{3(1+\omega)^2}$, ω:constant parameter

- Mimicking the background evolution of a perfect fluid universe with a constant equation of state.
  (Note: $b(\Psi) < 0$ for a time-like scalar velocity)
- Constraint: $6H^2 + 4\dot{H} = -2p$; the pressure (p) can not change sign.

Application - 2: Mimetic cubic Galileon

$$\mathcal{L} = c_2 X - 2c_3 X \Box\Psi + \frac{1}{2}R, \qquad S_m = 0$$

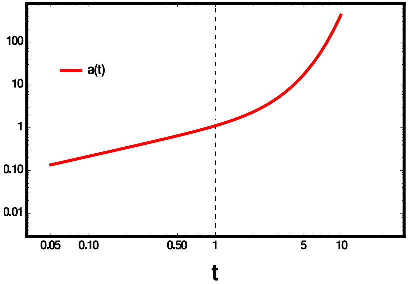
- Almost any desired expansion history: suitably choosing $b(\Psi)$.

Application - 2: Mimetic cubic Galileon

$$\mathcal{L} = c_2 X - 2c_3 X \Box\Psi + \frac{1}{2}R, \qquad S_m = 0$$

- Almost any desired expansion history: suitably choosing $b(\Psi)$.
- Consider the expansion history of a universe filled with dark matter and a positive cosmological constant Λ.

$a = a_\star \sinh^{\frac{2}{3}}(\mathcal{C}t)$, where $\mathcal{C} = \sqrt{3\Lambda/4}$.



$\mathcal{C} = c_2 = c_3 = a_\star = 1;$

By choosing a function $b(\Psi)$ with these asymptotic limits one can approximately reproduce the expansion history of a $\Lambda+$ dark matter universe.

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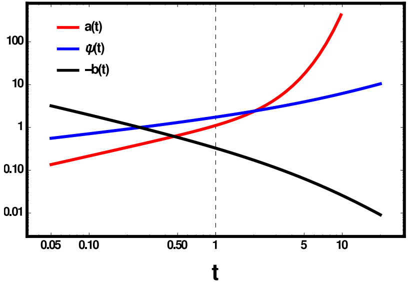
- Consider the expansion history of a universe filled with dark matter and a positive cosmological constant Λ.

$$a = a_\star \sinh^{\frac{2}{3}}(\mathcal{C}t), \qquad \text{where } \mathcal{C} = \sqrt{3\Lambda/4}. \quad \frac{4c_3}{c_2}\left[-\arctan\left(\pm\sqrt{\frac{3c_2}{8\mathcal{C}^2}}\dot{\Psi}\right) \pm \sqrt{\frac{3c_2}{8\mathcal{C}^2}}\dot{\Psi}\right] = t.$$



$\mathcal{C} = c_2 = c_3 = a_\star = 1$;

- Matter-dominated era ends $t \approx \mathcal{O}(1)$;
- After that the universe becomes dominated by the energy density of the Λ;
- $\dot{\Psi} \propto t^{1/3}$ for $\mathcal{C}t \ll 1$; $\quad \dot{\Psi} \propto t$ for $\mathcal{C}t \gg 1$
- $b(\Psi) \propto -\Psi^{-1/2}$ for $\mathcal{C}t \ll 1$ and $b(\Psi) \propto -\Psi^{-1}$ for $\mathcal{C}t \gg 1$

By choosing a function $b(\Psi)$ with these asymptotic limits one can approximately reproduce the expansion history of a $\Lambda+$ dark matter universe.

Take Home Message

- Very general scalar-tensor theories of gravity are generically invariant under disformal transformations.
- A special subset, which is non invertible under that transformation yields generalized "mimetic" gravity theories, i.e., the scalar field mimics almost any desired expansion history (including Dark Energy and Dark Matter dominated universe).
- Holds irrespective of whether the scalar field of the disformal transformation is the same or different from the one in the action.
- The generalized mimetic EOM can also be derived using the Lagrange multiplier method.
- The number of derivatives in the generalized mimetic-EOM is preserved (for generalized scalar-tensor theories).
- The simplest mimetic scalar-tensor model is able to mimic the cosmological background of a flat FLRW model with a barotropic perfect fluid with any constant equation of state.

Thank You