Cosmic Acceleration with a Negative Cosmological Constant in Higher Dimensions

Nobuyoshi Ohta (Kinki U. ⇒ Kindai U. (2016)) and Kei-ichi Maeda (Waseda U.)
JHEP 1406 (2014) 095 [arXiv:1404.0561 [hep-th]].

Abstract: In gravitational theories with a cosmological constant and the Gauss-Bonnet curvature squared term, we find the de Sitter solutions even for negative cosmological constant. The de Sitter solutions are stable (unstable) if both the curvature of the internal space and the cosmological constant are negative (positive).

1 Introduction

We believe that there is an inflationary epoch of the early stage of the evolution of our universe.
The present universe also exhibits accelerating expansion.
It is desirable if we can derive models which explain these from the first principle or fundamental theory of particle physics without artificial assumptions.

Fundamental theory ⇒ the ten-dimensional superstring or eleven-dimensional M theory.

However "No-go theorem": An accelerating universe is difficult to realize in such theories

Here we report our attempt to realize de Sitter expansion higher curvature terms and cosmological constant (CC).

- These are terms expected in heterotic string theories!!
- It turns out that de Sitter solutions are possible with the size of the internal space constant, both for positive and negative CC if the internal space is chosen to be hyperbolic.
- de Sitter solutions are common for positive CC, but here for negative CC as well.
- Furthermore stable solutions are obtained for negative CC!!

Solutions of special case:
F. Canfora, A. Giacomini and S. A. Pavluchenko, Phys. Rev. D 88 (2013) 064044 [arXiv:1308.1896 [gr-qc]].

We study such solutions systematically and examine stability.

Similar claim to turn a negative CC to a positive effective CC.
R. Cardenas, T. Gonzalez, Y. Leiva, O. Martin and I. Quiros, Phys. Rev. D 67 (2003) 083501 [astro-ph/0206315].
but this introduces not only a negative CC but also quintessence scalar field which has positive potential.

Ours does not introduce positive CC nor positive potential.

2 Einstein-Gauss-Bonnet system with a CC

The action:

$$S = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\left[R - 2\Lambda + \alpha_2 R_{\rm GB}^2\right],$$

κ_D^2: a D-dimensional gravitational constant, $\alpha_2 = \alpha'/8$. We neglect dilaton for simplicity.

The metric in D-dimensional space:

$$ds_D^2 = -e^{2u_0(t)}dt^2 + e^{2u_1(t)}ds_p^2 + e^{2u_2(t)}ds_q^2,$$

where $D = 1 + p + q$.

Field equations:

$$\begin{cases} F &:= F_1 + F_2 = 0, \\ F^{(p)} &:= f_1^{(p)} + f_2^{(p)} + X\left(g_1^{(p)} + g_2^{(p)}\right) + Y\left(h_1^{(p)} + h_2^{(p)}\right) = 0, \\ F^{(q)} &:= f_1^{(q)} + f_2^{(q)} + Y\left(g_1^{(q)} + g_2^{(q)}\right) + X\left(h_1^{(q)} + h_2^{(q)}\right) = 0, \end{cases}$$

where

$$\begin{aligned} F_1 &= p_1 A_p + q_1 A_q + 2pq\dot u_1 \dot u_2 - 2\Lambda e^{2u_0}, \\ f_1^{(p)} &= (p-1)_2 A_p + q_1 A_q + 2(p-1)q\dot u_1\dot u_2 - 2\Lambda e^{2u_0}, \\ f_1^{(q)} &= p_1 A_p + (q-1)_2 A_q + 2p(q-1)\dot u_1 \dot u_2 - 2\Lambda e^{2u_0}, \\ g_1^{(p)} &= 2(p-1), \quad g_1^{(q)} = 2(q-1), \quad h_1^{(p)} = 2q, \quad h_1^{(q)} = 2p, \\ A_p &:= \dot u_1^2 + \sigma_p e^{2(u_0-u_1)}, \quad A_q := \dot u_2^2 + \sigma_q e^{2(u_0-u_2)}, \\ X &:= \ddot u_1 - \dot u_0 \dot u_1 + \dot u_1^2, \qquad Y := \ddot u_2 - \dot u_0\dot u_2 + \dot u_2^2. \end{aligned}$$

and so on. (Other terms with suffix 2 are more complicated.)
There are not independent due to Bianchi identity; two are independent.

2.1 Solutions of Accelerating Universe – de Sitter

Assume

$$\dot{u}_1 = H\,, \quad \dot{u}_2 = 0$$

and choose the time coordinate as $u_0 = 0$. The Hubble parameter H: constant and the curvature of external space zero: $\sigma_p = 0$.

The basic equations turn to be algebraic:

$$-2\Lambda + p_1 H^2 + q_1 A_q + p_3 H^4 + 2p_1 q_1 H^2 A_q + q_3 A_q^2 = 0\,,$$
$$-2\Lambda + p(p+1)H^2 + (q-1)_2 A_q + (p+1)_2 H^4 + 2p(p+1)(q-1)_2 H^2 A_q + (q-1)_4 A_q^2 = 0\,.$$

Usually, for a given CC Λ, we obtain H^2 and A_q by solving these coupled quadratic equations.

We solve the equations for H^2 and Λ for given A_q, which are just a single quadratic (or linear) equation in H^2 and a linear equation in Λ.

$$2p_2 H^4 + pH^2[1 - 2(q-1)(p-q+1)A_q] - (q-1)A_q[1 + 2(q-2)_3 A_q] = 0,$$
$$2\Lambda = p_3 H^4 + p_1 H^2(1 + 2q_1 A_q) + q_1 A_q[1 + (q-2)_3 A_q]\,.$$

2.2 Solutions

Results without GB term,

We find [a $(p+1)$-dimensional de Sitter spacetime] $\times$ [a constant internal space with a positive curvature], if a CC is positive.
$\Rightarrow$ Unstable de Sitter solution.

Results with GB term:

♠ $q = 1$: $H = 0$ is a trivial solution $\Rightarrow$ locally a Minkowski spacetime ($\sigma_q = 0$).

♠ $q \geq 2$: the condition for the existence of the real positive solutions of H^2 are as follows:
(A_q is basically the curvature of internal space.)

$$A_q[1 + 2(q-2)_3 A_q] \geq 0 \quad \text{or} \quad \begin{cases} \text{Either} \quad A_q \geq 0, (q \geq 2) & \Rightarrow \text{CC is positive} \\ \text{or} \quad A_q \leq A_q^{(\mathrm{M})} := -\frac{1}{2(q-2)_3}, (q \geq 4) & \Rightarrow \text{CC is negative} \end{cases}$$

2.3 Stability of Accelerating Universe

Perturb the variables around the background solution

$$u_1(t) = Ht + \xi(t)\,, \quad u_2(t) = u_2^{(0)} + \eta(t)\,, \cdots \eta = \eta_0 e^{\omega t}$$

we obtain the perturbation equations. The eigenvalue ω:

$$\omega^2 + pH\omega + C = 0\,,$$

whose solutions are given by (C is a suitable functions of H etc.)

$$\omega = \omega_\pm := \frac{1}{2}\left(-pH \pm \sqrt{p^2H^2 - 4C}\right)$$

Both eigenvalues $\omega_\pm$ have negative real parts $\Rightarrow$ the solution for the expanding universe ($H > 0$) is stable.
$\Rightarrow$ the condition for stability is simply $C > 0$.

It is not so instructive to look at the general expression of C. Instead, it is easier to look at C at each dimension.

Example: $p = 3$ and $q = 6$

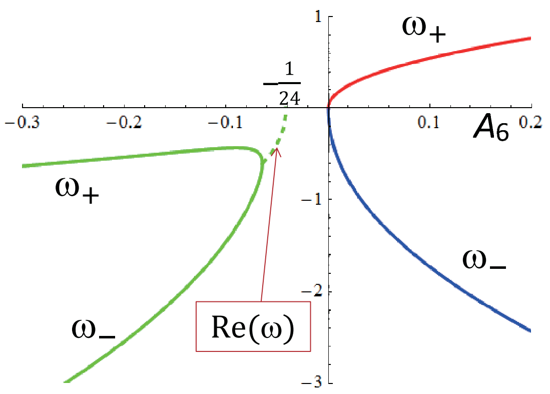

Figure 1: The eigenvalues $\omega_\pm$ and Re (ω) in terms of A_6.

The stable solution ($A_6 < -1/24$) has two real negative or a positive real part of two complex conjugate eigenvalues, which are shown by the green solid or dashed curves, respectively.

The unstable solution ($A_6 > 0$) has one real positive and one negative eigenvalues, which are shown by the red and blue solid curves, respectively.

Summary of the existence conditions for de Sitter solutions and their stability

q	branch	A_q	Λ	stability
$q=1$	-	No	No	-
$q=2,3$	(1)	$A_q \geq 0$	$\Lambda \geq 0$	unstable
$q \geq 4$	(1)	$A_q \geq 0$	$\Lambda \geq 0$	unstable
	(2)	$A_q \leq A_q^{(\mathrm{M})} = -\dfrac{1}{2(q-2)_3}$	$\Lambda \leq \Lambda^{(\mathrm{M})} = -\dfrac{q_1}{8(q-2)_3}$	stable

Table 1: The range of A_q where de Sitter solutions ($H^2 > 0$) exist.

3 de Sitter spacetimes with higher-order Lovelock terms and their stability

We can perform similar study for Lovelock gravity.

$$S = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}\sum_{n=0}^{n_{\max}}\alpha_n L_n\,,$$

where n-th order Lovelock terms L_n are given by

$$L_n := \frac{1}{2^n}\delta^{i_1\cdots i_{2n}}_{j_1\cdots j_{2n}} R^{j_1 j_2}{}_{i_1 i_2}\cdots R^{j_{2n-1}j_{2n}}{}_{i_{2n-1}i_{2n}}\,,$$

Assuming our spacetime is (de Sitter spacetime)$\times$(a static maximally symmetric space) and $p \leq 4$, we find

$$H^4\left[\sum_{n=2}^{n_{\max}}\alpha_n\frac{n(n-1)}{2}\frac{p_2\,[2D-n(p+1)]\,(q-1)!}{(q-2n+4)!}A_q^{n-2}\right] + H^2\left[\sum_{n=1}^{n_{\max}}\alpha_n\frac{np\,[D-n(p+1)]\,(q-1)!}{(q-2n+2)!}A_q^{n-1}\right] - \sum_{n=1}^{n_{\max}}\alpha_n\frac{n(q-1)!}{(q-2n)!}A_q^n = 0$$

which is quadratic in H^2.

For the obtained solution of H^2 in terms of A_q ($p \leq 4$), the cosmological constant is explicitly given by

$$2\Lambda = \sum_{n=1}^{n_{\max}}\alpha_n\sum_{k=0}^{n}{}_nC_k\,\frac{p!}{(p-2n+2k)!}\,\frac{q!}{(q-2k)!}\,H^{2(n-k)}A_q^k\,.$$

We studied the system up to 10 dimensions, with cubic and quartic Lovelock terms.

For $p=3$, the solution for the Hubble expansion parameter H is

$$H^2 \;=\; H_\pm^2 := \frac{-3\left[1+2(q-4)(q-1)A_q + 3\alpha_3(q-8)(q-1)_3A_q^2 + 4\alpha_4(q-12)(q-1)_5A_q^3\right] \pm \sqrt{\mathcal{D}}}{24\left[1+3\alpha_3(q-1)_2Aq + 6\alpha_4(q-1)_4Aq^2\right]}$$

with

$$\begin{aligned}\mathcal{D} \;:=\;& 48(q-1)A_q\left[1+2(q-2)_3A_q + 3\alpha_3(q-2)_5A_q^2\right]\left[1+3\alpha_3(q-1)_2A_q + 6\alpha_4(q-1)_4A_q^2\right]\\ &+9\left[1+2A_q(q-4)(-1+q) + 3\alpha_3A_q^2(q-8)(q-1)_3 + 4\alpha_4A_q^3(q-12)(q-1)_5\right]^2\,.\end{aligned}$$

The CC is given in terms of A_q:

$$\Lambda = \Lambda_\pm := 3H_\pm^2\left[1+2q_1A_q + 3\alpha_3q_3A_q^2 + 4\alpha_4q_5A_q^3\right] + \frac{q_1}{2}A_q\left[1+(q-2)_3A_q + \alpha_3(q-2)_5A_q^2\right]\,.$$

The stability condition for an expanding universe:

$$C = \frac{2A_q\left[6qH^2Y^2 - (q-1)_2A_qXZ\right]}{3Y\left[2H^2X - qA_qY\right]} \geq 0\,.$$

$$\begin{aligned} X \;&:=\; 1+2q_1A_q + 3\alpha_3q_3A_q^2 + 4\alpha_4q_5A_q^3,\\ Y \;&:=\; 1+2\Big(2H^2+(q-1)_2A_q\Big) + 3\alpha_3(q-1)_2\Big(4H^2+(q-3)_4A_q\Big)A_q + 4\alpha_4(q-1)_4\Big(6H^2+(q-5)_6A_q\Big)A_q^2,\\ Z \;&:=\; 1+2\Big(12H^2+(q-3)_4A_q\Big) + 3\alpha_3\Big(24H^4+24(q-3)_4H^2A_q+(q-3)_6A_q^2\Big) + 144\alpha_4(q-3)_4\Big(2H^2+(q-5)_6A_q\Big)H^2A_q. \end{aligned}$$

3.1 The effect of the quartic Lovelock term with α_4 ($\alpha_3 = 0$)

Example: $D = 10 (p=3, q=6)$

(A) $\alpha_4 > 0$

There exists stable de Sitter solutions with a negative CC for finite negative A_6. The solutions with positive A_6 or with large negative A_6 are unstable. There exists one stable Minkowski spacetime for $A_6 = -1/24$.

(B) $-196/45 < \alpha_4 < 0$

There exists de Sitter solution with a negative CC for a finite negative region of A_6. For the positive value of A_6, there are two de Sitter solutions: one is unstable and the other is stable, for which a CC is positive. Only one stable Minkowski spacetime is possible for $A_6 = -1/24$.

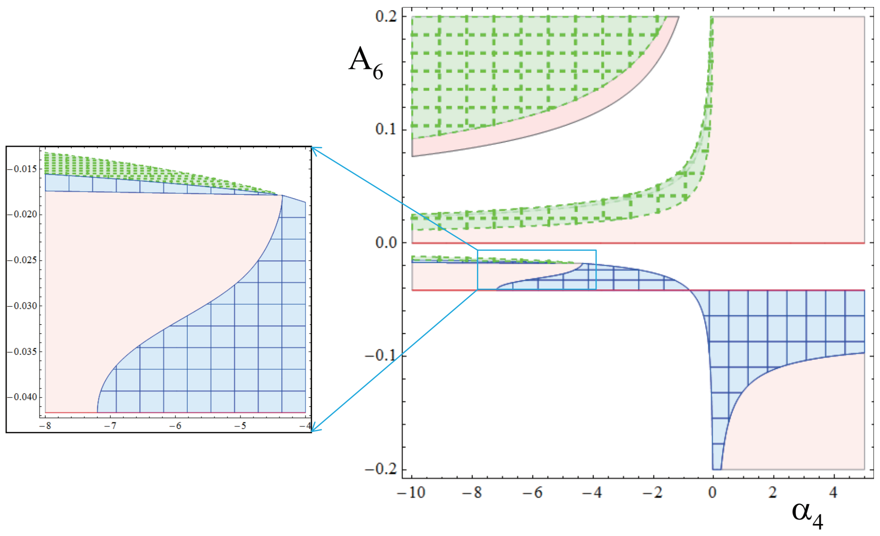

Figure 2: The de Sitter solution exists in the colored region on the α_4-A_6 plane for $D = 10$ ($\alpha_3 = 0$).

Example of $D = 10$ ($p = 3, q = 6$):

The meshed blue and meshed dotted green regions: the stable dS solutions with a negative and positive CCs.

The dS solution in the light-red shaded region is unstable.

The red lines at $A_6 = 0$ and at $A_6 = -\frac{1}{24}$ denote Minkowski spacetimes. The left small figure is the enlarged one of the part of the right figure.

(C) $-36/5 < \alpha_4 < -196/45$

This region is rather complicated. Changing the value of A_6, the stability and the sign of the CC changes frequently. There are three branches: (1) which includes a trivial Minkowski spacetime with $A_6 = 0$, (2) which include a stable Minkowski spacetime with $A_6 = -1/24$, and (3) which newly appears and does not involve a Minkowski spacetime.

There exists one stable de Sitter solution with a negative CC. The CC should be in a finite range of negative values.

(D) $\alpha_4 < -36/5$

The behaviour of the solutions is almost the same as the case (C), but de Sitter solution near Minkowski spacetime becomes unstable.

$D = 12 (p = 3, q = 8)$:

For $\alpha_4 > 0$, there exists a stable de Sitter solution with a negative CC for any negative A_8. If $\alpha_4 < 0$, we also find stable de Sitter solutions with negative values of A_8, but the CC must be positive.

3.2 The effect of the cubic Lovelock term with α_3 ($\alpha_4 = 0$)

(1) There exist stable de Sitter solutions with a negative A_q and a negative CC for $\alpha_3 < 0$.

(2) There exist stable de Sitter solutions with a negative A_q and a positive CC for $\alpha_3 > 0$ if $D \geq 10$.

(3) There exist a few stable de Sitter solutions with a positive A_q. Most solutions are unstable.

3.3 The effect of generic Lovelock terms ($\alpha_3, \alpha_4 \neq 0$)

We find a stable de Sitter solution with a negative CC when $\alpha_3 < 0$.
In this case, A_6 is always negative, but the existence region is restricted for $\alpha_4 < 0$.

On the other hand, there exists a stable de Sitter solution with a positive CC when $\alpha_3 > 0$. A_6 is always negative, but the existence region is restricted for $\alpha_4 > 0$. The solutions with $A_6 > 0$ are mostly unstable.

4 Conclusion

We studied gravitational theories with a CC and the Gauss-Bonnet curvature squared term systematically.

There are two branches of the de Sitter solutions:

Both the curvature of the internal space and the CC are (1) positive and (2) negative.

♠ The de Sitter solution of the branch (1) is unstable, while that in the branch (2) is stable.
♠ It is remarkable that we have de Sitter solutions even for a negative CC, which are the only stable ones.
♠ In this case, the curvature of the internal space is negative.

In more higher-order Lovelock curvature terms:

♣ There exist stable de Sitter solutions similar to the branch (2) for $\alpha_3 < 0$.
♣ We also find stable de Sitter solutions with positive CCs if $\alpha_3 > 0$.
♣ For most stable de Sitter solutions, the Hubble scale can be much smaller than the scale of a CC, which may explain a discrepancy between an inflation energy scale and the Planck scale.

Graceful exit:

Although the existence of a stable de Sitter spacetime with a negative CC is interesting, it is important to find a realistic cosmological model for the early universe, in which de Sitter exponential expansion must end at some stage.
⇒ de Sitter solution should be a marginally unstable state instead of an absolute stable state.

After more than 60 e-foldings, inflation must end and the universe must be reheated, finding a big bang initial state. Hence we have to find a graceful exist in the present model. Only after such a mechanism is found, we can discuss density perturbations and observational consequences.