General formalism for perturbations in bi-gravity

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Abstract

We introduce a new technique to study perturbations of Hassan-Rosen bi-gravity theory, around general backgrounds for the two dynamical metrics. In particular, we derive the general expression for the mass term and we explicitly compute it for some cosmological settings. Using this result, we study in detail tensor perturbations in branch-one bi-gravity. We show that the tensor sector is affected by a late-time instability, which sets in when the mass matrix becomes not positive definite.

Model: Hassan-Rosen bi-gravity [1]

- Massive bi-gravity theory: two interacting gravitons

$$S = \frac{M_g^2}{2}\int d^4x\sqrt{-g}\,\left[R(g) - U(g,f) + \mathcal{L}_m(g,\Phi)\right] + \frac{M_f^2}{2}\int d^4x\sqrt{-f}\,R(f)\,,$$

$$U(g,f) = -2m^2\sum_{n=0}^{4}\beta_n U_n(X)\,,\qquad X = \sqrt{g^{-1}f}\,,$$

where β_i are generic constant coefficients and the polynomials $U_i(X)$ are

$$U_0 = \mathbb{I},\quad U_1 = [X],\quad U_2 = \frac{1}{2}([X]^2 - [X^2])\,,\quad U_3 = \frac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3]\,,$$

$$U_4 = \frac{1}{24}([X]^4 - 6[X]^2[X^2] + 8[X][X^3] + 3[X^2]^2 - 6[X^4]) = \det X\,.$$

▷ It propagates 5 dofs around every backgrounds $\rightsquigarrow$ ghost-free massive gravity
▷ Dynamical dark energy density $\rightsquigarrow$ cosmological interest!

New formalism for perturbations [2][3]

We want to expand the bi-gravity action to 2nd order around generic backgrounds for g and f

$$g_{\mu\nu} = \bar g_{\mu\nu} + h_{\mu\nu}\,,\qquad f_{\mu\nu} = \bar f_{\mu\nu} + l_{\mu\nu}\,.$$

The expansion of the potential $U(\sqrt{g^{-1}f})$ is a non-trivial problem:

▷ for non-commutating matrices $\sqrt{AB} \neq \sqrt{A}\sqrt{B}$
▷ not correct to simply expand $\sqrt{g^{-1}f} = \sqrt{(\bar g\,(1+h))^{-1}\,\bar f\,(1+l)}$ in $h^\mu_\nu = h_{\alpha\nu}\bar g^{\alpha\mu}$ and $l^\mu_\nu = l_{\alpha\nu}\bar f^{\alpha\mu}$

How to proceed?

The idea is to express $U(\sqrt{g^{-1}f})$ as a function of $U(g^{-1}f)$ (λ_i eigenvalues of $g^{-1}f$)

$$t_1 \equiv U_1(\sqrt{g^{-1}f}) = \sum_i \lambda_i^{1/2}\,,\qquad s_1 \equiv U_1(g^{-1}f) = \sum_i \lambda_i\,,$$

$$t_2 \equiv U_2(\sqrt{g^{-1}f}) = \sum_{i<k} \lambda_i^{1/2}\lambda_k^{1/2}\,,\qquad s_2 \equiv U_2(g^{-1}f) = \sum_{i<j}\lambda_i\lambda_j\,,$$

$$t_3 \equiv U_3(\sqrt{g^{-1}f}) = \sum_{i<k<l}\lambda_i^{1/2}\lambda_k^{1/2}\lambda_l^{1/2}\,,\qquad s_3 \equiv U_3(g^{-1}f) = \sum_{i<j<k}\lambda_i\lambda_j\lambda_k\,,$$

$$t_4 \equiv U_4(\sqrt{g^{-1}f}) = \sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}\,,\qquad s_4 \equiv U_4(g^{-1}f) = \lambda_1\lambda_2\lambda_3\lambda_4\,.$$

$$\rightsquigarrow\quad t_1^2 = s_1 + 2t_2\,,\qquad t_2^2 = s_2 - 2\sqrt{s_4} + 2t_1t_3\,,\qquad t_3^2 = s_3 + 2t_2\sqrt{s_4}\,,\qquad t_4^2 = s_4\,.$$

We can write the perturbations of t_i in terms of those of s_i, which in turn can be obtained from

$$g^{-1}f = (1 - h + h^2)\bar g^{-1}\bar f\,(1+l)$$

We obtain, at second order in the perturbations $h_{\mu\nu}$ and $l_{\mu\nu}$

$$S_{(2)} = \frac{M_g^2}{2}\int d^4x\sqrt{-\bar g}\;h_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar g)h_{\alpha\beta} + \frac{M_f^2}{2}\int d^4x\,\sqrt{-\bar f}\;l_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}(\bar f)l_{\alpha\beta} -$$
$$-\frac{M_g^2}{2}\int d^4x\sqrt{-\bar g}\left[\mathcal{M}^{\mu\nu\alpha\beta}_{hh}(\bar f,\bar g)h_{\mu\nu}h_{\alpha\beta} + \mathcal{M}^{\mu\nu\alpha\beta}_{hl}(\bar f,\bar g)h_{\mu\nu}l_{\alpha\beta} + \mathcal{M}^{\mu\nu\alpha\beta}_{ll}(\bar f,\bar g)l_{\mu\nu}l_{\alpha\beta}\right]$$

▷ $\mathcal{E}^{\mu\nu\alpha\beta}(\bar g)$ and $\mathcal{E}^{\mu\nu\alpha\beta}(\bar f)$ are the Lichnerowicz operators in curved spacetime
▷ $\mathcal{M}^{\mu\nu\alpha\beta}_{\bullet\bullet}(\bar g,\bar f)$ are explicit functions of $\bar g$ and $\bar f$, calculated with the technology above

Application to cosmology: branch I bigravity

- Background metrics in conformal time

$$\bar g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^idx^j\right)\,,\quad \bar f_{\mu\nu}dx^\mu dx^\nu = b^2(\tau)\left(-c^2(\tau)d\tau^2 + \delta_{ij}dx^idx^j\right)\,.$$

- Conformal Hubble parameter ($\mathcal{H}$), standard one (H) and ratio between the two scale factors

$$H = \frac{\mathcal{H}}{a} = \frac{a'}{a^2}\,,\qquad H_f = \frac{\mathcal{H}_f}{b} = \frac{b'}{b^2\,c}\,,\qquad r = \frac{b}{a}\,.$$

- Bianchi constraint can be realized in two ways: two branches

$$m^2\underbrace{(\beta_3 r^2 + 2\beta_2 r + \beta_1)}_{\text{branch I}}\underbrace{(\mathcal{H} - \mathcal{H}_f)}_{\text{branch II}} = 0$$

- Branch I at the background level is equivalent to GR with an effective cosmological constant

$$H^2 = \frac{8\pi G}{3}\left(\rho + \Lambda_{eff}\right)\,,\qquad 3\,H_f^2 = \frac{m^2}{M_*^2}\left(\frac{\beta_1}{r^3} + \frac{3\beta_2}{r^2} + \frac{3\beta_3}{r} + \beta_4\right)\,,$$

where $\Lambda_{eff} = \frac{m^2}{8\pi G}\left(\beta_0 - 2\beta_3\,\bar r^3 - 3\,\beta_2\,\bar r^2\right)$ and $M_* \equiv M_f/M_g$.

- From the Friedmann equation for f, we can extract the lapse

$$c^2 = \frac{\rho + \Lambda_{eff}}{\Lambda_c}\,,\qquad \Lambda_c = \frac{m^2}{8\pi G}\,\frac{1}{M_*^2}\left(\beta_4\,\bar r^2 + 2\beta_3\,\bar r + \beta_2\right)\,.$$

- Scalar and vector perturbations evolve like in GR. Tensor perturbations have different evolution!

$\rightsquigarrow$ Branch I is cosmologically interesting!

Study of tensor perturbations in branch I [2]

Analysis of the mass matrix

- We specialize this formalism to parity-invariant tensor perturbations h and l (we set $M_* = 1$)

$$S^{(2)} = \frac{1}{2}M_g^2\int d^4x\,a^2\left\{(h')^2 + \frac{r^2}{c}\,(l')^2 - k^2\,h^2 - k^2c\,r^2\,l^2 + a^2\,\lambda(\beta_i,c)\left(2h\,l - h^2 - l^2\right)\right\}\,,$$

where $\lambda(\beta_i,c) = A(\beta_i) + B(\beta_i)\,c$.

- The mass matrix for the canonically-normalized variables $Q_h \equiv M_g\,a\,h$, $Q_l \equiv M_g\,a\,\frac{r}{\sqrt{c}}\,l$, is

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}(a) + k^2 + a^2\,m^2\,\lambda(\beta_i,c) & -a^2m^2\lambda(\beta_i,c)\frac{\sqrt{c}}{r} \\ -a^2m^2\lambda(\beta_i,c)\frac{\sqrt{c}}{r} & \mathcal{D}(a,c) + k^2c^2 + a^2m^2\lambda(\beta_i,c)\frac{c}{r^2}\end{pmatrix}\,,$$

with

$$\mathcal{C}(a) = -\mathcal{H}^2 - \mathcal{H}'\,,\qquad \mathcal{D}(a,c) = -\mathcal{H}^2 - \mathcal{H}' + \frac{c'}{c}\mathcal{H} + \frac{1}{2}\frac{c''}{c} - \frac{3}{4}\left(\frac{c'}{c}\right)^2\,.$$

- $\mathcal{M}$ is positive-definite (i.e. absence of tachyonic instabilities) for:

$$k^2 \gtrsim \left(\frac{1}{1+z}\right)^2\mathcal{H}_0^2\qquad\rightsquigarrow\qquad\text{tachyonic instability at low redshift!}$$

Cosmological dynamics of gravitational waves (GW)

- We analytically study the tensor sector during inflation and the following cosmological epochs

- The evolution of GW differs from the GR one only in the late de Sitter phase, where

$$h'' - \frac{2}{\tau}\,h' + k^2h + \frac{\mathcal{R}_\Lambda}{\tau^2}\,(h - l) = 0\,,\qquad \mathcal{R}_\Lambda \simeq \mathcal{O}(1)\,,$$
$$l'' - \frac{2}{\tau}\,l' + c^2\,k^2\,l - \frac{\mathcal{R}_\Lambda}{\tau^2}\,\frac{c}{r^2}\,(h - l) = 0\,.$$

Coupling relevant for:

$$\left(\frac{\mathcal{R}_\Lambda}{\tau^2}\,l\right)/\,(k^2\,h) \simeq \left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)^2\left(\frac{\mathcal{H}_0}{k}\right)^2\left(\frac{l}{h}\right) \gg 1$$

▷ for $h\,,l$ fixed, modes with smaller k experience earlier the effects of the coupling
▷ for a given k, the effect of the coupling at late times is proportional to (l/h)

Numerical results [2]

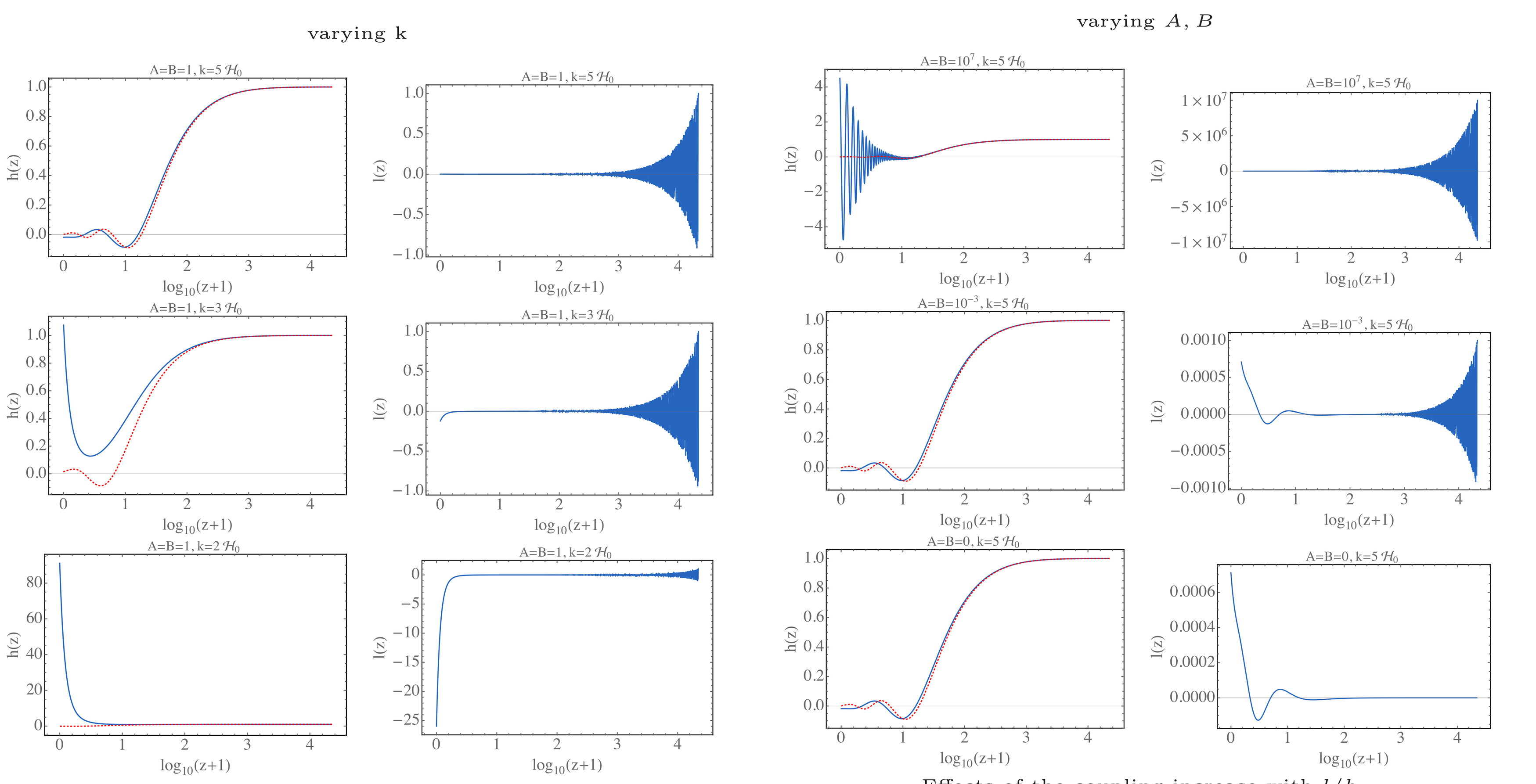
- We choose values for β_i such that the background evolution fits cosmological data

- We numerically evolve the system from redshift of equality with GR-like initial conditions for h

$$h(z_{eq}) = 1\,,\qquad h'(z_{eq}) = 0\,,\qquad l(z_{eq}) = A\,,\qquad l'(z_{eq}) = \mathcal{H}_0\,B\,.$$

(in red, GW evolution in ΛCDM)

varying k | varying A, B



Instability sets in earlier for smaller k | Effects of the coupling increase with l/h

Conclusions

- We have worked out the general expression for the bi-gravity action perturbed at 2nd order around general backgrounds for the two dynamical metrics

- We have specialized the result obtained to the study of tensor perturbations in branch I:

▷ tachyonic instability at late-times
▷ instability sets in earlier for smaller k
▷ deviation from the ΛCDM evolution increases with the ratio l/h after inflation
▷ we find after inflation $l \simeq h\sqrt{H_0/H_I}$ $\rightsquigarrow$ physical scales become unstable only in the future

References

[1] S. Hassan and R. A. Rosen, "Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity," *JHEP*, vol. 1204, p. 123, 2012.
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[3] P. Guarato and R. Durrer, "Perturbations for massive gravity theories," *Phys.Rev.*, vol. D89, p. 084016, 2014.