Direct measurement of $\alpha_{\text{QED}}(m_Z^2)$ @ FCC-ee

- **Outline**
  - Why measure $\alpha_{\text{QED}}(m_Z^2)$?
    - Physics behind precision
    - Impact of $\alpha_{\text{QED}}(m_Z^2)$ precision
  - Sensitivity of of $e^+e^-\rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$
    - Cross section
    - Forward-backward asymmetry
    - Statistical uncertainty and sensitivity
  - Determination of $\alpha_{\text{QED}}(m_Z^2)$
  - Systematic uncertainties
    - A few highlights and follow-ups
  - Conclusions

See arXiv:1512:05544
(Published as JHEP 2016(2) 1-22)
Physics behind precision

- EWPO measurements allow a prediction of $m_{\text{top}}$, $m_W$, $m_H$, $\sin^2 \theta_W$ in the SM
  - Compare with FCC-ee direct measurements
  - The standard model has nowhere to go

Without $m_Z$ @ FCC-ee, the SM line would have a 2.2 MeV width
Impact of $\alpha_{\text{QED}}(m_Z^2)$ precision

- Uncertainty on these predictions are of two origins
  - Parametric
  - Higher orders
- For $m_W$ and $\sin^2 \theta_W$ today (see Sven Heinemeyer’s talk on Tuesday)

$$M_W = 80.3593 \pm 0.0056_{m_t} \pm 0.0026_{M_Z} \pm 0.0018_{\Delta \alpha_{\text{had}}} \pm 0.0017_{\alpha_S} \pm 0.0002_{M_H} \pm 0.0040_{\text{theo}}$$
Exp: 0.015

$$= 80.359 \pm 0.011_{\text{tot}}$$

$$\sin^2 \theta_{\text{eff}} = 0.231496 \pm 0.000030_{m_t} \pm 0.000015_{M_Z} \pm 0.000035_{\Delta \alpha_{\text{had}}} \pm 0.000010_{\alpha_S} \pm 0.000002_{M_H} \pm 0.000047_{\text{theo}}$$
Exp: 0.00014

$$= 0.23150 \pm 0.00010_{\text{tot}}$$
Impact of $\alpha_{\text{QED}}(m_{Z}^2)$ precision

- Uncertainty on these predictions are of two origins
  - Parametric
  - Higher orders (QCD, EW, mixed)
- Reduced uncertainties at the FCC-ee

\[ M_W = 80.3593 \pm 0.0001 \pm 0.0001 \pm 0.0002 \pm 0.0000 = 80.359 \pm 0.005 \text{ tot} \]

\[ \sin^2 \theta_{\text{eff}} = 0.231496 \pm 0.000001 \pm 0.000001 \pm 0.000001 \pm 0.000000 = 0.23150 \pm 0.00006 \text{ tot} \]

- Must reduce current exp’tal uncertainty on $\alpha_{\text{QED}}(m_{Z}^2)$ by a factor $\sim 4-5$
  - New generation of theoretical calculations is necessary to gain a factor 10 in precision
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{QED}(m_Z^2)$

- The $e^+e^- \rightarrow \mu^+\mu^-$ cross section

\[ G = \frac{c^2_{\gamma}}{s}, \]
\[ Z = \frac{c^2_{Z}(v^2 + a^2)^2 \times s}{(s - m^2_Z)^2 + m^2_Z \Gamma^2_Z}, \]
\[ I = \frac{2c_{\gamma}c_{Z}v^2 \times (s - m^2_Z)}{(s - m^2_Z)^2 + m^2_Z \Gamma^2_Z}, \]

\[ c_{\gamma} = \sqrt{\frac{4\pi}{3}} \alpha_{QED}(s), \quad c_{Z} = \sqrt{\frac{4\pi}{3}} \frac{m^2_Z}{2\pi} \frac{G_F}{\sqrt{2}}, \quad a = -\frac{1}{2}, \quad v = a \times (1 - 4\sin^2 \theta_W), \]
\[ \sim -0.037 \]

- Photon exchange (G) proportional to $\alpha^2(s)$
- Z exchange (Z) proportional to $G_F^2$
- Interference term proportional to $\alpha(s) G_F$
  - Need to choose $\sqrt{s}$ judiciously to maximize sensitivity to $\alpha(s)$
  - If $\sqrt{s}$ is close to $m_Z$, the $\sqrt{s} \rightarrow m_Z$ extrapolation uncertainty is negligible
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$

- The $e^+e^- \rightarrow \mu^+\mu^-$ cross section

![Graph showing the cross section for $e^+e^- \rightarrow \mu^+\mu^-$ and its components.](image-url)
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$

- The $e^+e^- \rightarrow \mu^+\mu^-$ angular distribution
  - Absolute cross section measurement might be challenging to the required precision
  - Uncertainty of the integrated luminosity determination
  - Rely of a self-normalizing quantity, the forward-backward asymmetry $A_{FB}^{\mu\mu} = \frac{\sigma_F^{\mu\mu} - \sigma_B^{\mu\mu}}{\sigma_F^{\mu\mu} + \sigma_B^{\mu\mu}}$

\[
\frac{d\sigma_{\mu\mu}}{d\cos \theta}(s) \propto G_1(s) \times (1 + \cos^2 \theta) + G_3(s) \times 2 \cos \theta,
\]

\[
A_{FB}^{\mu\mu}(s) = \frac{3}{4} \frac{G_3(s)}{G_1(s)}.
\]

$G_1(s) = G + I + Z$ and $G_3(s) = \frac{a^2}{v^2} \left\{ I + \frac{4v^4/a^4}{(1 + v^2/a^2)^2} Z \right\}$

\[
A_{FB,0}^{\mu\mu} = (3/4) \times \frac{a^2}{4v^2} \frac{I}{G + Z}.
\]

\[
A_{FB}^{\mu\mu} = A_{FB,0}^{\mu\mu} + \frac{3}{4} \frac{a^2}{v^2} \frac{I}{G + Z}.
\]

$A_{FB,0}^{\mu\mu} = (3/4) \times \frac{a^2}{4v^2a^2/(a^2 + v^2)^2} \approx 0.016.$
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$

- **Variation of $A_{FB}$ with $\alpha_{\text{QED}}$**

\[
A_{FB}^{\mu\mu} = A_{FB,0}^{\mu\mu} + \frac{3}{4} S^2 \frac{3}{4} \frac{\mathcal{I}}{v^2 G + Z}.
\]

For a small variation $\Delta \alpha$

\[
\Delta A_{FB}^{\mu\mu} = \frac{\Delta \alpha}{\alpha} \times \frac{3}{4} S^2 \frac{3}{4} \frac{\mathcal{I}(Z - G)}{(G + Z)^2} = \left(A_{FB}^{\mu\mu} - A_{FB,0}^{\mu\mu}\right) \times \frac{Z - G}{Z + G} \times \frac{\Delta \alpha}{\alpha}.
\]

\[
\frac{\Delta \alpha}{\alpha} = \frac{\Delta A_{FB}^{\mu\mu}}{A_{FB}^{\mu\mu} - A_{FB,0}^{\mu\mu}} \times \frac{Z + G}{Z - G} \approx \frac{\Delta A_{FB}^{\mu\mu}}{A_{FB}^{\mu\mu}} \times \frac{Z + G}{Z - G},
\]

- **Statistical uncertainty on $A_{FB}$**

\[
\sigma(A_{FB}^{\mu\mu}) = \sqrt{\frac{1 - A_{FB}^{\mu\mu}^2}{L \sigma_{\mu\mu}}},
\]
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$

- Comparison of the sensitivity and the statistical uncertainty
  - Statistical uncertainty: one year with target luminosities (86 ab$^{-1}$ at the Z pole)
  - Compared with $\Delta A_{FB}$ for current $\alpha_{\text{QED}}$ uncertainty

- $\Delta A_{FB}$ changes sign for $\sqrt{s} = m_Z$ and when $Z = G$ (78 and 112 GeV): no sensitivity to $\alpha_{\text{QED}}$
Sensitivity of $e^+e^- \rightarrow \mu^+\mu^-$ to $\alpha_{\text{QED}}(m_Z^2)$

- Turning the previous plot in a $\sigma(\alpha)/\alpha$ plot, for a year of running at any $\sqrt{s}$

- Optimal centre-of-mass energies for a $3 \times 10^{-5}$ uncertainty on $\alpha_{\text{QED}}$
  - One year at $\sqrt{s_-} = 87.9$ GeV or one year at $\sqrt{s_+} = 94.3$ GeV
  - Even better: six months at $\sqrt{s_-}$ and six months at $\sqrt{s_+}$
Determination of $\alpha_{\text{QED}}(m_Z^2)$

- Two measurements
  - Two asymmetries at two $\sqrt{s}$: $A_{FB}(s_-)$ and $A_{FB}(s_+)$
    \[ \alpha_- \equiv \alpha_{\text{QED}}(s_-) \quad \text{and} \quad \alpha_+ \equiv \alpha_{\text{QED}}(s_+) \]

- Running from $\sqrt{s}_{\pm}$ to $m_Z$ gives two determinations of $\alpha_0$
  \[ \frac{1}{\alpha_0} = \frac{1}{\alpha_{\pm}} + \beta \log \frac{s_\pm}{m_Z^2} \]

- Solve for $\alpha_0 = \alpha_{\text{QED}}(m_Z^2)$
  \[ \frac{1}{\alpha_0} = \frac{1}{2} \left( \frac{1 - \xi}{\alpha_-} + \frac{1 + \xi}{\alpha_+} \right), \quad \text{where} \quad \xi = \frac{\log s_- s_+ / m_Z^4}{\log s_- / s_+} \approx 0.045 \]

- With potential of almost exact cancellations for correlated effects at $s_\pm$
  \[ \frac{\Delta \alpha_0}{\alpha_0} \approx 0.528 \frac{\Delta A_{FB}}{A_{FB}}(s_-) \oplus 0.563 \frac{\Delta A_{FB}}{A_{FB}}(s_+) \]
  \[ \Delta A_{FB}/A_{FB}(s_-) > 0 \quad \Delta A_{FB}/A_{FB}(s_+) < 0 \]
## Systematic uncertainties

### Summary of the study

<table>
<thead>
<tr>
<th>Type</th>
<th>Source</th>
<th>Uncertainty</th>
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<tbody>
<tr>
<td><strong>Experimental</strong></td>
<td>$E_{\text{beam}}$ calibration</td>
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<tr>
<td></td>
<td>$E_{\text{beam}}$ spread</td>
<td>$&lt; 10^{-7}$</td>
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<td>Acceptance and efficiency</td>
<td>negl.</td>
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<td>Charge inversion</td>
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<td></td>
<td>Backgrounds</td>
<td>negl.</td>
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<td><strong>Parametric</strong></td>
<td>$m_Z$ and $\Gamma_Z$</td>
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<td></td>
<td>$\sin^2 \theta_W$</td>
<td>$5 \times 10^{-6}$</td>
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<td></td>
<td>$G_F$</td>
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<tr>
<td><strong>Theoretical</strong></td>
<td>QED (ISR, FSR, IFI)</td>
<td>$&lt; 10^{-6}$</td>
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<td>Missing EW higher orders</td>
<td>few $10^{-4}$</td>
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<td>New physics in the running</td>
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<td><strong>Total</strong> (except missing EW higher orders)</td>
<td>Systematics</td>
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<tr>
<td></td>
<td>Statistics</td>
<td>$3 \times 10^{-5}$</td>
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</table>
Knowledge of the beam energy is crucial

- Strong dependence of $A_{FB}^{\mu\mu}$ with $\sqrt{s}$ (and $m_Z$)

$$A_{FB}^{\mu\mu}(s, m_Z) \propto (s - m_Z^2)/(sm_Z^2).$$

- Uncertainty of the beam energy measurements
  - Total $\sim 50$ keV
    - of which 45 keV correlated and 23 keV uncorrelated between energy points
  - Uncorrelated variables
    - $D = \sqrt{s - m_Z}$ and $\Sigma = (\sqrt{s} + m_Z)/2$.
    - $\sigma_D = 46$ keV and $\sigma_\Sigma = 94$ keV.

- Propagation to $A_{FB}$:
  $$\frac{\sigma(A_{FB}^{\mu\mu})}{A_{FB}^{\mu\mu}} \approx \frac{1}{\sqrt{sm_Z}} \sqrt{(s + m_Z^2 - \sqrt{sm_Z})^2 \frac{\sigma_D^2}{D^2} + (s + m_Z^2 + \sqrt{sm_Z})^2 \frac{\sigma_\Sigma^2}{\Sigma^2}},$$

- Propagation to $\alpha_{\text{QED}}$:
  $$\frac{\sigma(\alpha_\pm)}{\alpha_\pm} \approx \frac{\sigma_D}{D},$$  with  $\quad D_\pm = \sqrt{s_\pm - m_Z}$,  \quad (uncorrelated)

$$\frac{\sigma(\alpha_0)}{\alpha_0} \approx \frac{1}{2} \sqrt{(1 - \xi)^2 \frac{\sigma^2(\alpha_-)}{\alpha_-^2} + (1 + \xi)^2 \frac{\sigma^2(\alpha_+)}{\alpha_+^2}} \approx 1 \times 10^{-5}.$$
Missing EW higher orders: few $10^{-4}$

- See presentation on Tuesday (Sven Heinemeyer)
  - Current status: full one-loop calculation available
    - Box EW correction: $8 \times 10^{-4}$
    - Vertex EW correction: $\sim 10^{-3}$
    - Uncertainty: few $10^{-4}$
  - Higher-order corrections computable with current techniques
    - At the level of a few $10^{-4}$
    - Uncertainty: few $10^{-5}$ - This is what we need.
  - New techniques might be needed for 3-loop corrections
    - At the level of a few $10^{-5}$
    - Uncertainty: $< 10^{-5}$

- Significant precision improvement needed for all EWPO predictions
  - Need to set up a consistent international effort now
    - To benefit from the experience of our experts
    - To train a new generation of theorists
Efficiency and acceptance: negligible

- At lowest order in QED (no ISR/FSR/IFI)
  - Any $\cos \theta$-dependent efficiency and acceptance $\varepsilon(\cos \theta)$ can be unfolded as follows
    \[
    \frac{dN^\pm}{d \cos \theta} \propto \left\{ 1 + \cos^2 \theta \mp \frac{8}{3} A_{FB}^{\mu\mu} \cos \theta \right\} \times \varepsilon(\cos \theta) \quad \Rightarrow \quad A_\pm(\cos \theta) \equiv \frac{N_- (\cos \theta) - N_+ (\cos \theta)}{N_- (\cos \theta) + N_+ (\cos \theta)} = \frac{4}{3} \frac{2 \cos \theta}{1 + \cos^2 \theta} A_{FB}^{\mu\mu}
    \]
  - $A_{FB}$ is thus obtained from $\mu^+/\mu^-$ asymmetry in each $\cos \theta$ bin

- Effect of QED initial-state radiation (no FSR/IFI)
  - ISR changes the muon angular distribution
    - Because of the longitudinal boost
    - Because of the reduction of $\sqrt{s}$ ($DA_{FB} < 0$)
  - Both effect can be dealt with in case of one ISR photon
    - Measure $A_\pm(s'/s, \cos \theta^*)$ in each $\cos \theta^*$ bin
      - And fit for $A_{FB}^{\mu\mu}(s'/s)$
The $s'/s$ and $\cos \theta^*$ determination is no longer exact when

- Two ISR photons are radiated
- The muon angular resolution of the detector is not perfect
- The beam energy spread is not negligible (comment from Mogens Dam)

- All generate important biases on the measurement of $A_{FB}(s'/s)$

- Changes the asymmetry by up to 2% at $\sqrt{s} = 87.9$ GeV!
The $s'/s$ and $\cos\theta^*$ determination is no longer exact when
- Two ISR photons are radiated
- The muon angular resolution of the detector is not perfect
- The beam energy spread is not negligible (comment from Mogens Dam)
  - All generate important biases on the measurement of $A_{FB}(s'/s)$

Changes the asymmetry by up to 2% at $\sqrt{s} = 87.9$ GeV!
- But the absolute effect on $A_{FB}$ is universal: it is ~identical at $\sqrt{s} = 94.3$ GeV

\[ \Delta A_{FB}/A_{FB}(s^+) < 0 \text{ and } \Delta A_{FB}/A_{FB}(s^-) > 0 \]
Initial-state radiation (no FSR/IFI)

- Cancellation of the biases when combining the two measurements

\[
\frac{\Delta \alpha_0}{\alpha_0} \simeq 0.528 \frac{\Delta A_{FB}}{A_{FB}} (s_-) + 0.563 \frac{\Delta A_{FB}}{A_{FB}} (s_+) \]

- Total bias on \(\alpha_{QED}(m_Z^2)\) of the order of \(8 \times 10^{-6}\)
  - And is well predictable by QED soft photon prediction: negligible uncertainty

- Note: \(\sqrt{s}\)-spread also changes the average \(\sqrt{s}\) due to non flat cross section
  - Additional bias of \(8 \times 10^{-5}\) on \(\alpha_{QED}(m_Z^2)\): must know \(\sqrt{s}\)-spread to better than 10%
See also Staszek Jadach’s talk on Tuesday

- Angular distribution modified with another totally asymmetric function

\[
\text{IFI} \approx Q_{\mu^-} \times Q_{e^-} \text{ or } Q_{\mu^+} \times Q_{e^+} > 0
\]

\[
\frac{d\sigma_{\mu\mu}(s')}{d\cos\theta^*} \propto \left\{ 1 + \cos^2\theta^* + \frac{8}{3} A_{\text{FB}}(s') \cos\theta^* \right\} \times \left\{ 1 + f \left( \frac{s'}{s} \right) \log \frac{1 + \cos\theta^*}{1 - \cos\theta^*} \right\}, (+\text{FSR})
\]

- Effect is large (unlike what I inferred in arXiv:1512:05544)
  - It is larger for tighter cuts on s'/s

- Cancellation still occurs between s_ and s_+
  - But the simple trick with \( A_\perp^\pm(\cos\theta) \) needs to be refined
Plot from Staszek’s talk
- In this plot, \( A_{FB} \) is determined as

\[
A_{FB}^{\mu\mu} = \frac{\sigma_F^{\mu\mu} - \sigma_B^{\mu\mu}}{\sigma_F^{\mu\mu} + \sigma_B^{\mu\mu}},
\]

- Relative effect on \( A_{FB}(s_2) \) of the order of 10% for a tight cut on \( s'/s \)
- Relative effect on the difference of the order of 0.2% [\text{used to determine} \( \alpha_{QED}(m_Z^2) \)]
- Requires IFI to be predicted to a precision better than 1% to reach the required precision on \( \alpha_{QED}(m_Z^2) \)
Best would be to fit IFI effect away from the data – at least partially.

Characteristic effect similar at all $\sqrt{s}$: simultaneous fit may bring an elegant solution.

$A_{\pm}(\cos \theta) \times \frac{3}{4} (1 + \cos^2 \theta) / 2 \cos \theta$

$10^7$ events produced with KKMC by Scott Yost and Staszek Jadach.
Once IFI is fit from the data (to be done)

\[ N_{\pm}(\cos\theta) \approx N_0 \times \varepsilon(\cos\theta) \times \left\{ 1 + \cos^2 \theta^* \pm \frac{8}{3} A_{FB}(s') \cos \theta^* \right\} \times \left\{ 1 \pm f \left( \frac{s'}{s} \right) \log \frac{1 + \cos \theta^*}{1 - \cos \theta^*} \right\}, \]

\[ A_{FB} = \frac{3}{4} \frac{1 + \cos^2 \vartheta}{2 \cos \vartheta} \frac{A_\pm - F}{1 - A_\pm F} \quad \text{with} \quad A_\pm = \frac{N_-(\cos \vartheta) - N_+(\cos \vartheta)}{N_-(\cos \vartheta) + N_+(\cos \vartheta)} \]

- I still need to find my way (analytically or not) towards a satisfactory fit
  - And benefit from the full cancellation of the IFI bias between \( s_+ \) and \( s_- \)

- The IFI functional form can also be taken from Monte Carlo
  - In this case, it has to be predicted with a precision of a few \( 10^{-3} \)
    - Towards a precision on \( \alpha_{QED}(m_Z^2) \) of few \( 10^{-5} \)
Conclusions

- The FCC-ee can deliver a direct measurement of $\alpha_{\text{QED}}(m_Z^2)$ to $3 \times 10^{-5}$
  - With an accuracy 4-5 times smaller than current measurements of $\Delta\alpha_{\text{had}}$
    - Needed to exploit EWPO measurements at ANY $e^+e^-$ collider

- The FCC-ee is the only $e^+e^-$ collider able to do so (because it is circular)
  - First key breakthrough: The large integrated luminosities
    - Target luminosities with 4 IP’s allow this measurement to be done in a year
  - Second key breakthrough: The ultra-precise measurement of the beam energy
    - Two orders of magnitude better than at linear colliders
      - Still the dominant experimental systematic uncertainty at the FCC-ee
      - Note: The beam energy spread must also be measured to ~10% or better

- The only obstacle today (beyond building the collider) is THEORY
  - Pure QED corrections (in particular IFI) can probably be fit from the data
    - Or will need to be predicted with a precision of a few $10^{-3}$
  - Electroweak corrections to $A_{FB}$ need to be computed to higher orders
    - Today, full one-loop calculation available, uncertainty ~ few $10^{-4}$
  - A consistent international effort must be setup for all EWPO predictions.