

Lévy sorfejtések

T. Novák
KRF

T. Csörgő
KRF, Wigner RCP

H.C. Eggers and M.B. De Kock
University of Stellenbosh

1. Magyar Femtoszkópai Műhely
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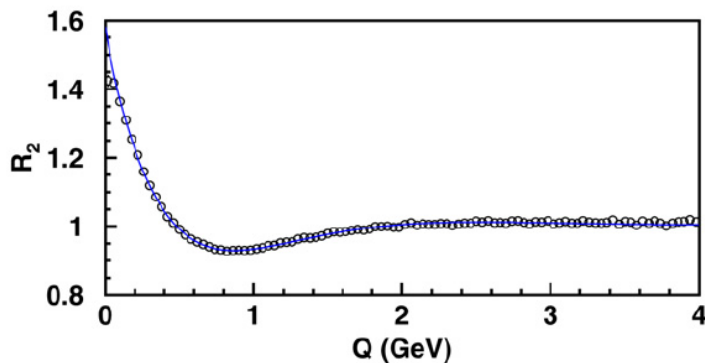


Fig. 1. The Bose-Einstein correlation function R_2 for events generated by PYTHIA. The curve corresponds to a fit of the one-sided Lévy parametrization, Eq. (13).

Model-independent shape analysis:

- General introduction
- Edgeworth,
- Laguerre,
- Lévy expansions

Summary

MODEL - INDEPENDENT SHAPE ANALYSIS I.

experimental properties:

i) The correlation function tends to a constant for large values of the relative momentum Q .

ii) The correlation function has a non-trivial structure at a certain value of its argument.

The location of the non-trivial structure in the correlation function is assumed for simplicity to be close to $Q = 0$.

Model-independent but experimentally testable:

- $w(t)$ measure in an abstract H-space
- approximate form of the correlations
- t : dimensionless scale variable

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n h_n(t),$$

$$f_n = \int dt w(t) f(t) h_n(t).$$

e.g. $t = Q_I R_I$

MODEL - INDEPENDENT SHAPE ANALYSIS II.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)},$$

$$R_2(\mathbf{k}_1, \mathbf{k}_2) = C_2(\mathbf{k}_1, \mathbf{k}_2) - 1.$$

Let us assume, that the function $g(t) = R_2(t)/w(t)$ is also an element of the Hilbert space H . This is possible, if

$$\int dt w(t)g^2(t) = \int dt [R_2^2(t)/w(t)] < \infty, \quad (6)$$

Then the function g can be expanded as

$$g(t) = \sum_{n=0}^{\infty} g_n h_n(t),$$
$$g_n = \int dt R_2(t) h_n(t).$$

From the completeness of the Hilbert space and from the assumption that $g(t)$ is in the Hilbert space:

$$R_2(t) = w(t) \sum_{n=0}^{\infty} g_n h_n(t).$$

MODEL - INDEPENDENT SHAPE ANALYSIS III.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N} \left\{ 1 + \lambda_w w(t) \sum_{n=0}^{\infty} g_n h_n(t) \right\}$$

Model-independent AND experimentally testable:

- method for any approximate shape $w(t)$
- the core-halo intercept parameter of the CF is
- coefficients by numerical integration (fits to data)
- condition for applicability: experimentally testable

$$\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$$

$$g_n = \int dt R_2(t) h_n(t)$$

$$\int dt [R_2^2(t)/w(t)] < \infty$$

EDGEWORTH EXPANSION: ~ GAUSSIAN

$$t = \sqrt{2}QR_E,$$

$$w(t) = \exp(-t^2/2),$$

$$\int_{-\infty}^{\infty} dt \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2}QR_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2}QR_E) + \dots \right] \right\}.$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt} \right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$

$$H_2(t) = t^2 - 1,$$

$$H_3(t) = t^3 - 3t,$$

$$H_4(t) = t^4 - 6t^2 + 3, \dots$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb?)

LAGUERRE EXPANSIONS: ~ EXPONENTIAL

Model-independent but experimentally tested:

- $w(t)$ exponential
- t dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$

$$\int_0^{\infty} dt \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} (-t)^n \exp(-t).$$

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \dots \right] \right\}$$

First successful tests

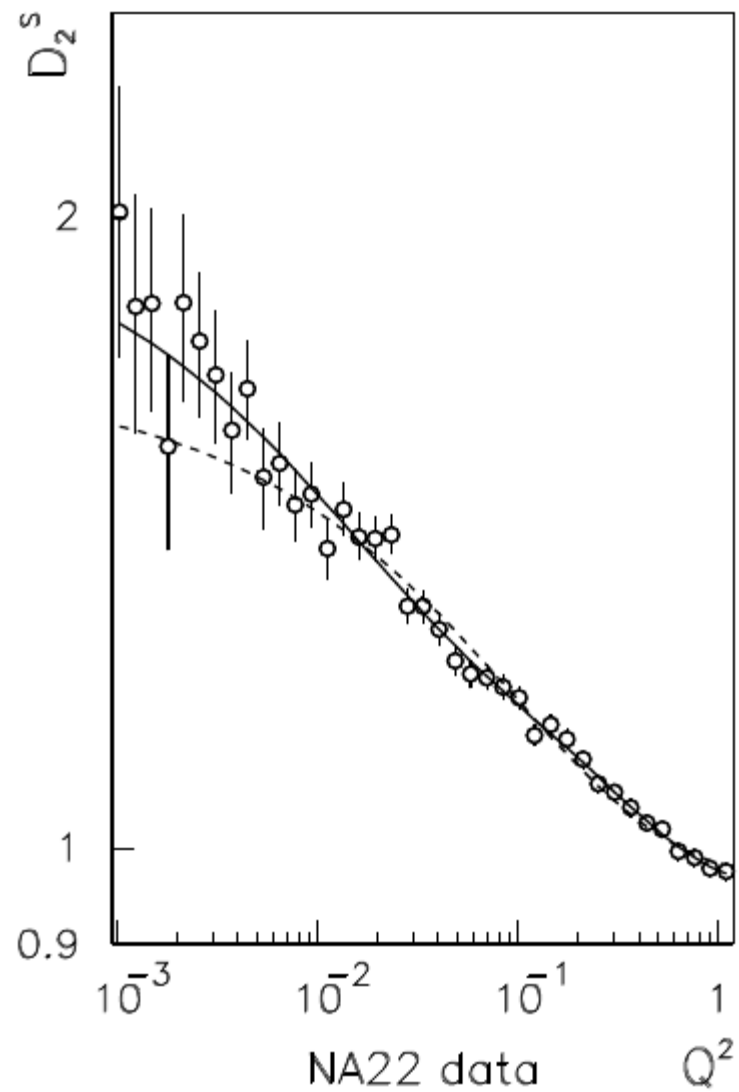
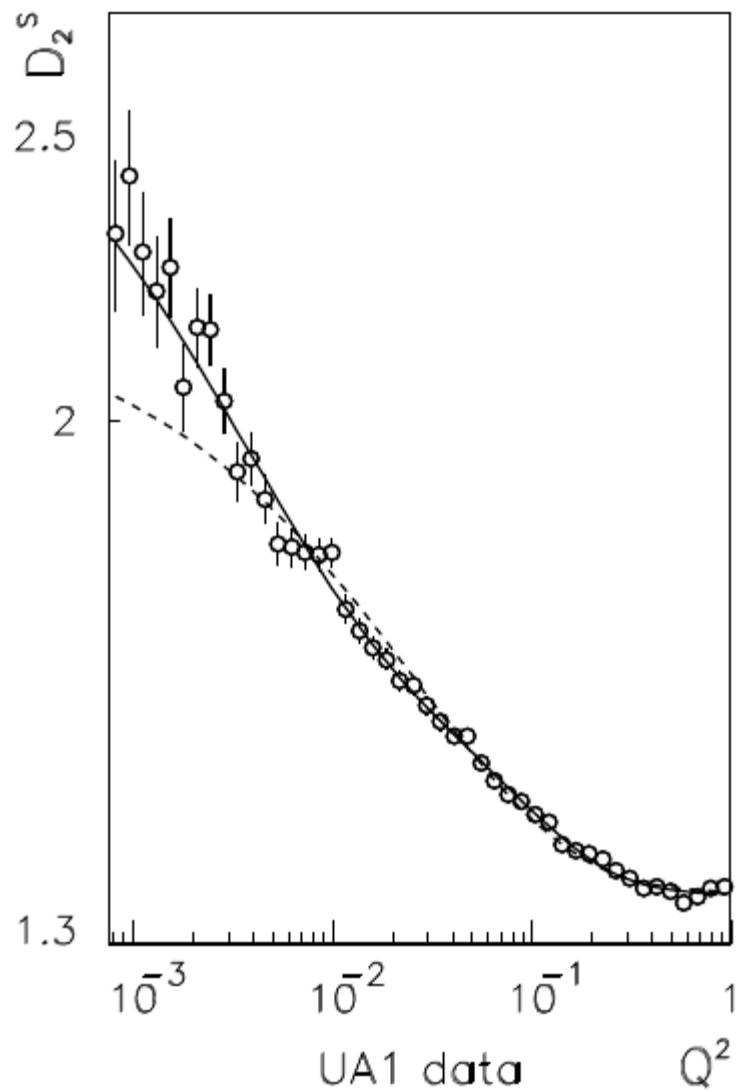
- NA22, UA1 data
- convergence criteria satisfied
- intercept parameter ~ 1

$$\int_0^{\infty} dt R_2^2(t) \exp(+t) < \infty,$$

$$\lambda_* = \lambda_L [1 - c_1 + c_2 - \dots],$$
$$\delta^2 \lambda_* = \delta^2 \lambda_L [1 + c_1^2 + c_2^2 + \dots] + \lambda_L^2 [\delta^2 c_1 + \delta^2 c_2 + \dots]$$

LAGUERRE EXPANSIONS: ~ superEXPONENTIAL

Laguerre expansion fit



LEVY EXPANSIONS: ~ 1d LEVY

Model-independent but:

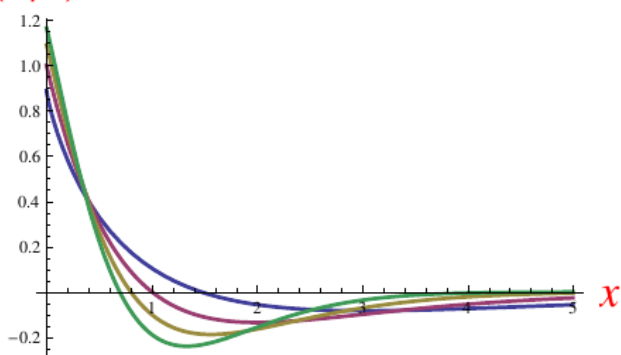
- Levy generalizes Gaussians and exponentials
- ubiquitous in nature
- How far from a Levy?
- Need new set of polynomials orthonormal to a Levy weight

$$L_1(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

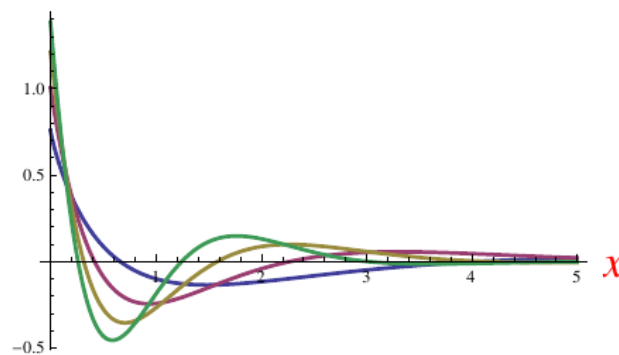
$$L_2(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx x^r f(x | \alpha) = \frac{1}{\alpha} \Gamma\left(\frac{r+1}{\alpha}\right)$$

$L_1(x|\alpha)e^{-x^\alpha}$



$L_3(x|\alpha)e^{-x^\alpha}$



Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

1st-order Lévy polynomial $\gamma \left[1 + \lambda e^{-R^\alpha Q^\alpha} [1 + c_1 L_1(Q|\alpha, R)] \right]$

3rd-order Lévy polynomial $\gamma \left[1 + \lambda e^{-R^\alpha Q^\alpha} [1 + c_1 L_1(Q|\alpha, R) + c_3 L_3(Q|\alpha, R)] \right]$

MINIMAL MODEL ASSUMPTION: LEVY

experimental conditions:

(i) The correlation function tends to a constant for large values of the relative momentum Q .

(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable x k
- Normalizations :
 - density
 - multiplicity
 - single-particle spectra

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)}$$

$$S(x, k) = f(x) g(k)$$

$$\int dx f(x) = 1, \quad \int dk g(k) = \langle n \rangle,$$

$$N_1(k) = \int dx S(x, k) = g(k).$$

MINIMAL MODEL ASSUMPTION: LEVY/GAUSS

$$\begin{aligned}\psi_{k_1, k_2}(x_1, x_2) \\ = \frac{1}{\sqrt{2}} [\exp(ik_1x_1 + ik_2x_2) + \exp(ik_1x_2 + ik_2x_1)].\end{aligned}$$

$$\begin{aligned}N_2(k_1, k_2) \\ = \int dx_1 dx_2 S(x_1, k_1) S(x_2, k_2) |\psi_{k_1, k_2}(x_1, x_2)|^2,\end{aligned}$$

Model-independent but:

- Assumes plane-wave propagation
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Analyticity assumptions is an extra!
- Resulting correlations Gaussian
- Radius interpreted as variances

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int dx \exp(iq_{12}x) f(x),$$

$$q_{12} = k_1 - k_2.$$

$$\tilde{f}(q) \approx 1 + iq\langle x \rangle - q^2\langle x^2 \rangle/2 + \dots,$$

$$C(q) = 1 + |\tilde{f}(q)|^2 \approx 2 - q^2(\langle x^2 \rangle - \langle x \rangle^2) \approx 1 + \exp(-q^2 R^2),$$

$$R = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}.$$

MINIMAL MODEL ASSUMPTION: LEVY

Model-independent but:

- not assumes analyticity
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Levy: Generalized Central Limit Theorems
- Stable for adding one more small random source
- Correlations non-Gaussian
- Radius not a variance
- $0 < \alpha \leq 2$

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int dx \exp(iq_{12}x) f(x),$$

$$x = \sum_{i=1}^n x_i,$$

$$f(x) = \int \prod_{i=1}^n dx_i \prod_{j=1}^n f_j(x_j) \delta(x - \sum_{k=1}^n x_k).$$

$$\tilde{f}_i(q) = \exp(iq\delta_i - |\gamma_i q|^\alpha), \quad \prod_{i=1}^n \tilde{f}_i(q) = \exp(iq\delta - |\gamma q|^\alpha),$$

$$\gamma^\alpha = \sum_{i=1}^n \gamma_i^\alpha,$$

$$\delta = \sum_{i=1}^n \delta_i.$$

$$\tilde{f}(q) = \prod_{i=1}^n \tilde{f}_i(q)$$

$$C(q; \alpha) = 1 + \lambda \exp(-|qR|^\alpha).$$

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- $\alpha = 2$

- Gaussian source, Gaussian C_2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp \left[-\frac{(x - x_0)^2}{2R^2} \right]$$

$$C(q) = 1 + \exp(-q^2 R^2)$$

- $\alpha = 1$

- Lorentzian source, exponential C_2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x - x_0)^2},$$

$$C(q) = 1 + \exp(-|q R|).$$

- asymmetric Levy:

- asymmetric support
- Stretched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x - x_0)^{3/2}} \exp \left(-\frac{R}{8(x - x_0)} \right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp \left(-\sqrt{|q R|} \right).$$

- Radius not a variance,

- Variance infinite for Lorentz

- infinite for any $\alpha < 2$

MULTIVARIATE LEVY DISTRIBUTIONS

The characteristic function is $f(t) = e^{-t^\alpha}$, where

$$t = \left(\sum_{i,j=1,3} R_{i,j}^2 q_i q_j \right)^{1/2}$$

$$C_2(k_1, k_2) = 1 + \lambda \exp \left[- \left(\sum_{i,j=1}^3 R_{ij}^2 q_i q_j \right)^{\alpha/2} \right]$$

Model-independent but:

- A new parameter alpha generalizes Gauss
- Solved only for symmetric Levy distributions ($R_{i,j}^2 = R_{j,i}^2$)
- Deep open problems in mathematical statistics

MULTIVARIATE LEVY EXPANSIONS

$$L_1(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$L_1(t|\alpha) = \frac{t}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) - \frac{1}{\alpha} \Gamma\left(\frac{2}{\alpha}\right)$$

$$L_2(t|\alpha) = \frac{1}{\alpha} \left[\Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\frac{4}{\alpha}\right) - \frac{1}{\alpha} \Gamma^2\left(\frac{3}{\alpha}\right) \right] - \frac{t}{\alpha} \left[\Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{4}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\frac{3}{\alpha}\right) \right] + \frac{t^2}{\alpha} \left[\Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{3}{\alpha}\right) - \frac{1}{\alpha} \Gamma^2\left(\frac{2}{\alpha}\right) \right]$$

$$\mu_{r,\alpha} = \int_0^\infty dx x^r f(x|\alpha) = \frac{1}{\alpha} \Gamma\left(\frac{r+1}{\alpha}\right)$$

1st-order Levy expansion

$$t = \left(\sum_{i,j=1}^3 R_{i,j}^2 q_i q_j \right)^{1/2}$$

$$C_2(Q) = N \left\{ 1 + \lambda \exp \left(- \left(\sum_{i,j=1}^3 R_{i,j}^2 q_i q_j \right)^{\alpha/2} \right) \left[1 + c_1 \frac{\left(\sum_{i,j=1}^3 R_{i,j}^2 q_i q_j \right)^{1/2}}{\alpha} \left(\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{2}{\alpha}\right) \right) \right] \right\}$$

SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in H to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Levy: Generalized Central Limit Theorems
- Levy expansions $0 < \alpha \leq 2$
- New directions: multivariate Levy expansions