Dispersive Treatment of $K_S \longrightarrow \gamma\gamma$ and $K_S \longrightarrow \gamma l^+ l^-$

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[and with assistance from Bachir Moussallam (Orsay)]

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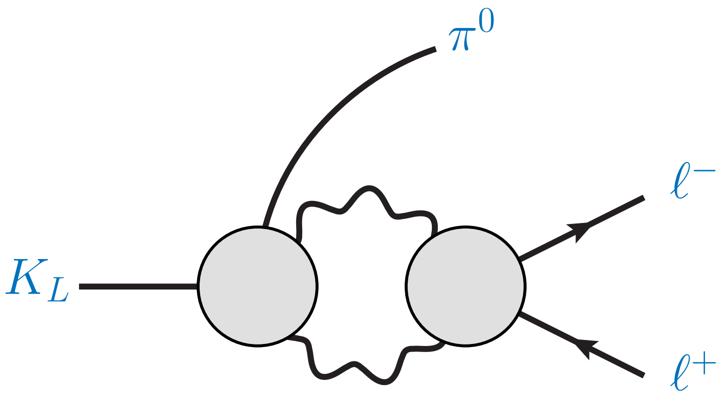
The bad: CP-violating decays like $K_L \to \pi^0\ell^+\ell^-$ where short- and **long-distance (LD)** effects come in equal measure

$$A(K_L \to \pi^0\ell^+\ell^-)\big|_{\rm CPV-ind} = \epsilon A(K_S \to \pi^0\ell^+\ell^-)$$



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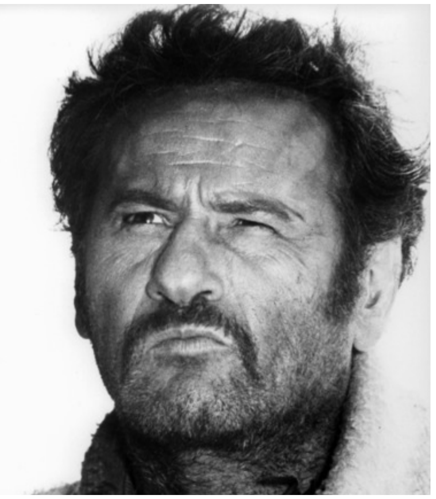

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Dominated by long-distance contributions $\Rightarrow$ require **non-perturbative** methods to determine e.g. $\langle \pi\pi | Q_i | K \rangle$

[See talks by Buras (large N_c) & Sachrajda / Feng / Garron (lattice)]

Necessary to make sense of long-standing puzzles like the **ΔI=1/2 rule** (assumed to be **exact** for purposes of this talk)

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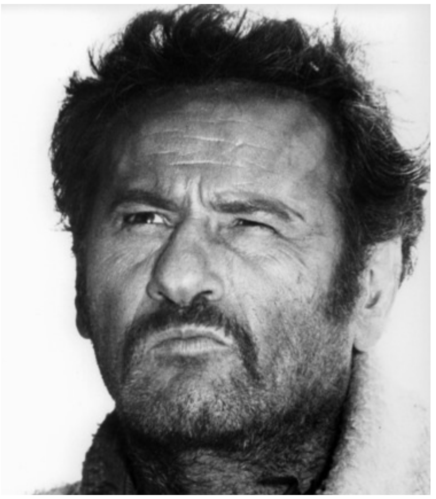

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In between the trio are decays where **separation of SD and LD effects** can be achieved with varying degree of success

A **systematic** analysis is possible within 3-flavour χPT: [Status reviewed by Cirigliano et al. (12)]

$$\mathcal{A} = \{\mathcal{A}_{\rm LO} + \mathcal{A}_{\rm NLO} + \mathcal{A}_{\rm NNLO} + \ldots\}_{\chi{\rm PT}_3}$$

Expansion in powers of $p = O(m_K)$ momentum and $m_{u,d,s} = O(m_K^2)$

1 | Chiral perturbation theory

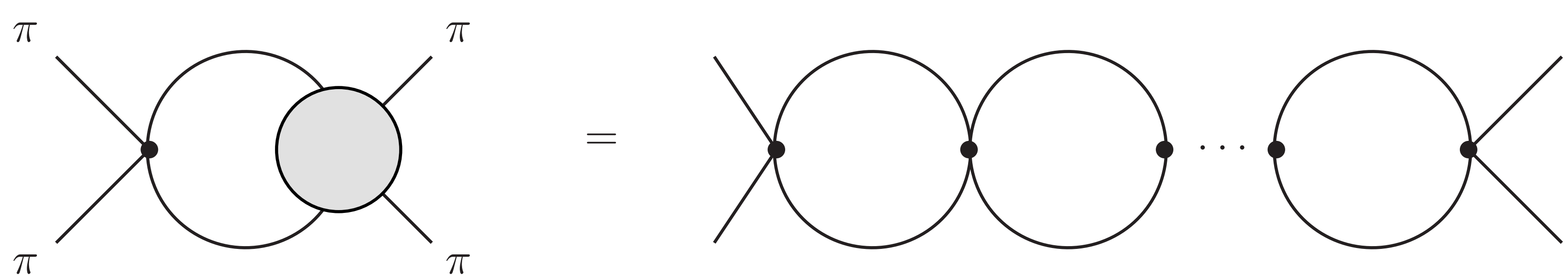
Two features determine the **quality** of predictions arising from $\chi\mathrm{PT}_3$:

1 hadronic uncertainties ⇔ low-energy constants (**LECs**), e.g. F_π
not fixed by chiral symmetry, so need data or lattice to pin down

- leptonic and semi-leptonic kaon decays 😃 [See Peter Stoffer's talk]
- non-leptonic and weak radiative decays 😰

2 chiral expansion **poorly convergent** above $\pi\pi$ threshold [Meißner (91)]

⇒ final-state $\pi\pi$ interactions (**FSI**) important [Truong (84 & 88)]



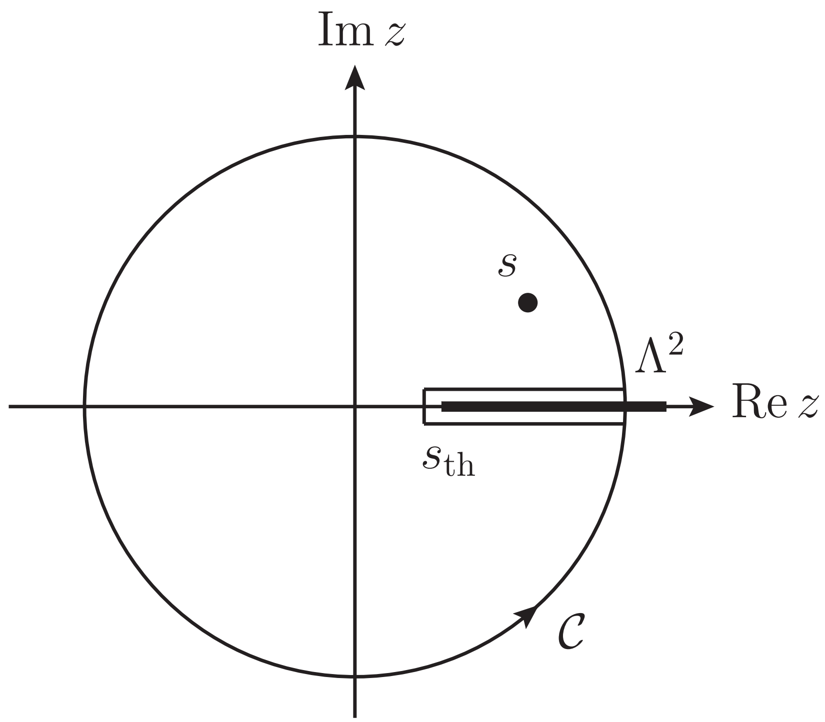
chiral perturbative methods	$\ll$	non-pert. methods based on **unitarity**, **analyticity**, and **crossing symmetry**

2 | Dispersion relations demystified

Dispersion relations address 1 & 2 in **model-independent** framework

How? Consider e.g. some form factor

$$F(z) = \left\{ \begin{array}{c} \text{real } z < s_{\text{th}} \\ \text{branch cut } z > s_{\text{th}} \\ \text{analytic for complex } z \end{array} \right\}$$



Cauchy theorem then gives:

$$F(s) = \frac{1}{2\pi i}\oint_{\mathcal{C}} dz \frac{F(z)}{z-s} = \frac{1}{\pi}\int_{s_{\text{th}}}^{\Lambda^2} dz \frac{\operatorname{Im} F(z)}{z-s-i\epsilon} + \frac{1}{2\pi i}\oint_{|z|=\Lambda^2} dz \frac{F(z)}{z-s}$$

If boundary terms vanishes for $\Lambda \to \infty$ get **unsubtracted** dispersion rel.

$$F(s) = \frac{1}{\pi}\int_{s_{\text{th}}}^{\infty} dz \frac{\operatorname{Im} F(z)}{z-s}$$

$\Rightarrow$ can reconstruct real part if imaginary part known (usually from unitarity)

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

This talk: **dispersive treatment** of $K_S \to \gamma\gamma^*$ transitions

$\Rightarrow$ determine impact of **FSI** on predictions from LO χPT_3

1. for both photons on-shell compare

$$\mathrm{BR}(K_S \to \gamma\gamma)_{\chi\mathrm{PT}_3} = 2.0 \times 10^{-6}$$

[D'Ambrosio & Espriu (86); Goity (87)]

vs.

$$\mathrm{BR}(K_S \to \gamma\gamma)_{\mathrm{expt}} = (2.63 \pm 0.17) \times 10^{-6}$$

2. the chiral predictions for the leptonic modes [Ecker, Pich & de Rafael (88)]

$$\left.\frac{\Gamma(K_S \to \gamma\ell^+\ell^-)}{\Gamma(K_S \to \gamma\gamma)}\right|_{\chi\mathrm{PT}_3} = \begin{cases} 1.6 \times 10^{-2} & (\ell = e) \\ 3.8 \times 10^{-4} & (\ell = \mu) \end{cases}$$

have not yet been tested by experiment but may lie within the projected sensitivity $\mathrm{BR}(K_S) \sim 10^{-9}$ of KLOE-2 (or LHCb?)

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

Problem: kinematics **completely fixed** in two-body decay amplitudes

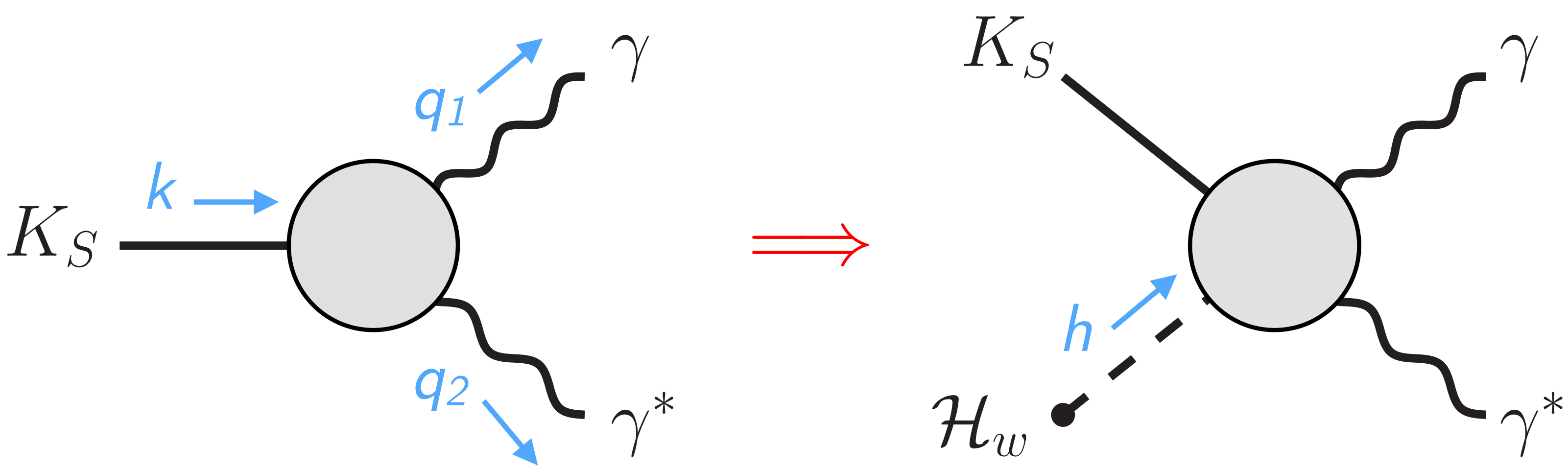
Promote $m_K^2 \to$ kinematic variable "s" and construct dispersion relation?

e.g. $$A(K_S \to \pi\pi)\big|_{\chi\mathrm{PT}_3} = \{\mathrm{LECs}\} \times (s - m_\pi^2)$$

∞ # ways to go **off-shell** $\Rightarrow$ ∞ arbitrariness [Büchler et al. (01)]

Key idea: let weak Hamiltonian $\mathcal{H}_w$ **inject momentum** in $\langle \gamma\gamma^* | \mathcal{H}_w | K_S \rangle$

[Büchler et al. (01)]



$A_{\gamma\gamma^*}(m_K^2, q_2^2)$ $\qquad$ $A_{\gamma\gamma^*}(s, t, u)$

Mandelstam variables

$$s = (q_1 + q_2)^2$$
$$t = (k - q_1)^2$$
$$u = (k - q_2)^2$$

NB. Physical decay amplitude recovered in limit $h \to 0$

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

The cookbook

Several steps & ingredients needed to construct the dispersion relations:

1 tensor decomposition into basis free from kin. **zeros** and **singularities**

$$A_{\mu\nu}(k, q_1, q_2) = g_{\mu\nu} A_1 + \sum_{i,j=1}^{3} q_{i\mu} q_{j\nu} A_2^{ij}$$

e'mag Ward identities + suitable linear combos [Bardeen & Tung (71); Tarrach (75); Colangelo et al. (14 & 15)]

$$\Rightarrow \quad A_{\mu\nu}(k, q_1, q_2) = \sum_{i=1}^{3} T^i_{\mu\nu} B_i(s, t, u, q_2^2)$$

free from kinematic zeros and singularities

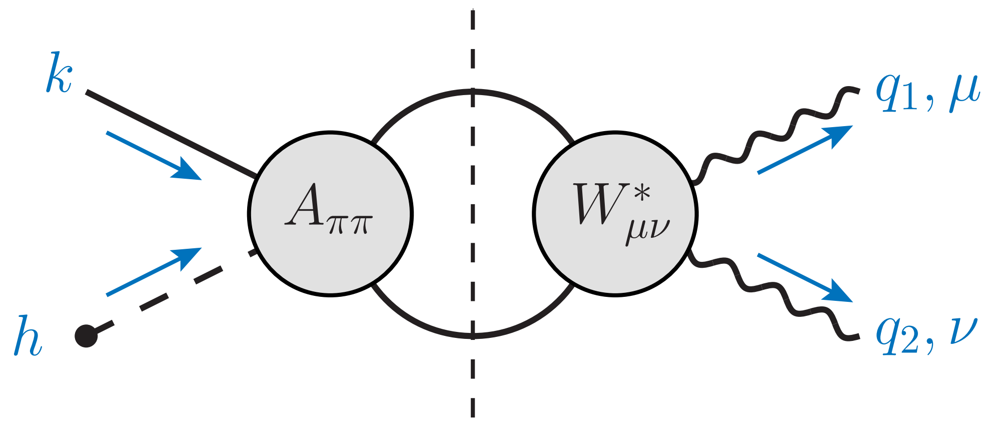
$\Rightarrow$ Determination of scalar functions B_i **completely fixes** prediction for $K_S \to \gamma\gamma^*$

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

A **complete** dispersive treatment of $K_S \to \gamma\gamma^*$ $\Leftrightarrow$ analysis of **all** possible states ππ, 4π, KK,... in **all** three channels s,t,u

This is **hard** $\Rightarrow$ simplify and neglect contributions to discontinuities coming from D-waves and higher

2 first intermediate state due to ππ $\Rightarrow$ unitarity relation



$$\text{disc}_s\, A_{\mu\nu} = \int d\{\text{phase}\} \times A_{\pi\pi} \times W^*_{\mu\nu}$$

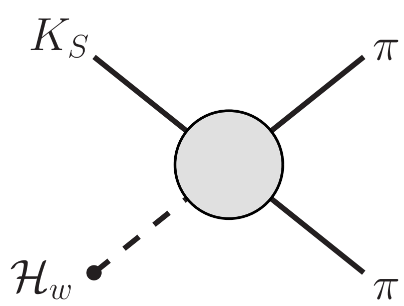
Dominant effect from FSI expected in S-wave $\Rightarrow$ integration is **simple**:

$$\text{disc}_s\, B_1(s, q_2^2) = \{\text{phase space}\} \times A_{\pi\pi}(s) \times \left(\frac{h^0_{++}(s, q_2^2)}{s - q_2^2}\right)^*$$

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

3 need input for **subprocesses** $K_S \to \pi\pi$ and $\gamma\gamma^{(*)} \to \pi\pi$

$K_S \to \pi\pi$ use dispersive representation of Büchler et al. (01)



$$A_{\pi\pi}(s,t',u') = \langle(\pi\pi)_{I=0}|\mathcal{H}_w|K_S\rangle = M_0(s) + C(s,t',u')$$

angular dep.

- **FSI** fully accounted for in terms of Omnès factors such as

$$\Omega_\ell^I(s) = \exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} dz\,\frac{\delta_\ell^I(z)}{z(z-s-i\epsilon)}\right)$$

scattering phase shift

- convergence $\Rightarrow$ two subtraction constants $a_{\pi\pi}$ & $b_{\pi\pi}$ required

not fixed by data or lattice

3 | Dispersive framework for $K_S \to \gamma\gamma^*$

- match to χPT$_3$ at soft-pion point $p_\pi \to 0$ to eliminate $b_{\pi\pi}$:

$$b_{\pi\pi} = \frac{3a_{\pi\pi}(1+X)}{m_K^2 - m_\pi^2(4+3X)} + O(m_K^4)$$

parametrises effects from O(p^6): X=±0.3

$$\Rightarrow \qquad A_{\pi\pi}(s) \simeq a_{\pi\pi}\left[1 + E(X)s/m_K^2\right]\Omega_0^0(s)$$

fix by matching to physical K→ππ amp

$\gamma\gamma^{(*)} \to \pi\pi$ for helicity PW use data from two dispersive analyses

- $h^0_{++}(s)$ coupled-channel $\left\{ \begin{array}{c} \gamma\gamma \to \pi\pi \\ \gamma\gamma \to KK \end{array} \right\}$ [Garcia-Martin & Moussallam (10)]
- $h^0_{++}(s, q_2^2)$ single-channel [Moussallam (13)]

4 | Dispersion relations for $K_S \to \gamma\gamma$

Putting everything together and defining $A_{\gamma\gamma}(s) \equiv e^2 B_1(s)$ gives **once-subtracted** dispersion relation:

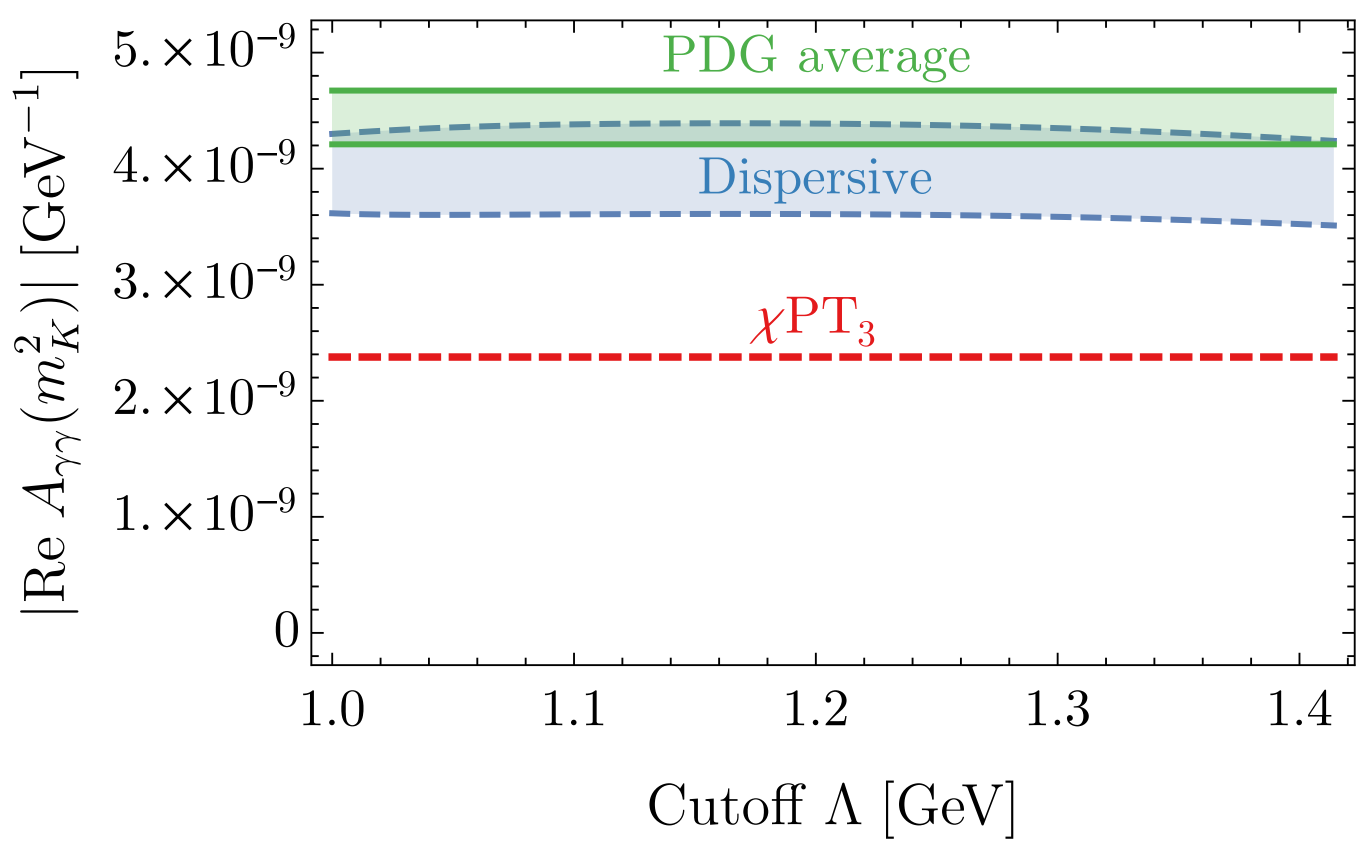
$$A_{\gamma\gamma}(s) = a_{\gamma\gamma} + \frac{s - s_0}{\pi} \int_{4m_\pi^2}^{\infty} dz \frac{\text{Im}_s\, A_{\gamma\gamma}(z)}{(z - s_0)(z - s - i\epsilon)}$$

fix by matching to χPT_3 at chiral zero s_0=-0.098 GeV^2

Cutoff dependence?

Range of validity on $h^0_{++}(s)$ for $s \lesssim 2\ \text{GeV}^2 \Rightarrow$ UV cutoff

Dependence is **very mild** so take Λ=1.2 GeV as benchmark



4 | Dispersion relations for $K_S \to \gamma\gamma$

Results

At physical point $s = m_K^2$ the effects from FSI **distort** the amplitude

$\mathrm{Re}\, A_{\gamma\gamma}$ **enhanced**

$\mathrm{Im}\, A_{\gamma\gamma}$ **suppressed**

[confirms obs. of Kambor & Holstein (94)]

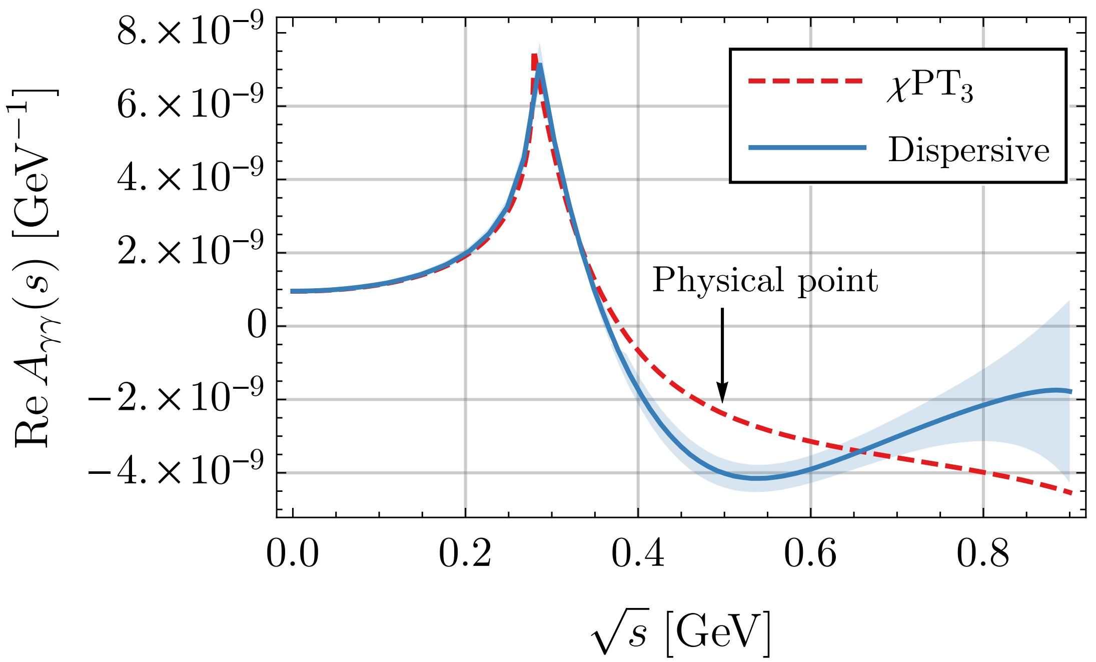
⇒ **enhanced** prediction for rate:

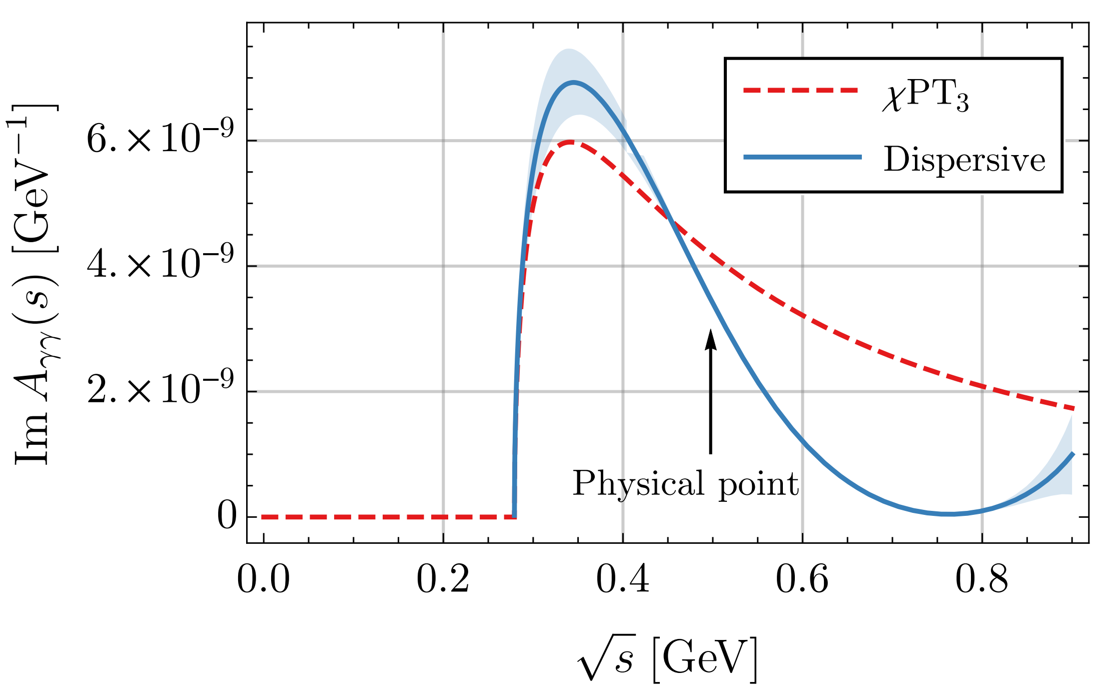
$$\mathrm{BR}^{\mathrm{disp}}_{\gamma\gamma} = (2.34 \pm 0.26) \times 10^{-6}$$

uncertainty from X=±0.3 & Omnès input

⇒ SM in much better agreement with experiment:

$$\mathrm{BR}^{\mathrm{expt}}_{\gamma\gamma} = (2.63 \pm 0.17) \times 10^{-6}$$





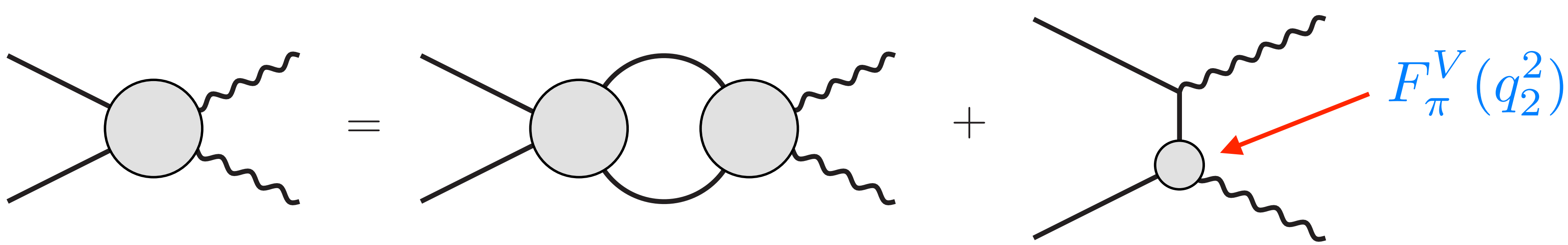
5 | Dispersion relations for $K_S \to \gamma\ell^+\ell^-$

Now allow one γ to be off-shell. Define $A_{\gamma\gamma^*}(s, q_2^2) \equiv e^2 B_1(s, q_2^2)$ and consider once-subtracted dispersion relation at $s_0=0$:

$$A_{\gamma\gamma^*}(s, q_2^2) = a_{\gamma\gamma^*}(q_2^2) + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} dz\, \frac{\mathrm{disc}_s\, A_{\gamma\gamma^*}(z, q_2^2)}{z(z-s-i\epsilon)}$$

fix by matching to χPT_3 at $s_0=0$

New feature: in addition to FSI get effects from pion vector form factor

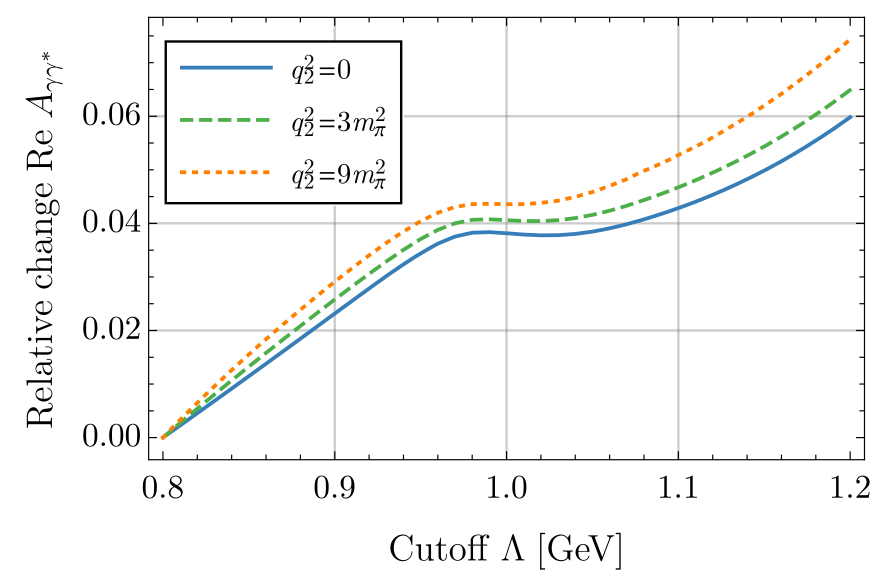


Cutoff dependence?

Comparison of $h^0_{++}(s)$ and $h^0_{++}(s, q_2^2 = 0)$

$\Rightarrow$ range of validity $s \lesssim 0.8$ GeV^2

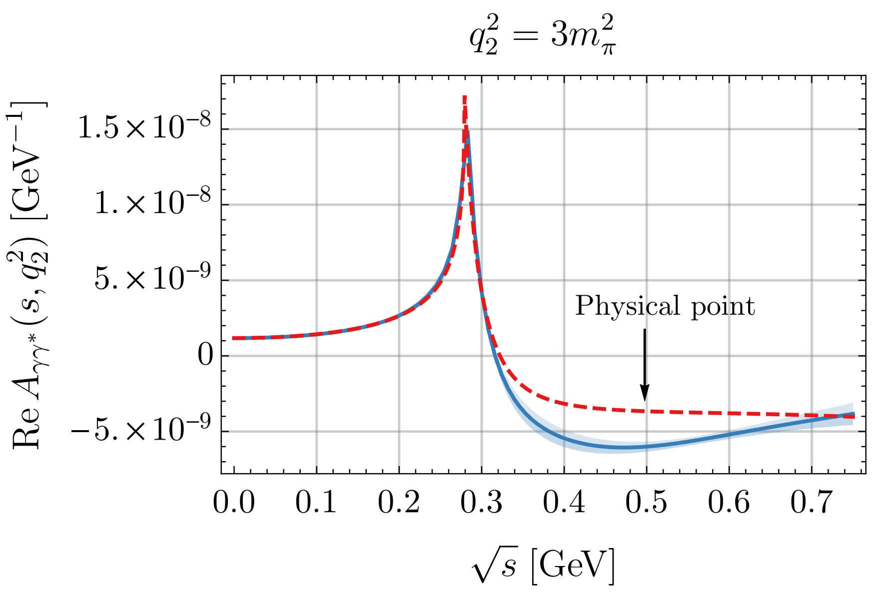
Taking Λ=1.2 GeV only leads to ≈ 7% shift

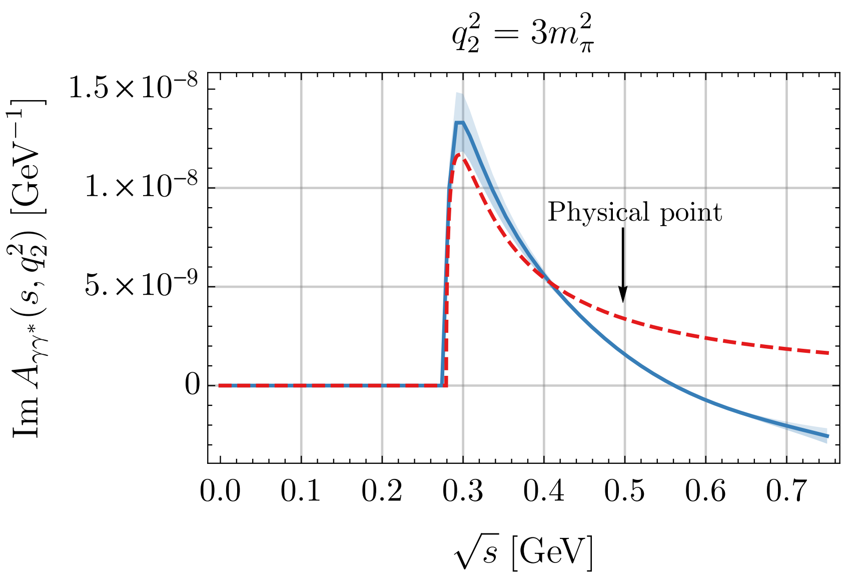


5 | Dispersion relations for $K_S \to \gamma\ell^+\ell^-$

Results

Consider **energy dependence** for fixed values of γ momentum





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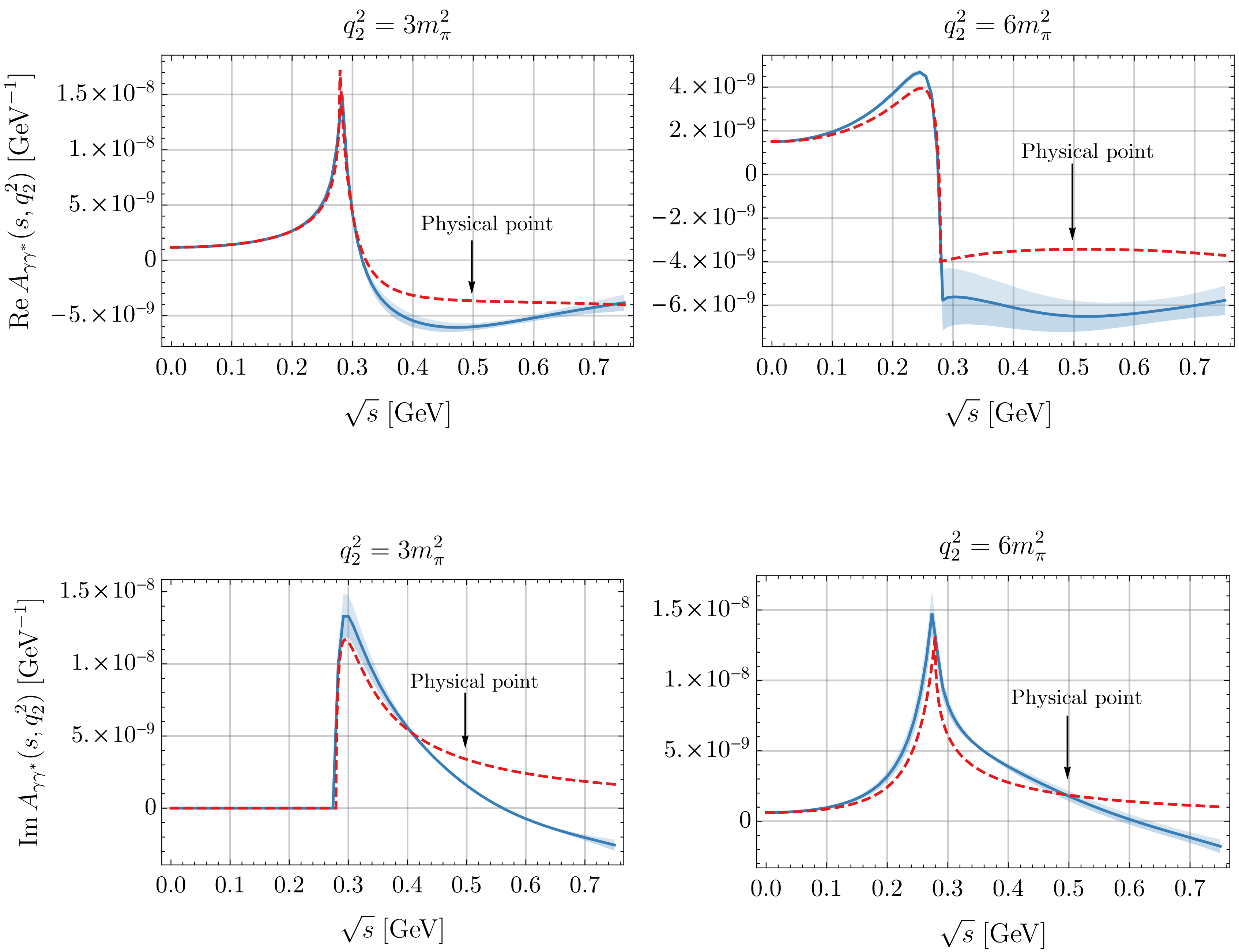

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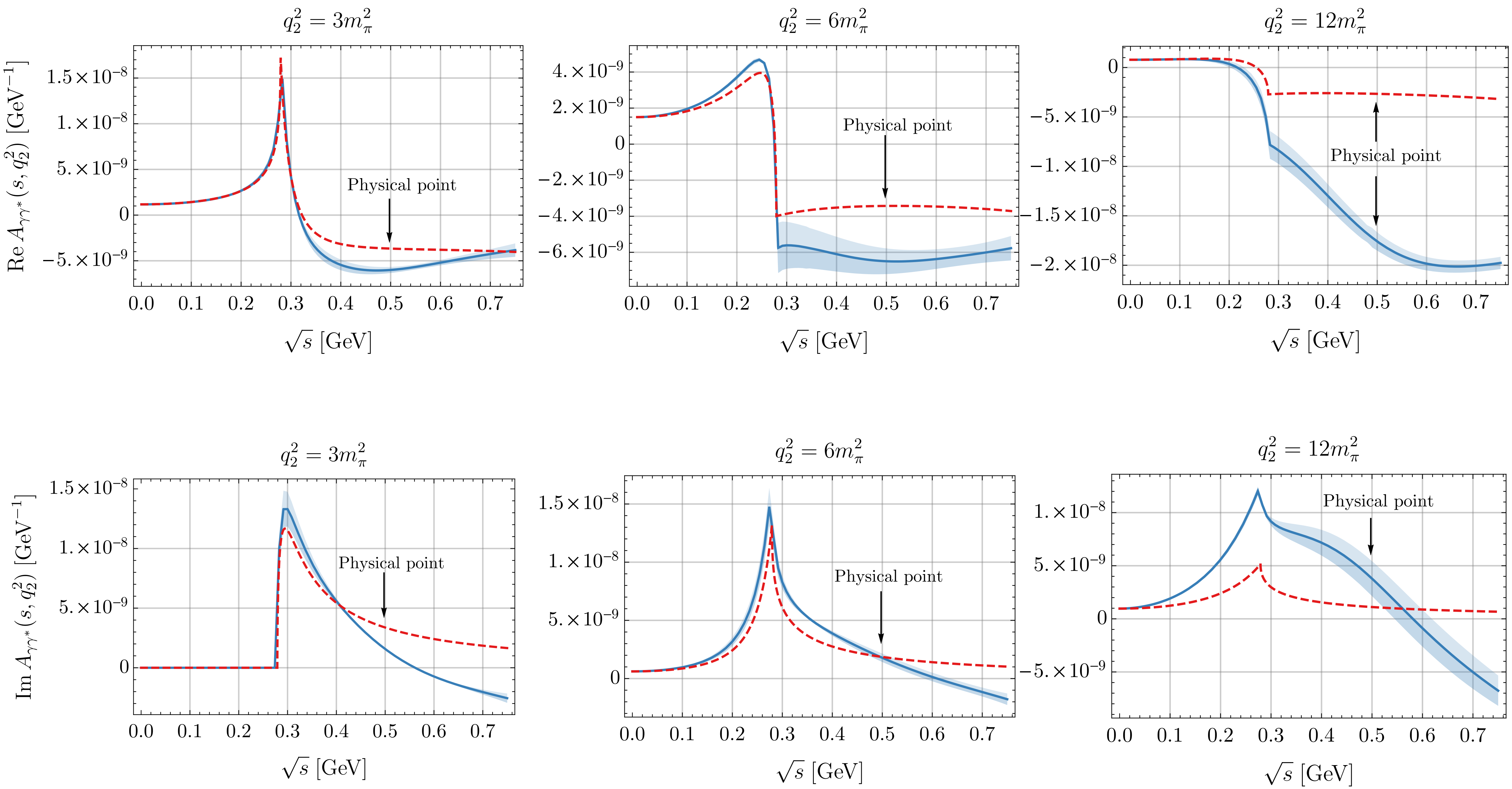
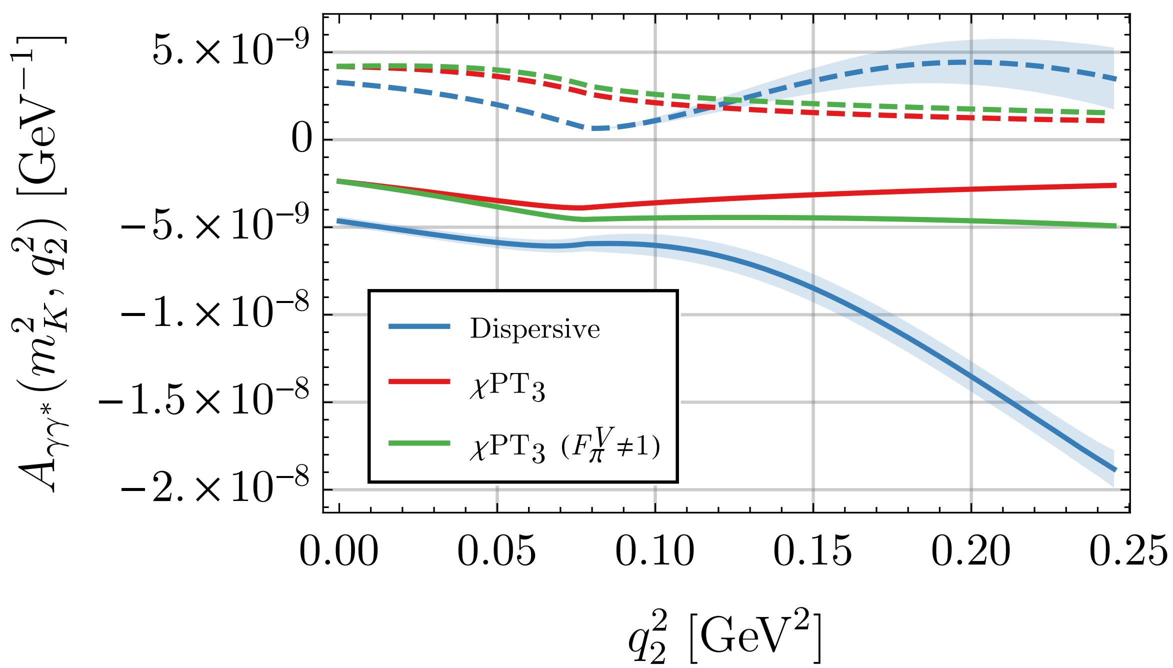
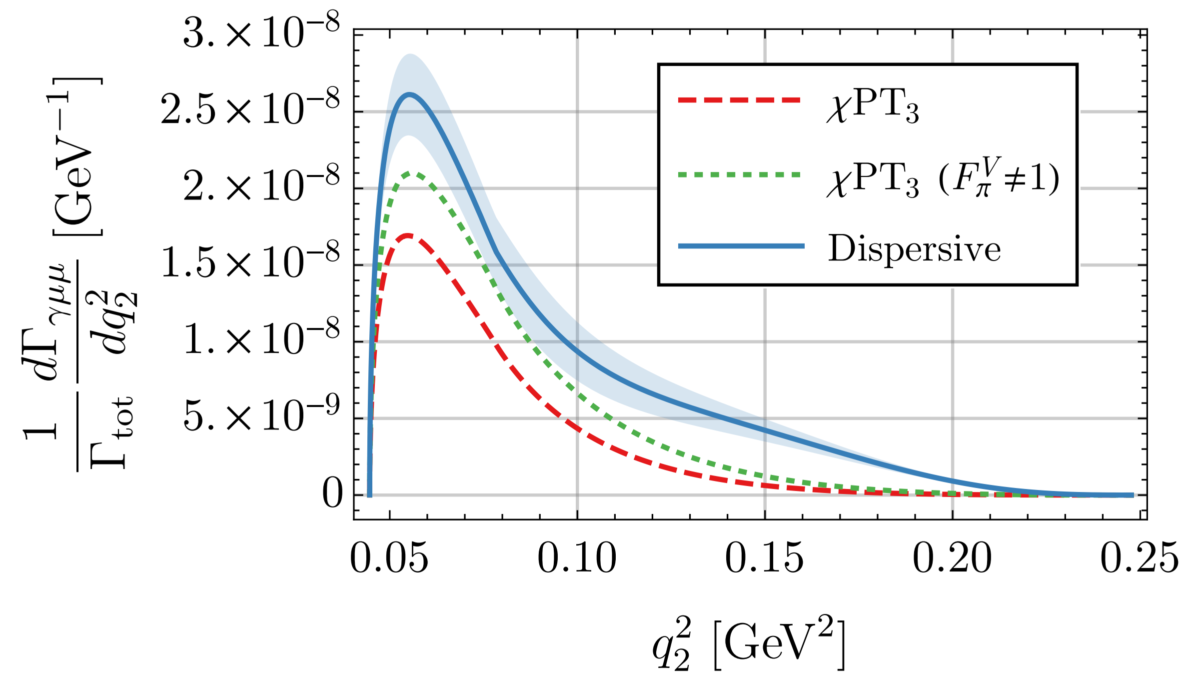


5 | Dispersion relations for $K_S \to \gamma\ell^+\ell^-$

Results

Now fix $s = m_K^2$ and vary γ momentum: FF effects **large** for $q_2^2 > 4m_\pi^2$





Corrections from FSI and FF ⇒ **sizeable enhancements** in the rates

Input	BR($K_S \to \gamma e^+e^-$)	BR($K_S \to \gamma\mu^+\mu^-$)
χPT$_3$	3.09×10^{-8}	7.25×10^{-10}
χPT$_3$ ($F_\pi^V \neq 1$)	3.17×10^{-8}	9.97×10^{-10}
This work	$(4.38 \pm 0.33) \times 10^{-8}$	$(1.45 \pm 0.21) \times 10^{-9}$
	$O(50\%)$	$O(100\%)$

6 | Summary and future prospects

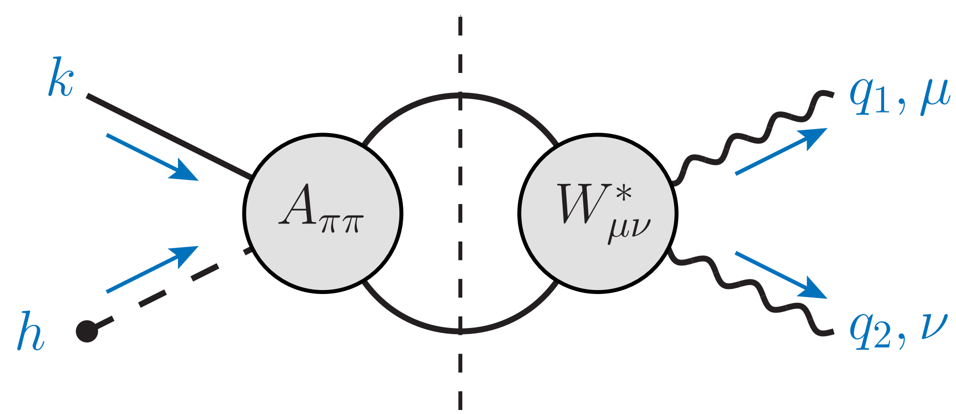
Dispersion relations offer a **complementary** approach to χPT and $\ell_{\rm QCD}$

$$\left\{ \begin{array}{c} \text{unitarity} \\ + \\ \text{analyticity} \end{array} \right\} \Rightarrow$$

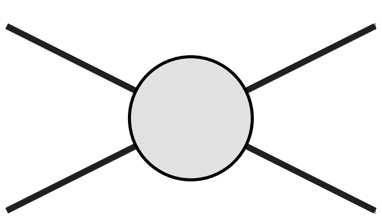
much better control over effects due to **ππ rescattering** in final state (**FSI**)

For two-body decays, **off-shell** extrapolations in m_K^2 are **ambiguous**

$\Rightarrow$ let $\mathcal{H}_w$ carry momentum and analyse **on-shell** amplitudes



$K_S \to \gamma\gamma$

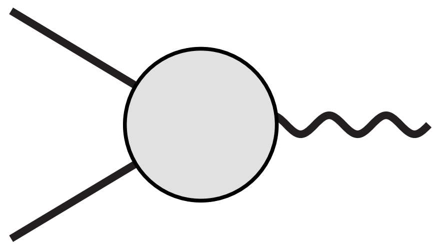


- FSI **significantly** distorts the amplitude $\mathrm{Re}\, A_{\gamma\gamma} \Leftrightarrow \mathrm{Im}\, A_{\gamma\gamma}$
- agreement between SM and experiment is **improved**

$\mathrm{BR}^{\rm disp}_{\gamma\gamma} = (2.34 \pm 0.26) \times 10^{-6} \qquad \mathrm{BR}^{\rm exp}_{\gamma\gamma}(2.63 \pm 0.17) \times 10^{-6}$

6 | Summary and future prospects

$K_S \to \gamma \ell^+ \ell^-$



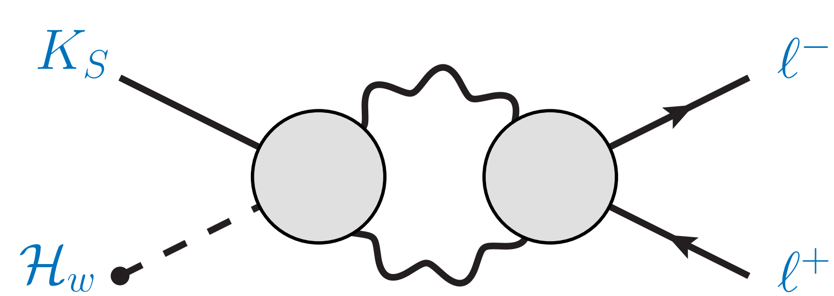
- pion vector form factor $\Rightarrow$ additional source of enhancement over LO $\chi\mathrm{PT}_3$

Input	$\mathrm{BR}(K_S \to \gamma e^+ e^-)$	$\mathrm{BR}(K_S \to \gamma \mu^+ \mu^-)$
$\chi\mathrm{PT}_3$	3.09×10^{-8}	7.25×10^{-10}
$\chi\mathrm{PT}_3$ ($F_\pi^V \neq 1$)	3.17×10^{-8}	9.97×10^{-10}
This work	$(4.38 \pm 0.33) \times 10^{-8}$	$(1.45 \pm 0.21) \times 10^{-9}$

- effect **largest** in μμ mode … within reach of KLOE-2?

In progress: extend dispersive framework to $K_S \to \gamma^* \gamma^*$

- dominant **long-distance** contribution to $K_S \to \ell^+ \ell^-$
- can we expect large corrections to $\chi\mathrm{PT}_3$?



$$\mathrm{BR}^{\chi\mathrm{PT}_3}_{\mu^+\mu^-} = 5.1 \times 10^{-12} \quad \textbf{vs.} \quad \mathrm{BR}^{\mathrm{LHCb}}_{\mu^+\mu^-} < 6.9(5.8) \times 10^{-9}$$

[See talk by Ramos Pernas]

- disentangle New Physics at $\mathrm{BR}^{\mathrm{NP}}_{\mu\mu} \gtrsim 10^{-11}$? [Isidori & Unterdorfer (03)]

Back up slides

B1 | What happened to the weak mass term?

In principle, chiral and CPS symmetry permits an **octet operator** Q_{mw} to be present in the effective theory; e.g. at $O(p^2)$ one has [Bernard et al. (85)]

$$\mathcal{L}_{\text{weak}}^{\chi\text{PT}_3} \supset \text{Tr}\lambda_{6-i7}(g_M M U^\dagger + \bar{g}_M U M^\dagger)$$

Tadpole cancellation $\Rightarrow$ Q_{mw} **completely removed** by chiral rotation

$$U \to \tilde{U} = RUL^\dagger\,, \qquad \langle\tilde{U}\rangle_{\text{vac}} = I$$

[Crewther (86)]

- vacuum alignment can be extended to $O(p^4)$ [Kambor et al. (90)]
- remains valid when $\mathcal{H}_w$ carries momentum (chiral symmetry **local**)

Conclude that Q_{mw} has **no effect** on chiral low-energy theorems, esp.

$$b_{\pi\pi} = \frac{3a_{\pi\pi}(1+X)}{m_K^2 - m_\pi^2(4+3X)} + O(m_K^4)$$

B2 | Omnès factors and inelasticities

Phases of Ω_0^0 and h_{++}^0 have to match in order for $\mathrm{Im}\, A_{\gamma\gamma} \in \mathbb{R}$

True in elastic region (Watson thm) but how does phase behave at $s > 4m_K^2$?

Define phase with **"dip"** behaviour:

$$\phi_0^0(s) = \begin{cases} \delta_0^0(s)\,, & s \leq s_\pi \\ \delta_0^0(s) - \pi\,, & s > s_\pi \end{cases}$$

Comparison against **"non-dip"** phase

$$\psi_0^0(s) = \arg h_{0,++}^0(s)$$

then estimates systematic uncertainty