Dispersive Treatment of $K_{\ell 4}$ Decays

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Definition of the $K_{\ell 4}$ decay

Decay of a kaon into two pions and a lepton pair:

$$K^+(p) \to \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu)$$

$\ell \in \{e, \mu\}$ is either an electron or a muon.

(Other modes involving neutral pions can be related by isospin symmetry.)

Importance of the $K_{\ell 4}$ decay

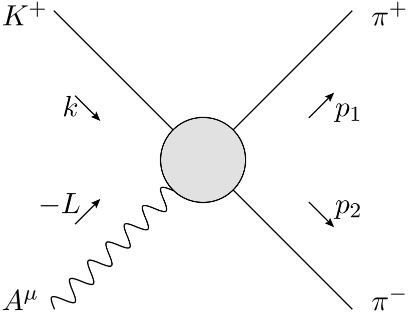
- provides information on $\pi\pi$-scattering lengths a_0^0, a_0^2
- K_{e4} very precisely measured $\Rightarrow$ test of χPT

  $\rightarrow$ Geneva-Saclay, E865, NA48/2
- best source of information on the χPT low-energy constants L_1^r, L_2^r and L_3^r
- happens at very low energy, where χPT is expected to converge best

Advantages of dispersion relations

- based on analyticity and unitarity $\Rightarrow$ model independence
- resummation of rescattering
- connect different energy regions

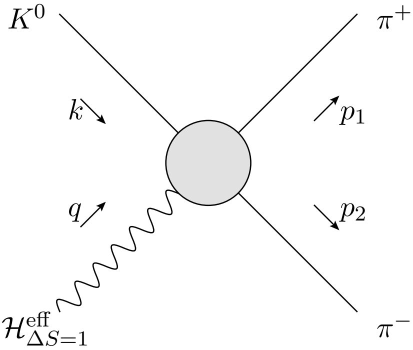
Hadronic part of $K_{\ell 4}$ as $2 \to 2$ scattering



Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (k - p_1)^2, \quad u = (k - p_2)^2$$

Similar to $K \to 2\pi$



$\to$ Büchler, Colangelo, Kambor, Orellana (2001)

Application in rare K_S decays: see Lewis' talk on Friday

$\to$ Colangelo, Stucki, Tunstall (2016)

Form factors

- Lorentz structure allows four form factors in the hadronic matrix element ($P = p_1 + p_2$, $Q = p_1 - p_2$):

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|A_\mu(0)\middle|K^+(k)\right\rangle = -i\frac{1}{M_K}\left(P_\mu \boldsymbol{F} + Q_\mu \boldsymbol{G} + L_\mu \boldsymbol{R}\right)$$

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|V_\mu(0)\middle|K^+(k)\right\rangle = -\frac{\boldsymbol{H}}{M_K^3}\epsilon_{\mu\nu\rho\sigma}L^\nu P^\rho Q^\sigma$$

- contribution of R invisible in the electron mode
- H chirally suppressed
- concentrate here on F and G
- form factors are functions of the Mandelstam variables s, t and u

Analytic properties

- $F(s,t,u)$ and $G(s,t,u)$ have a right-hand branch cut in the complex s-plane, starting at the $\pi\pi$-threshold
- left-hand cut present due to crossing
- analogous situation in t- and u-channel

Reconstruction theorem

→ Stern, Sazdjian, Fuchs (1993), Ananthanarayan, Buettiker (2001), . . .

- define a function that has just the right-hand cut of f_0, the first partial wave of F:

$$M_0(s) := P(s) + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\operatorname{Im} f_0(s')}{(s' - s - i\epsilon){s'}^2} ds'$$

- similar functions take care of the right-hand cuts of all the other S- and P-waves (also crossed channels)
- all the discontinuities are split up into functions of a single variable
- neglect imaginary parts of D- and higher waves

Reconstruction theorem

Form factors decomposed into functions of one Mandelstam variable only:

$$F(s,t,u) = M_0(s) + \frac{u-t}{M_K^2} M_1(s) + (\text{functions of } t \text{ or } u),$$

$$G(s,t,u) = \tilde{M}_1(s) + (\text{functions of } t \text{ or } u).$$

Omnès representation

Function M_0 contains only right-hand cut of the partial wave f_0: difference is the 'inhomogeneity' $\hat{M}_0$:

$$f_0(s) = M_0(s) + \hat{M}_0(s)$$

Inhomogeneous Omnès problem:

$$\mathrm{Im} M_0(s) = (M_0(s) + \hat{M}_0(s)) e^{-i\delta_0^0(s)} \sin \delta_0^0(s)$$

Watson's theorem: δ_0^0 is elastic $\pi\pi$ phase shift

Omnès representation

Omnès function takes care of rescattering:

$$\Omega_l^I(s) := \exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta_l^I(s')}{(s'-s-i\epsilon)s'}ds'\right\}$$

δ_l^I: elastic $\pi\pi$ or $K\pi$ phase shifts

Write dispersion relation for $\frac{M_0(s)}{\Omega_0^0(s)}$

Omnès representation

Omnès solution for the functions $M_0(s)$, $M_1(s)$, $\tilde{M}_1(s)$, etc.:

$$M_0(s) = \Omega_0^0(s) \left\{ P(s) + \frac{s^3}{\pi} \int_{4M_\pi^2}^{\Lambda^2} \frac{\hat{M}_0(s') \sin \delta_0^0(s')}{|\Omega_0^0(s')|(s' - s - i\epsilon) {s'}^3} ds' \right\},$$

P: subtraction polynomial

$\hat{M}_i$: inhomogeneities, angular averages of all the functions M_i

Intermediate summary

- problem parametrised by 9 subtraction constants
- input: elastic $\pi\pi$- and $K\pi$-scattering phase shifts
- energy dependence fully determined by the dispersion relation

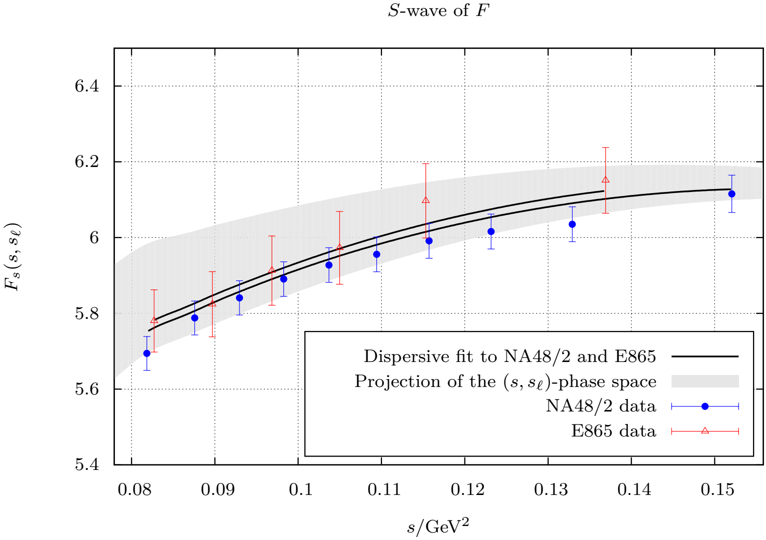
Intermediate summary

- set of coupled integral equations:
  $\Rightarrow M_0(s)$, $M_1(s)$, $\ldots$: DR involving $\hat{M}_0(s)$, $\hat{M}_1(s)$, $\ldots$
  $\Rightarrow \hat{M}_0(s)$, $\hat{M}_1(s)$, $\ldots$: angular integrals over $M_0(s)$, $M_1(s)$, $\ldots$
- system solved by iteration
- problem linear in the subtraction constants
  $\Rightarrow$ construct 9 basic solutions

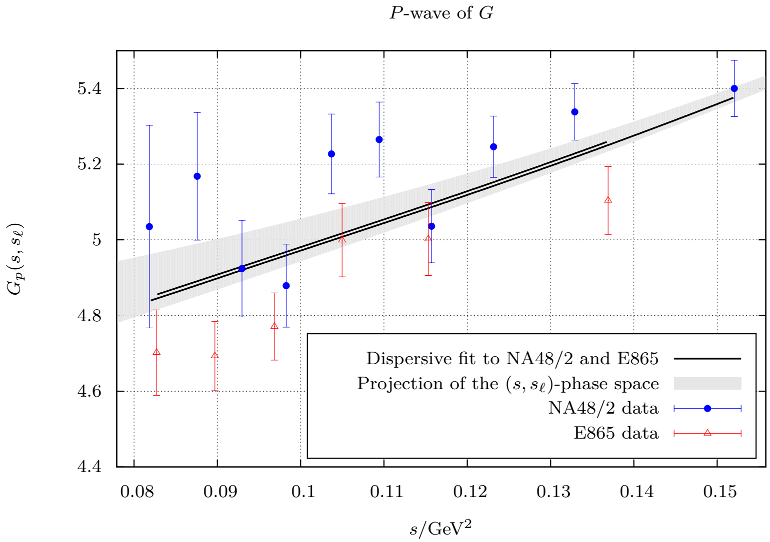
Determination of the subtraction constants

- fit to data of the high-statistics experiments E865 and NA48/2
- soft-pion theorems as additional constraints
- chiral input for the subtraction constants that are not well determined by data

Fit results for partial waves



Fit results for partial waves



Matching to χPT

- matching to χPT at the level of subtraction constants in Omnès form: separate rescattering effects
- fit to 2-dimensional data set of NA48/2
- L_9^r can be determined from dependence on s_ℓ

Matching at NNLO

- many poorly known LECs C_i^r at NNLO
- include additional constraints in the fit: require good chiral convergence
- input: C_i^r contribution to subtraction constants with $\pm 50\%$ uncertainty
- fit the C_i^r contribution
- not all sets of C_i^r input lead to a good chiral convergence: prefer BE14 $\rightarrow$ Bijnens, Ecker (2014)

Low-energy constants

Results for the LECs using χPT at NLO and NNLO.

	NLO	NNLO	Bijnens, Ecker (2014)
$10^3 \cdot L_1^r$	$0.51(2)(6)$	$0.69(16)(8)$	$0.53(6)$
$10^3 \cdot L_2^r$	$0.89(5)(7)$	$0.63(9)(10)$	$0.81(4)$
$10^3 \cdot L_3^r$	$-2.82(10)(7)$	$-2.63(39)(24)$	$-3.07(20)$
χ^2/dof	$141/116 = 1.2$	$124/122 = 1.0$	

Overview

What could be done in NA62?

- $K_{\mu 4}$ (i.e. $K^+ \to \pi^+\pi^-\mu^+\nu_\mu$) may come through trigger
- K_{e4} does not, but is background for $K \to \pi\nu\bar{\nu}$
  $\to$ maybe a special run is planned?

Electron mode K_{e4}

What could be done with higher statistics?

- s_ℓ-dependence of F and G can be used to extract L_9^r
- determination of L_1^r, L_2^r, L_3^r with even higher precision
- (better) determination of linear combinations of C_i^r
- include 1-loop radiative corrections for $K_{e4}(\gamma)$ in PHOTOS Monte Carlo $\rightarrow$ EPJC **74** (2014) 2749

Muon mode $K_{\mu 4}$

- larger values of s_ℓ
- form factor R is accessible
- s-dependence of R contains L_4^r, L_5^r and L_9^r
- information on $K\pi$ scattering

Summary

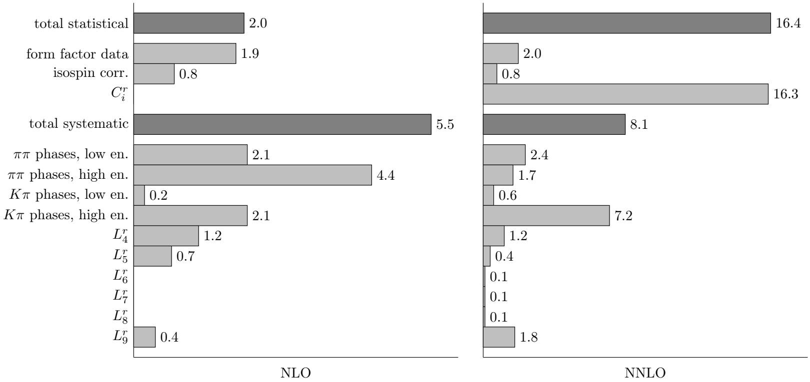
- parametrisation valid up to and including $\mathcal{O}(p^6)$
- model independence
- resummation of rescattering effects
- very precise data available
- determination of LECs from matching to χPT

Summary

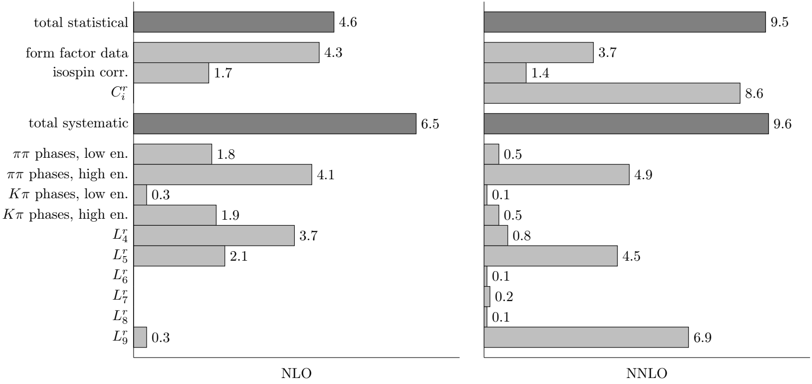
- even higher statistics could be useful for better determination of L_i^r and combinations of C_i^r
- better data on s_ℓ-dependence would enable independent determination of L_9^r
- radiative corrections should be included in Monte Carlo
- new form factor and further LECs accessible in $K_{\mu 4}$

Backup

Error budget: L_1^r



Error budget: L_2^r



Error budget: L_3^r