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Pendulum Equations and Low Gain Regime

CERN Accelerator School – FELs and ERLs

Interaction with Radiation Field in Helical Undulator

- Superimpose a circular co-propagating plane wave with frequency ω and wavenumber *k:*

$$\vec{E}(z,t)=\vec{e}_x E_0 \cos\left(kz-\omega t+\phi\right)\pm\vec{e}_y E_0 \sin\left(kz-\omega t+\phi\right)$$

For now we are leaving the polarization undefined

- To change the Energy *T*, the particle has to move with or against the field lines.

$$\frac{dT}{dt}=e\vec{v}\cdot\vec{E} \quad\Rightarrow\quad \frac{d\gamma}{dz}=\frac{e}{\bar{\beta}_z mc^2}\vec{\beta}\cdot\vec{E}$$

$$\frac{d\gamma}{dz}=\frac{eE_0K}{\bar{\beta}_z\gamma mc^2}\left[\cos(k_u z)\cos(kz-\omega t+\phi)\pm\sin(k_u z)\sin(kz-\omega t+\phi)\right]$$

$$=\frac{eE_0K}{\bar{\beta}_z\gamma mc^2}\cos\left(\left[k\mp k_u\right]z-\omega t+\phi\right)$$

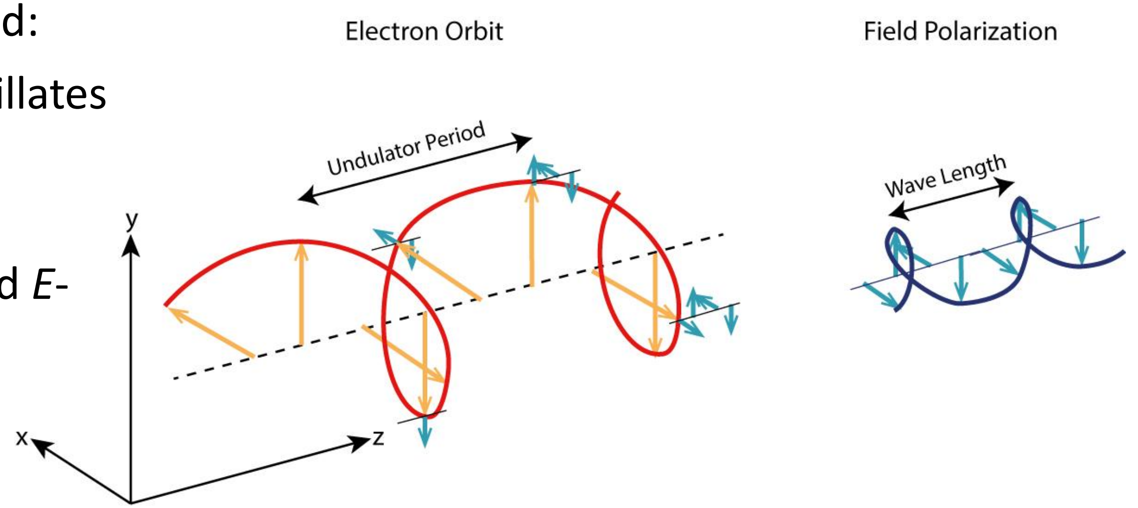
So far the wave number k has been undefined. Is there a preferred wave number to maximize the energy change?

The Resonance Condition

- Energy change is resonant if the phase of the cosine function does not change:

$$\frac{d\gamma}{dz} = \frac{eE_0K}{\bar{\beta}_z\gamma mc^2}\cos\left(\left[k \mp k_u\right]z - \omega t + \phi\right) \quad \Rightarrow \quad \frac{d}{dt}\left(\left[k \mp k_u\right]z - \omega t + \phi\right) = c\left(\left[k \mp k_u\right]\bar{\beta}_z - k\right) = 0$$

- Because the normalized velocity is always smaller than unity only the case with the plus-sign can be fulfilled:
  - The minus-sign term oscillates as cos($k_u z$) or faster and averages out to zero
  - The polarization of *B*- and *E*-field are opposite



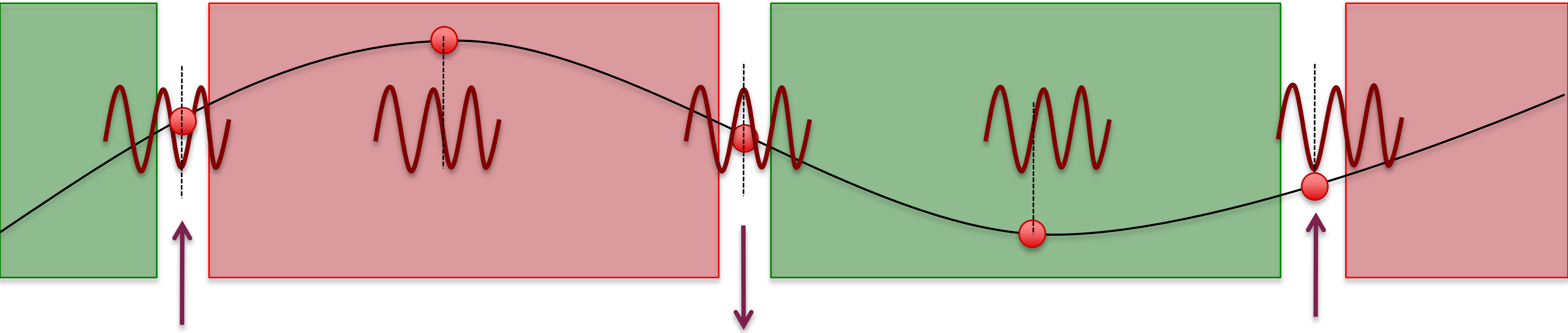
- The resonance condition is:

$$\frac{k}{k+k_u} = \bar{\beta}_z = 1 - \frac{1+K^2}{2\gamma^2} \approx 1 - \frac{k_u}{k}$$

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+K^2\right)}$$

The Resonance Condition

- The transverse oscillation allows to couple with a co-propagating field
- The electron moves either with or against the field line, depending on the radiation phase and the injection time of the electron
- After half undulator period the radiation field has slipped half wavelength. Both, velocity and field, have changed sign and the direction of energy transfer remains.



The net energy change can be accumulated over many period.

Ponderomotive Phase

- Because the phase in the cosine function is considered a “slowly” changing parameter, we define it as a new variable, the ponderomotive phase:

$$\boxed{\theta = \left(k + k_u\right) z - \omega t}$$

- The resonant energy change becomes:

$$\frac{d\gamma}{dz} = \frac{eE_0 K}{\bar{\beta}_z \gamma m c^2} \cos\left(\theta + \phi\right)$$

- The resonance condition links an electron energy for a given wavenumber

$$\boxed{\gamma_r = \sqrt{\frac{\lambda_u}{2\lambda}\left(1 + K^2\right)}}$$

Because we must allow the case that an electron might not be in resonance, we have energy and resonant energy as two independent parameters

- For small deviation from the resonant energy the ponderomotive phase changes slowly with

$$\frac{d}{dz}\theta = k + k_u - \frac{k}{\bar{\beta}_Z} \approx k_u - k\frac{1+K^2}{2\left(\gamma_r + \Delta\gamma\right)^2} = k_u - \frac{k_u}{\left(1 + \frac{\Delta\gamma}{\gamma_r}\right)^2} \approx 2k_u \frac{\Delta\gamma}{\gamma_r}$$

The Pendulum Equations

- The particle motion in a helical undulator with presence of an external field is:

$$\frac{d}{dz}\theta = 2k_u \frac{\Delta\gamma}{\gamma_r} \qquad \frac{d}{dz}\gamma = \frac{eE_0K}{\bar{\beta}_z\gamma mc^2}\cos(\theta+\phi)$$

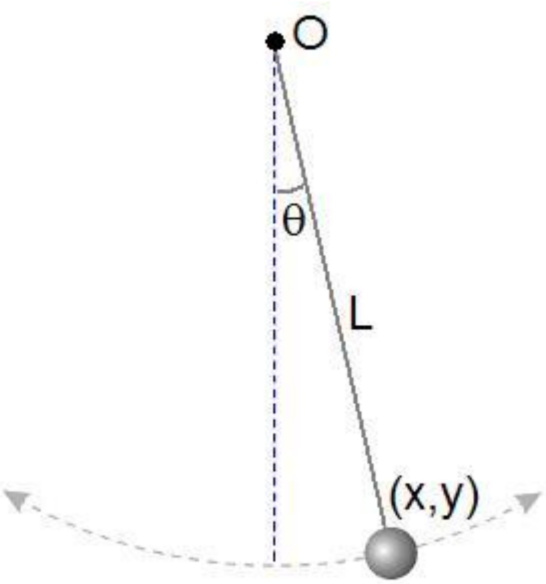
- Transform the equations in a more handy-form:
  - Assume only small deviation from the resonant energy: $\gamma \approx \gamma_r$
  - Use new variables:

$$\Gamma = 2k_u \frac{\Delta\gamma}{\gamma_r} \qquad \Theta = \theta + \phi - \pi / 2$$

$$\frac{d}{dz}\Theta = \Gamma \qquad \frac{d}{dz}\Gamma = -2k_u \frac{eE_0K}{\bar{\beta}_z\gamma_r^2 mc^2}\sin\Theta$$

Classical Pendulum equations

$$\frac{d}{dt}\theta = I \qquad \frac{d}{dt}I = -\frac{g}{L}\sin\theta$$



Electron Trajectories

- There are two fix points:
  - Stable: $\Gamma=0, \Theta=0$
  - Unstable: $\Gamma=0, \Theta=\pi$

$$\frac{d}{dz}\Theta = \Gamma \qquad \frac{d}{dz}\Gamma = -\Omega_s^2 \sin\Theta$$

- For small amplitude around stable point the motion is a simple oscillation with the frequency

$$\Omega_s = \sqrt{2k_u \frac{eE_0K}{\overline{\beta}_z \gamma_r^2 mc^2}}$$

Subscript "s" indicates the similarity to synchrotron oscillation in storage rings

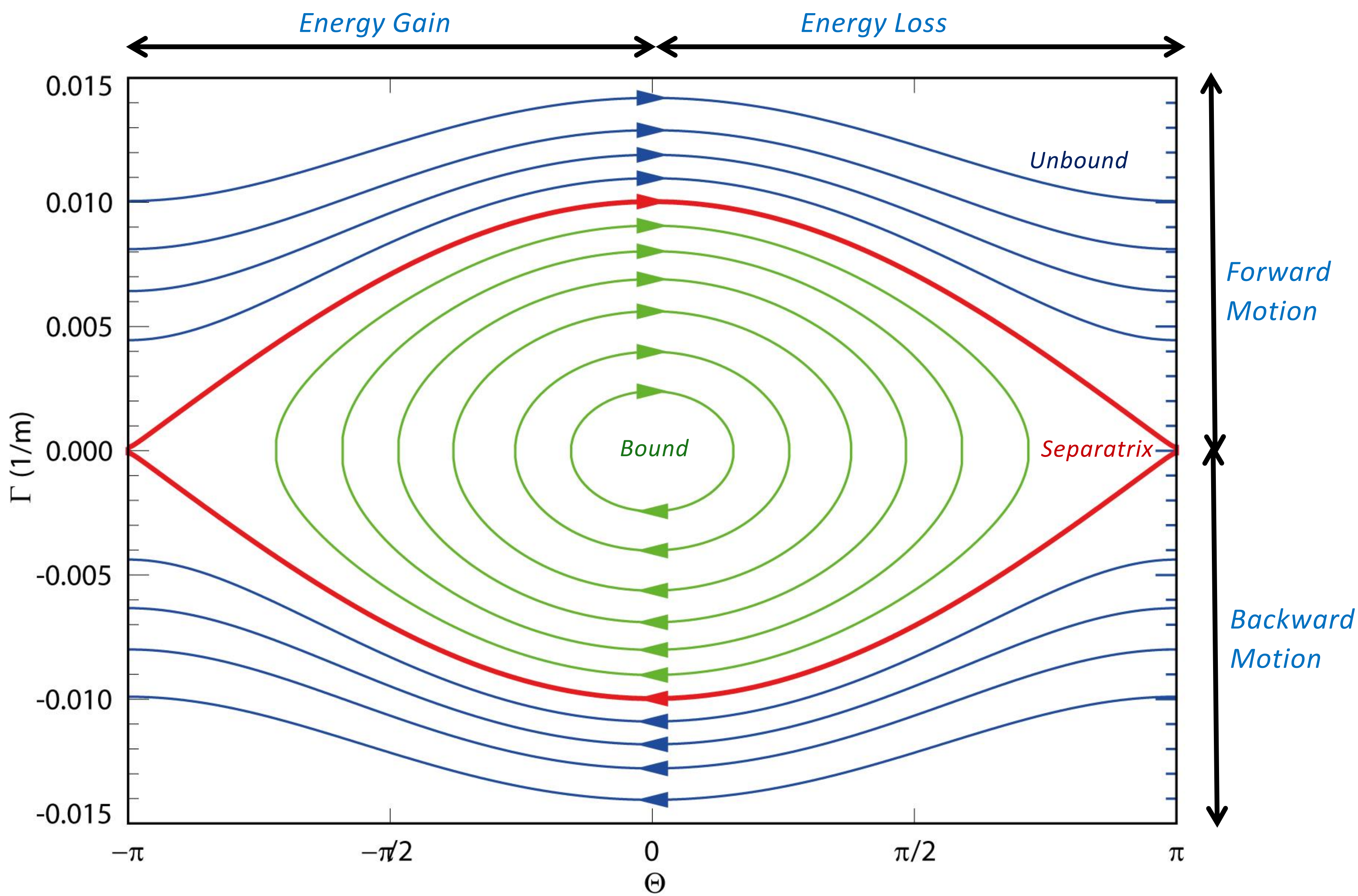
- "Energy" (Hamiltonian) of the system is:

$$H = \frac{1}{2}\Gamma^2 - \Omega_s^2 \cos\Theta$$

- Bound and unbound motion are split by the separatrix:

$$\Gamma_{sep} = \Omega_s \sqrt{2\left(1+\cos\Theta\right)}$$

Trajectories in Longitudinal Phase Space



Ensemble of Electrons

- The dynamic of a system with many electrons is best described by a phase space density function *f*.
- We know the Hamilton function. Therefore the density function is a constant of motion according to Liouville's theorem:

$$\frac{d}{d} f(z,\Theta,\Gamma) = \frac{\partial}{\partial z} f + \Theta' \frac{\partial}{\partial \Theta} f + \Gamma' \frac{\partial}{\partial \Gamma} f = 0$$

- We know the expression for Θ' and Γ' from the Hamilton function

$$\boxed{\frac{\partial}{\partial z} f + \Gamma \frac{\partial}{\partial \Theta} f - \Omega_s^2 \sin\Theta \frac{\partial}{\partial \Gamma} f = 0}$$

- With the distribution function we can calculate the mean energy of the system:

$$\langle \Gamma \rangle = \iint \Gamma f(z,\Theta,\Gamma) d\Gamma d\Theta$$

Small Signal Approximation I

$$\frac{\partial}{\partial z} f + \Gamma \frac{\partial}{\partial \Theta} f - \Omega_s^2 \sin\Theta \frac{\partial}{\partial \Gamma} f = 0$$

- We assume that the electric field is small and therefore the synchrotron frequency $\Omega_s{}^2$ can be use as a perturbation parameter:

$$f(z,\Theta,\Gamma) = f_0(z,\Theta,\Gamma) + \Omega_s^2 f_1(z,\Theta,\Gamma) + \Omega_s^4 f_2(z,\Theta,\Gamma) + \ldots$$

- The initial beam is uniformly distributed in Θ, the first order term includes the initial energy distribution g(Γ):

$$f_0 = \frac{g(\Gamma)}{2\pi}$$

Function g is unspecified so far

- Sorting powers of $\Omega_s{}^2$ yields recursive equation:

$$\boxed{\frac{\partial}{\partial z} f_n + \Gamma \frac{\partial}{\partial \Theta} f_n = \sin\Theta \frac{\partial f_{n-1}}{\partial \Gamma}}$$

- Formal 1st order solution is:

$$f_1(z,\Theta,\Gamma) = \left[\cos\left(\Theta - \Gamma z\right) - \cos\Theta\right] \frac{1}{\Gamma} \frac{\partial f_0}{\partial \Gamma}$$

Small Signal Approximation II

- The first order term does not show any change in the mean energy:

$$<\Gamma_1>=\Omega_s^2\iint \Gamma f_1 d\Theta d\Gamma=\Omega_s^2\int\frac{\partial f_0}{\partial\Gamma}\int\left[\cos\left(\Theta-\Gamma z\right)-\cos\Theta\right]d\Theta d\Gamma=0$$

- We have to go to second order. Here we are only interested in the terms which have no explicit dependence on Θ (zero order in Fourier series):

$$\frac{\partial}{\partial z}f_2+\Gamma\frac{\partial}{\partial\Theta}f_2=\sin\Theta\frac{\partial}{\partial\Gamma}\left\{\left[\cos\left(\Theta-\Gamma z\right)-\cos\Theta\right]\frac{1}{\Gamma}\frac{\partial f_0}{\partial\Gamma}\right\}$$

$$\frac{\partial}{\partial z}\tilde{f}_{2,0}=\frac{z\cos\left(\Gamma z\right)}{2}\frac{1}{\Gamma}\frac{\partial f_0}{\partial\Gamma}+\frac{\sin\left(\Gamma z\right)}{2}\frac{\partial}{\partial\Gamma}\left\{\frac{1}{\Gamma}\frac{\partial f_0}{\partial\Gamma}\right\}$$

$$\tilde{f}_{2,0}=\frac{\partial}{\partial\Gamma}\left\{\frac{\left[1-\cos\left(\Gamma z\right)\right]}{2\Gamma^2}\frac{\partial f_0}{\partial\Gamma}\right\}$$

The term (1-cosΓz) enforces that at z=0 there is no contribution from this coeficient

Small Signal Approximation III

- Calculating the mean energy change (2nd order term):

$$<\Gamma_2> = -\Omega_s^4 \iint \Gamma f_2 d\Theta d\Gamma = \Omega_s^4 \int \Gamma \frac{\partial}{\partial \Gamma} \left\{ \frac{\left[1-\cos(\Gamma z)\right]}{2\Gamma^2} \frac{\partial g}{\partial \Gamma} \right\} d\Gamma = -\Omega_s^4 \int \left\{ \frac{\left[1-\cos(\Gamma z)\right]}{2\Gamma^2} \frac{\partial g}{\partial \Gamma} \right\} d\Gamma$$

- Integration by parts:

$$<\Gamma_2> = -z^3 \Omega_s^4 \int \frac{1-\cos(\Gamma z) - \Gamma z \sin(\Gamma z)/2}{(\Gamma z)^3} g(\Gamma) d\Gamma$$

$$<\Gamma_2> = \frac{z^3 \Omega_s^4}{8} \int \frac{d}{d(\Delta)} \boxed{\left[\frac{\sin(\Delta)^2}{(\Delta)^2} \right]} g(\Gamma) d\Gamma \qquad \text{with} \qquad \Delta = \frac{\Gamma z}{2}$$

Intensity Spectrum of spontaneous undulator radiation

$$\Delta = \frac{\Gamma z}{2} = 2k_u \frac{\Delta\gamma}{\gamma_r} \frac{N_u \lambda_u}{2} = 2\pi N_u \frac{\Delta\gamma}{\gamma_r} = \pi N_u \frac{\Delta\omega}{\omega_r}$$

Small Signal Gain

- Converting back the gain formula
  - Transfer energy from electron beam to radiation field:

$$\Delta U = -n_e mc^2 \Delta\gamma = -\frac{\gamma_r mc^2}{2k_u} n_e \left\langle \Delta\Gamma \right\rangle = -\frac{\gamma_r mc^2}{2k_u} n_e \left\langle \Gamma_2 \right\rangle$$

  - Express electric field E_0 by energy density: $\left|E_0\right|^2 = \frac{U}{\varepsilon_0}$

  - Assume no energy spread: $g(\Gamma)=\delta(\Gamma-<\Gamma>)$

$$\Delta U = -\frac{mc^2}{2k_u} n_e z^3 \Omega_s^4 \frac{d}{d(\Delta)}\left[\frac{\sin(\Delta)^2}{(\Delta)^2}\right] = -\frac{e^2 n_e}{4\varepsilon_0} \frac{k_u K^2 N_u^3 \lambda_u^3}{\gamma_r^3 mc^2} \frac{d}{d(\Delta)}\left[\frac{\sin(\Delta)^2}{(\Delta)^2}\right] U$$

with $\Delta = \pi N_u \frac{\Delta\omega}{\omega_r}$

FEL Gain

$$\boxed{G(\Delta) = -\frac{e^2 n_e}{4\varepsilon_0} \frac{k_u K^2 N_u^3 \lambda_u^3}{\gamma_r^3 mc^2} \frac{d}{d(\Delta)}\left[\frac{\sin(\Delta)^2}{(\Delta)^2}\right]}$$

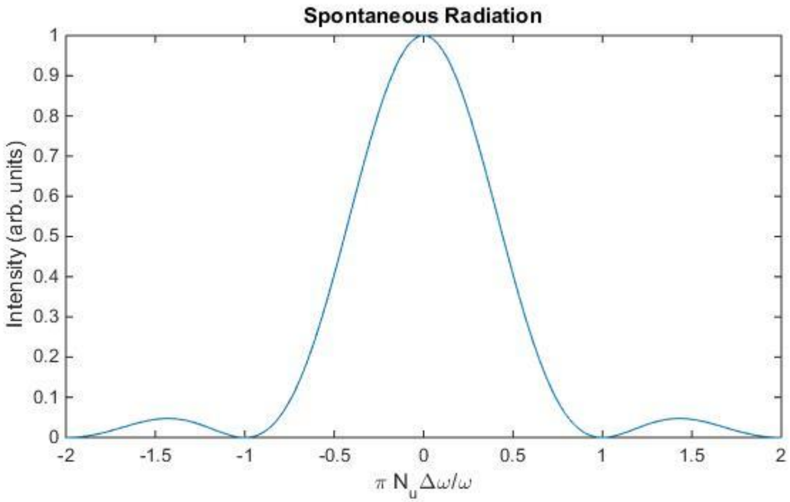
Madey's Theorems

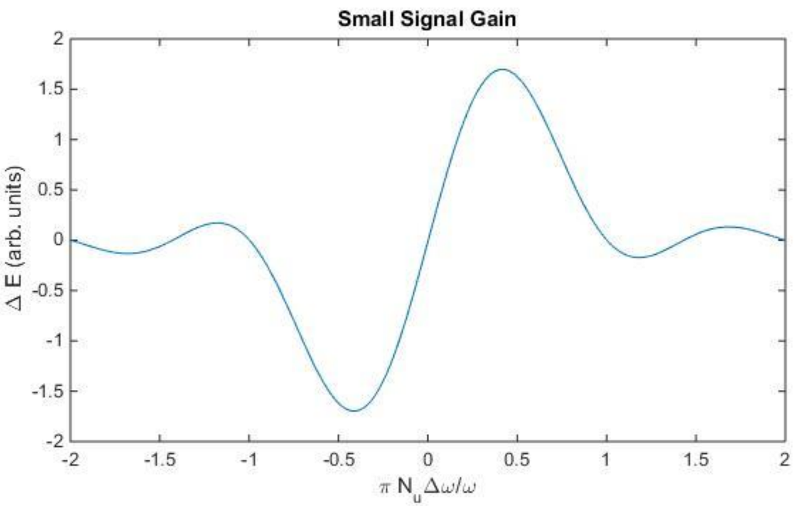
- 1st Theorem: Small signal FEL gain is proportional to derivative of spontaneous undulator spectrum

$$G \propto -\frac{d}{d(\Delta)}\left[\frac{\sin(\Delta)^2}{(\Delta)^2}\right]$$

- 2nd Theorem: Connects the induced energy spread to the energy spread

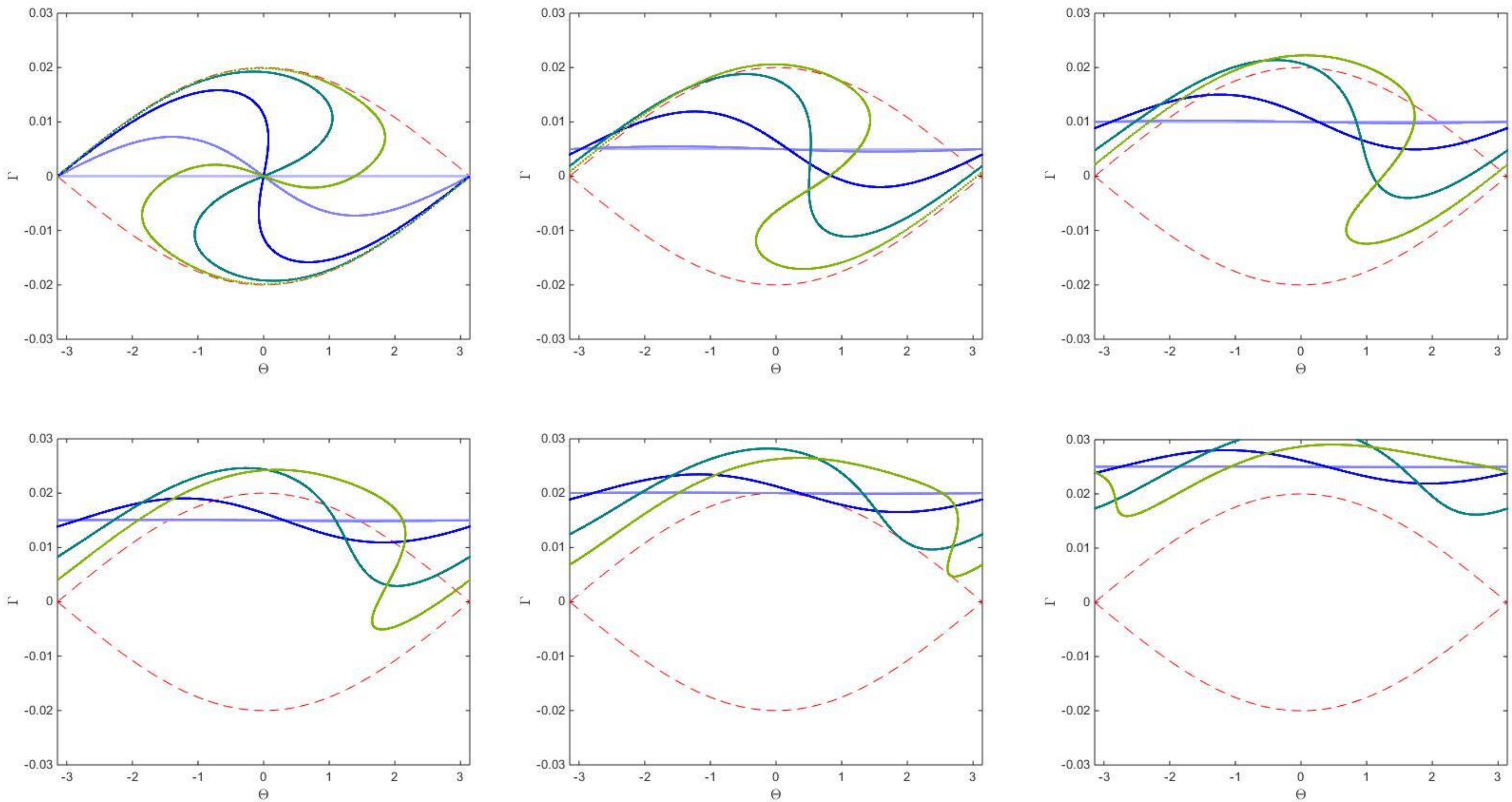
$$\langle \gamma - \gamma_0 \rangle = \frac{1}{2}\frac{\partial}{\partial \gamma_0}\left\langle (\gamma - \gamma_0)^2 \right\rangle$$





Numerical Examples

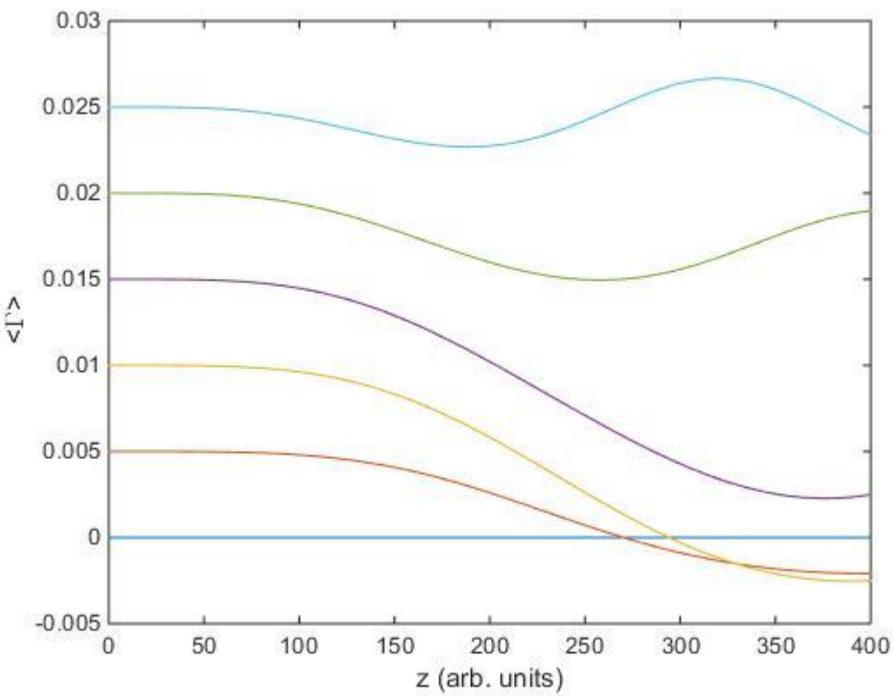
- Various initial mean energies



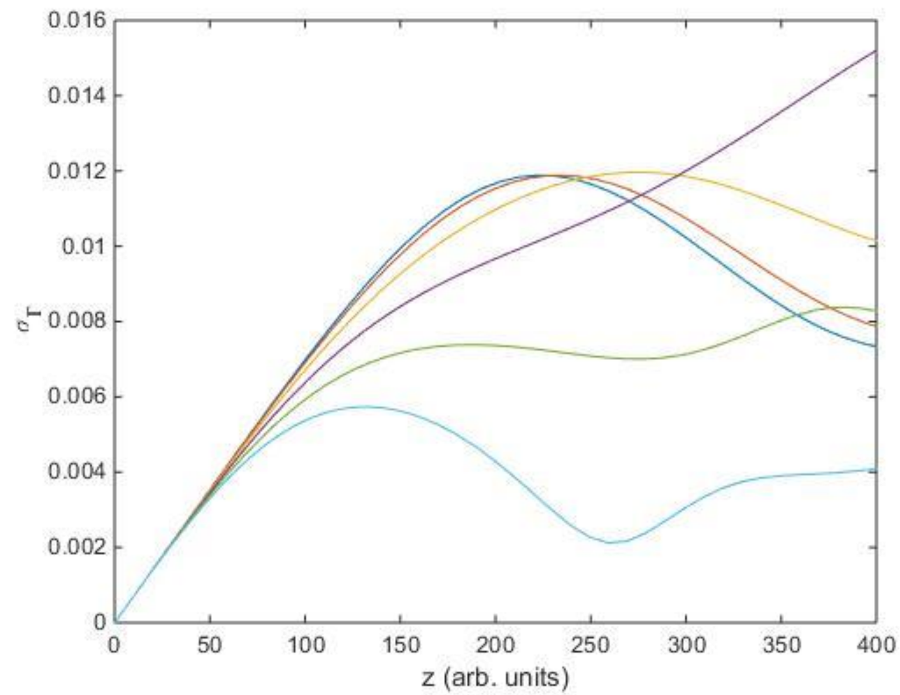
Mean Energy and Energy Spread

- Optimum detuning at around half the separatrix height
- Even particles outside of separatrix can transfer energy to field if undulator is short enough
- Energy change is cubic with undulator length in start up.

Mean Energy



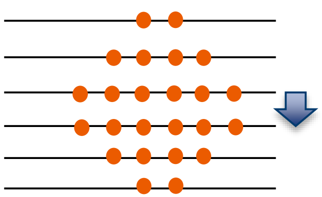
Energy Spread



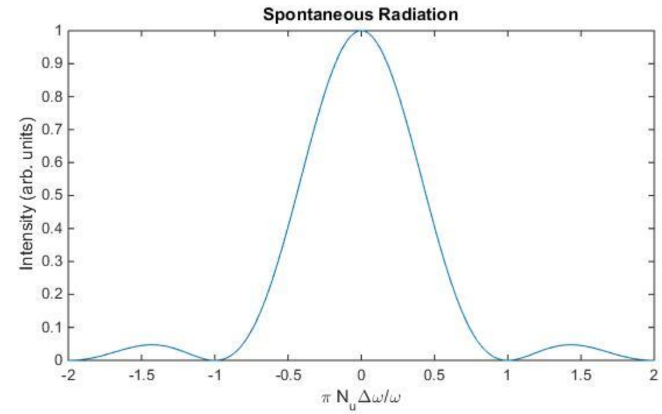
A Quantum Level Point of View

- A very crude argumentation, why the FEL can be considered a laser.
- The intensity distribution can be considered as the population density of the virtual states

$$\begin{array}{r} \dots \\ \Delta E = \hbar(\omega_r + \Delta\omega) \\ \Delta E = \hbar\omega_r \\ \Delta E = \hbar(\omega_r - \Delta\omega) \\ \dots \end{array}$$

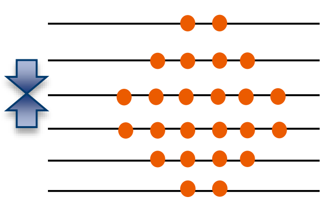


Spontaneous emission
~ density of state $N_{density}$



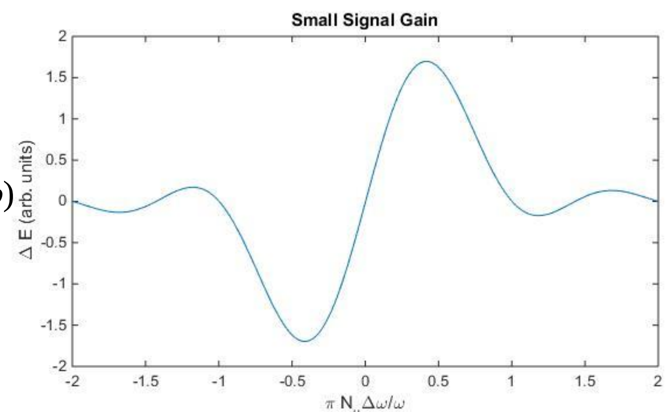
- In presence of external field the photon can either be absorbed or stimulate the emission of another one.

Emission
Absorption



$$\dot{N}_{ph} \propto N_{density}(E_0 + \hbar\omega) - N_{density}(E_0 - \hbar\omega)$$

$$\approx 2\hbar\omega \cdot \left.\frac{dN_{density}}{dE}\right|_{E_0}$$



Assuming photon energy much smaller than electron energy

Planar Undulator

- The interaction occurs with a radiation field of same polarization as the undulator

$$\vec{E}(z,t) = \vec{e}_x E_0 \cos\left(kz - \omega t + \phi\right)$$

- The maximum phase change is given by

$$\frac{d\gamma}{dz} = \frac{e}{\bar{\beta}_z mc^2} \vec{\beta} \cdot \vec{E} = \frac{eE_0 K}{\bar{\beta}_z \gamma mc^2} \cos(k_u z) \cos(kz - \omega t + \phi)$$

$$= \frac{eE_0 K}{2\bar{\beta}_z \gamma mc^2} \left[\cos\left(\left[k + k_u \right] z - \omega t + \phi \right) + \cos\left(\left[k - k_u \right] z - \omega t + \phi \right) \right]$$

- As in the helical case, the cos((k-k_u)z-ωt) term cannot be fully in resonance (β_z<1). However we cannot drop it yet due to the longitudinal oscillation

- The resonance condition is:

$$\frac{k}{k+k_u} = \bar{\beta}_z = 1 - \frac{1 + K^2/2}{2\gamma^2} \approx 1 - \frac{k_u}{k}$$

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)}$$

Planar Undulator

- The longitudinal oscillation has to be taken into account:

$$\frac{d\gamma}{dz} = \frac{eE_0K}{2\bar{\beta}_z\gamma mc^2}\Big[\cos\big([k+k_u]\bar{z}-\chi\sin(2k_u\bar{z})-\omega t+\phi\big)+\cos\big([k-k_u]\bar{z}-\chi\sin(2k_u\bar{z})-\omega t+\phi\big)\Big]$$

with $\chi = \frac{(k+k_u)K^2}{k_u 8\gamma^2\bar{\beta}_z} = \frac{1}{\bar{\beta}_z^2}\frac{K^2}{4+2K^2} \approx \frac{K^2}{4+2K^2}$

$$\frac{d\gamma}{dz} = \frac{eE_0K}{2\bar{\beta}_z\gamma mc^2}\Re e\left(e^{i(\theta+\phi)}\left[1+e^{-2ik_u\bar{z}}\right]\cdot e^{-i\chi\sin(2k_u\bar{z})}\right)$$

$$= \frac{eE_0K}{2\bar{\beta}_z\gamma mc^2}\Re e\left(e^{i(\theta+\phi)}\left[1+e^{-2ik_u\bar{z}}\right]\sum_{m=-\infty}^{\infty}(-1)^m J_m(\chi)e^{i2mk_u\bar{z}}\right)$$

$$= \frac{eE_0K}{2\bar{\beta}_z\gamma mc^2}\Re e\left(e^{i(\theta+\phi)}\sum_{m=-\infty}^{\infty}(-1)^m\left[J_m(\chi)-J_{m+1}(\chi)\right]e^{i2mk_u\bar{z}}\right)$$

$$= \frac{eE_0K}{2\bar{\beta}_z\gamma mc^2}\sum_{m=-\infty}^{\infty}(-1)^m[J_m(\chi)-J_{m+1}(\chi)]\cos(\theta+\phi+2mk_u\bar{z})$$

Harmonics

- For the fundamental wavelength on the term for m=0 is resonant

$$\frac{d\gamma}{dz} = \frac{eE_0 K}{2\bar{\beta}_z \gamma m c^2} \sum_{m=-\infty}^{\infty} (-1)^m [J_m(\chi) - J_{m+1}(\chi)] \cos(\theta + \phi + 2mk_u \bar{z})$$

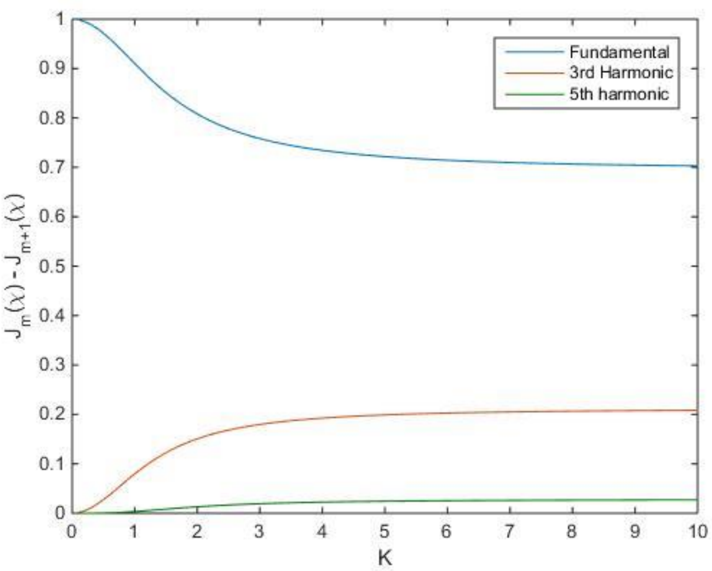
- However if we choose a harmonic of the fundamental with $k_n = nk$:

$$\theta_{n,m} = (k_n + k_u) z - \omega_n t + 2mk_u z = (nk + (2m+1)k_u) z - n\omega t$$

- The resonance condition is then:

$$\bar{\beta}_z = \frac{k}{k + \frac{2m+1}{n} k_u}$$

Odd harmonics can be in resonance as well, though with a reduced coupling $J_m(\chi)$-$J_{m+1}(\chi)$

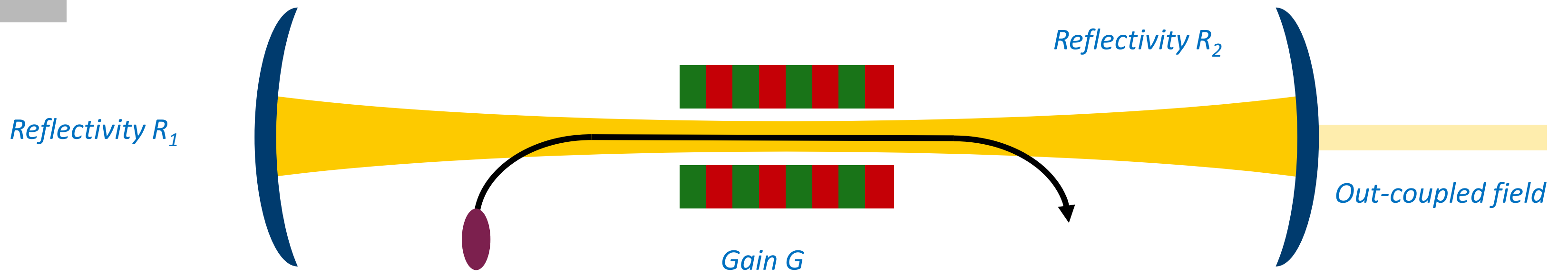


Comparison Helical vs Planar Undulator

	Helical	Planar
Polarization B- and E-Field	Opposite, circular	Same, planar
Harmonics	None	Odd harmonics
Resonance Condition	$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+K^2\right)$	$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$
Coupling with E-Field	1	$\frac{J_0(\chi)-J_1(\chi)}{2}$
Synchrotron Frequency	Ω_s	$\frac{\Omega_s}{\sqrt{2}}$
Low Gain Function	$G(\Delta)\propto -\frac{d}{d\Delta}\left(\frac{\sin\Delta}{\Delta}\right)^2$	$G(\Delta)\propto -\frac{\left(J_0(\chi)-J_1(\chi)\right)^2}{4}\frac{d}{d\Delta}\left(\frac{\sin\Delta}{\Delta}\right)^2$

FEL Oscillator

- Because the gain can be rather small it needs to be accumulated in an optical cavity:



- The bunch repetition rate needs to match the round trip frequency of the cavity or a harmonic of it.

- For amplification it requires:

$$\boxed{\eta = R_1 R_2 (1+G) > 1} \Rightarrow P_n = P_0 \eta^n$$

Saturation-Effects

- Saturation effects occurs once the length of the undulator matches the period length of the synchrotron oscillation:

$$\Omega_s L_u = 2\pi \quad \Rightarrow \quad 4\pi^2 = L_u^2 2k_u \frac{eE_0 K}{\overline{\beta}_z \gamma_r^2 mc^2} = 16\pi^2 N_u^2 \boxed{\frac{eE_0}{kmc^2}} \frac{K}{1+K^2}$$

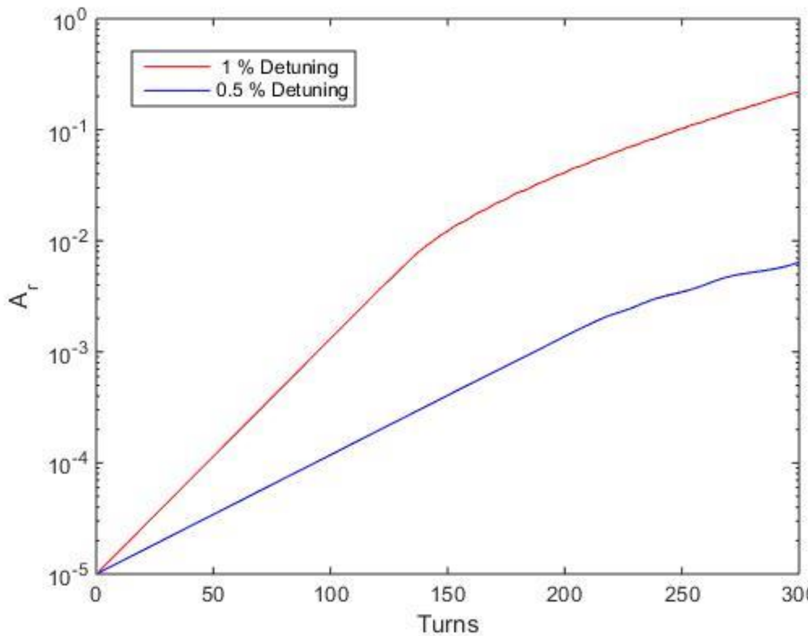
Normalized Vector Potential of Radiation Field: A_r

$$\boxed{\Rightarrow \quad A_r = \frac{1+K^2}{4K} \frac{1}{N_u^2}}$$

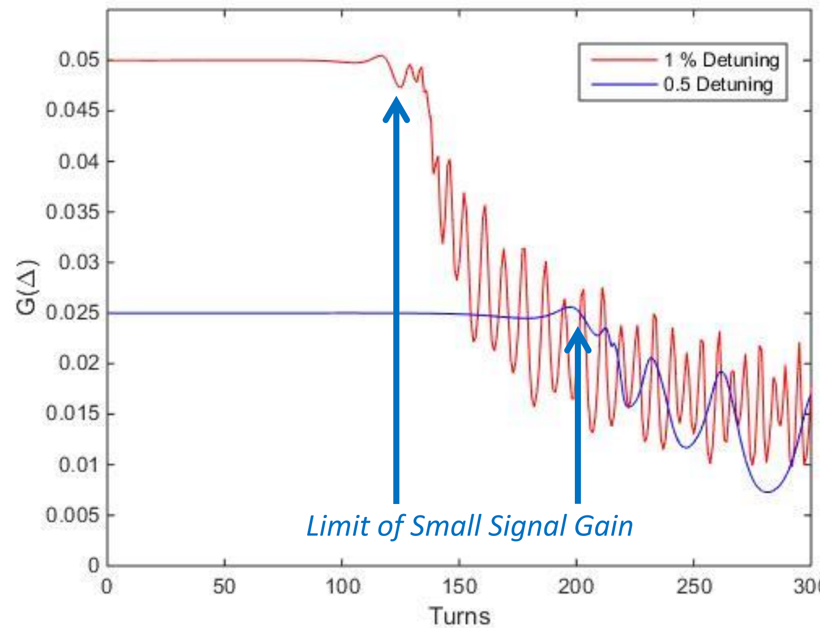
- More radiation power can be achieved if the undulator is shorter.
- However it makes the FEL oscillator more difficult to laser because the small signal gain drops.

Numerical Simulations

Field Growth



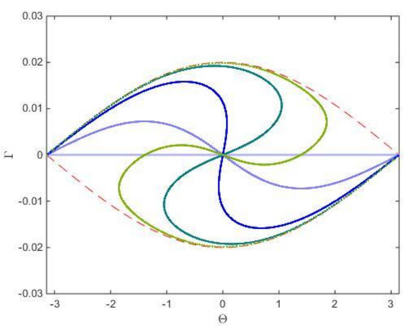
Growth Rate



- FEL Oscillator model gets more complicated in reality:
  - Slippage of the radiation field with respect to electron beam
  - Detuning of the optical cavity
  - Mode selection in the cavity

Model Limitations

- The small signal low gain FEL model is limited in its applicability by:
  - Scaling with more number of particles the transferred energy can be become larger than the assumption that the electric field is constant.
  - On resonance the beam can be strongly bunched and should emit coherently.



  - For large field values the analytical solution is not valid anymore. In addition the requirement of being in resonance is diminished.

Model must include Maxwell equation as well
High Gain FEL Model

Outlook

Continuation with

- High Gain FEL Theory by K.-J. Kim (tomorrow and Monday)

More info on FEL Oscillators

- XFELO (Monday)