# Vladimir Sudakov and double-logarithmic asymptotics of amplitudes in QED, QCD and gravity

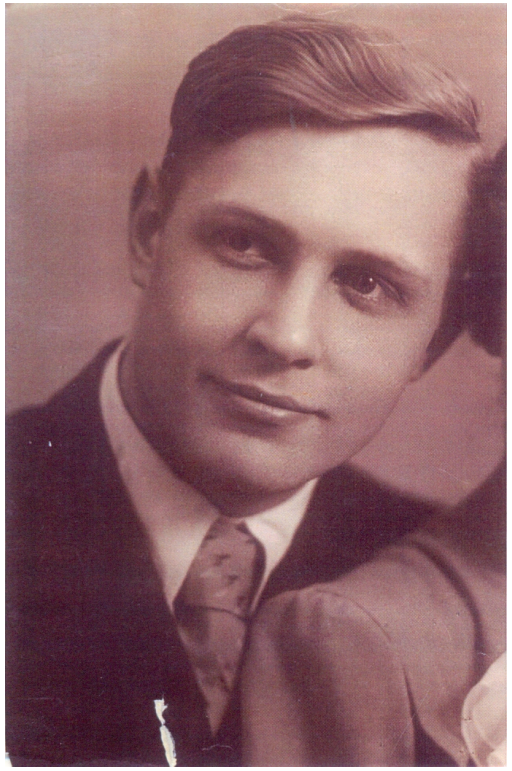
L. N. Lipatov
Petersburg Nuclear Physics Institute

## Content

Guido Altarelli


Vladimir Gribov

Vladimir Sudakov



# 1 Sudakov one-loop vertex

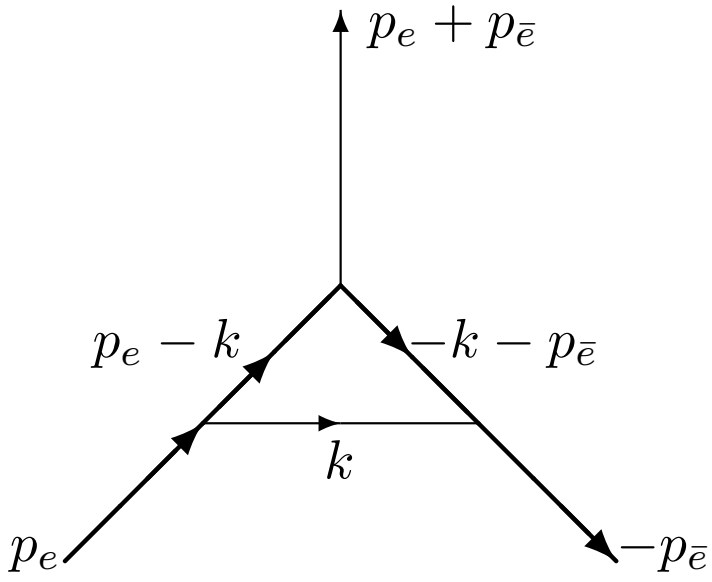



Figure 1: One loop diagram for the Sudakov vertex

Region of applicability of double logarithmic approximation

$$\alpha = \frac{e^2}{4\pi} \ll 1\,, \;\; \frac{\alpha}{\pi}\,\ln^2\frac{s}{m_e^2} \sim 1\,, \;\; s = (p_e + p_{\bar{e}})^2 \gg m_e^2$$

# 2 Electron form-factor in DLA

Sudakov parametrization of the photon momentum

$$k = \alpha\, p'_e + \beta\, p'_{\bar{e}} + k_\perp\,,\ p'^2_e = p'^2_{\bar{e}} = 0\,,\ k_\perp p'_e = k_\perp p'_{\bar{e}} = 0$$

$$d^4k = \frac{s'}{2}\, d\,\alpha\, d\,\beta\, d^2k_\perp\,, |\alpha| \ll 1\,,\ |\beta| \ll 1\,,\ -k_\perp^2 \ll \sqrt{s}$$

One loop vertex $\gamma^{(1)}/\gamma_B$ after integration over $k_\perp$

$$-\frac{e^2}{8\pi^2}\int_0^1\int_0^1 \frac{d\,\alpha\, d\beta\ \ \theta(s\alpha\beta - \lambda^2)}{\left(\alpha + \frac{m_e^2}{s}\beta\right)\left(\beta + \frac{m_e^2}{s}\alpha\right)} = -\frac{e^2 \ln\frac{s}{m_e^2}}{8\pi^2}\left(\frac{\ln\frac{s}{m_e^2}}{2} + \ln\frac{m_e^2}{\lambda^2}\right)$$

Evolution equation for $\gamma$ in DLA and its solution

$$\frac{\partial\gamma}{\partial \ln m_e^2/\lambda^2} = -\frac{e^2}{8\pi^2}\ln\frac{s}{m_e^2}\ \gamma\,,\ \gamma = \gamma_B\ \exp\left(-\frac{e^2\ln\frac{s}{m_e^2}}{8\pi^2}\left(\frac{\ln\frac{s}{m_e^2}}{2} + \ln\frac{m_e^2}{\lambda^2}\right)\right)$$

Application: production of $j/\psi$-mesons in $e^+e^-$ collisions (AKLV)

# 3 $\gamma e$ scattering and electron reggeization

High energy kinematics of the backward $\gamma e$ scattering

$$s = (k+p)^2 \gg u = (k-p')^2 \sim m_e^2$$

Regge asymptotics of the amplitude

$$A(s,u) = A_B \left(\frac{s}{m_e^2}\right)^{\omega(u)} \ , \ u = -|q|_\perp^2$$

Regge trajectory in one loop

$$\omega(u) = \frac{\alpha}{2\pi}\,(\hat{q} - m_e) \int \frac{d^2 k_\perp}{\pi} \frac{1}{k_\perp^2 - \lambda^2} \frac{\hat{q}_\perp - \hat{k}_\perp + m_e}{(q-k)_\perp^2 - m_e^2}$$

DLA of the $\gamma e$-scattering amplitude (GGF, 1967)

$$A_{DL}(s,u) = A_B \exp\left(-\frac{\alpha}{2\pi} \ln\frac{s}{m_e^2} \ln\frac{|u|}{\lambda^2}\right)$$

# 4 Forward annihilation $e^+e^- \to \mu^+\mu^-$

Regge kinematics of the forward scattering

$$s = (p_1 + p_2)^2 \gg t = (p_1 - p_1')^2 \sim m_e^2$$

Factorization of the amplitude in DLA

$$A_{DL}(s) = A_B\, f(s)\,, \;\; f(s) = f(s, p^2)_{p^2=m^2}$$

Bethe-Salpeter equation for the ladder diagrams

$$f(s, p^2) = 1 + \frac{\alpha}{2\pi} \int_{m^2}^{s} \frac{ds'}{s'} \int_{\max(m^2, p^2 \frac{s'}{s})}^{s'} \frac{dp'^2}{p'^2}\, f(s', p'^2)$$

Forward scattering amplitude in DLA (GGLF, 1968)

$$f(s) = \int_{a-i\infty}^{a+i\infty} \frac{dj}{2\pi i} \left(\frac{s}{m^2}\right)^j \frac{2}{j + \sqrt{j^2 - \frac{2\alpha}{\pi}}} = I_0\left(\sqrt{\frac{\alpha}{\pi}} \ln \frac{s}{m^2}\right)$$

# 5 $e^+e^-$ backward scattering

Regge kinematics of the backward scattering

$$s = (p_1 + p_2)^2 \gg u = (p_1 - p_2')^2 \sim m_e^2$$

Factorization of the amplitude in DLA

$$A_{DL}(s) = A_B\, f(s)\,, \;\; f(s) = f(s, p^2)_{p^2=m^2}$$

Solution of the Bethe-Salpeter equation (GGLF)

$$f(s) = 4 \int_{a-i\infty}^{a+i\infty} \frac{d\,l}{2\pi i}\, e^{l\rho}\, \frac{d}{d\,l}\, \ln D_{-\frac{1}{4}}(l)\,, \;\; \rho = \sqrt{\frac{2\alpha}{\pi}}\, \ln \frac{s}{m^2}$$

Parabolic cylinder function

$$D_p(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(-p)} \int_0^\infty \frac{dt}{t^{1+p}}\, e^{-xt-\frac{t^2}{2}}$$

# 6 Infrared evolution equations in DLA

Cut-off at small transverse momenta

$$|k_\perp|^2 > \mu^2$$

$\mu^2$-evolution equation for forward $e^+e^-$ scattering (KL)

$$f(s,\mu^2) = 1 + e^2 \int_{-\infty}^{\infty} \frac{ds\alpha}{s\alpha} \frac{ds\beta}{s\beta} \int_\mu \frac{d^2k_\perp}{i(2\pi)^4} f(s\alpha, |k_\perp|^2) \frac{|k_\perp|^2}{(s\alpha\beta - |k_\perp|^2)^2} f(s\beta, |k_\perp|^2)$$

Solution of the evolution equation

$$f(s,\mu^2) = \int_{a-i\infty}^{a+i\infty} \frac{dj}{\pi i} \left( \left( -\frac{s}{\mu^2} \right)^j + \left( \frac{s}{\mu^2} \right)^j \right) \frac{f_j}{j} \ , \quad f_j = 1 + \frac{\alpha}{2\pi} \frac{f_j^2}{j^2}$$

Evolution equation for the backward $e^+e^-$ scattering (KL)

$$f_j = 1 + \frac{2\alpha}{\pi} \frac{d}{dj} \frac{f_j}{j} - \frac{\alpha}{2\pi} \frac{f_j^2}{j^2} \ , \quad \frac{f_j}{j} = 4\sqrt{\frac{\pi}{2\alpha}} \frac{d}{dl} \ln \left( e^{l^2/4} D_{-1/4}(l) \right)$$

# 7 DGLAP equation in QCD and DLA

DGLAP equations for non-singlet distributions at $x \to 0$

$$\frac{d}{d\ln Q^2}\, n_q(Q^2, x) = \frac{n_c^2 - 1}{2N_c^2}\, \frac{\alpha_c(Q^2)}{2\pi} \int_x^1 \frac{d\beta}{\beta}\, n_q(Q^2, \beta)$$

DL equations for non-singlet quark distributions (KL)

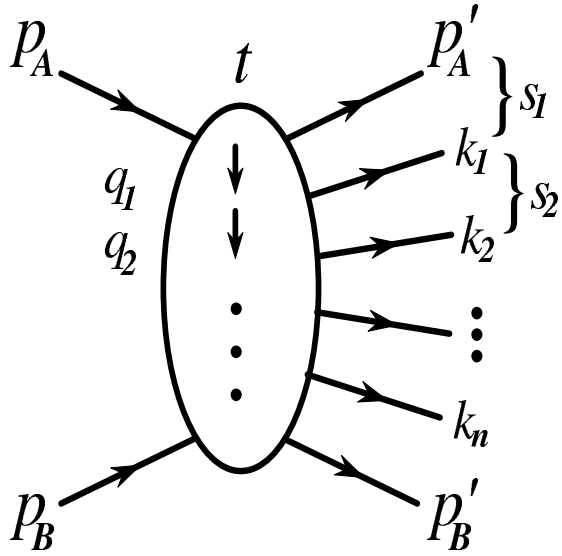
$$n_q(Q^2, x) = \delta(x-1) + \frac{n_c^2 - 1}{2N_c^2}\, \frac{\alpha_c}{2\pi} \int_x^1 \frac{d\beta}{\beta} \int_{m^2}^{Q^2 \frac{\beta}{x}} \frac{dk^2}{k^2}\, n_q(k^2, \beta)$$

The non-singlet quark distributions in DLA (KL)

$$n_q(Q^2, x) = \int_{a-i\infty}^{a+i\infty} \frac{dj}{2\pi} \left(\frac{1}{x}\right)^j \frac{j\gamma(j)}{\frac{N_c^2-1}{2N_c}\frac{\alpha}{2\pi}} \left(\frac{Q^2}{m^2}\right)^{\gamma(j)}, \; j\gamma(j) = \frac{N_c^2 - 1}{2N_c}\frac{\alpha}{2\pi} + \gamma^2(j)$$

Singlet quark and gluon distributions in DLA (BREM)

# 8 Production in multi-Regge kinematics




$$M^{BFKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1} c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$C_\mu = -q^\perp_{1\mu} - q^\perp_{2\mu} + n^+_\mu \left(k_1^- + \frac{q_1^2}{k_1^+}\right) - n^-_\mu \left(k_1^+ + \frac{q_2^2}{k_1^-}\right) \to C(q_2,q_1) = \frac{q_2^\perp\, q_1^{\perp *}}{k_1^{\perp *}}$$

# 9 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2\to n} = -s^2 \Gamma^{\mu'\nu'}_{\mu\nu} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} ... \Gamma^{\rho'\sigma'}_{\rho\sigma}$$

Graviton-reggeized graviton vertices

$$\Gamma^{\mu'\nu'}_{\mu\nu} = \frac{\kappa}{4} \left( \Gamma_{\mu\mu'}\Gamma_{\nu\nu'} + \Gamma_{\mu\nu'}\Gamma_{\nu\mu'} \right) , \; \Gamma^{RRG}_{\rho\sigma} = \frac{\kappa}{4} \left( C_\rho C_\sigma - N_\rho N_\sigma \right)$$

Gluon-reggeized gluon vertices

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p^A_{\mu'} p^B_\mu + p^{A'}_\mu p^B_{\mu'}}{p^A p^B} + \frac{q^2}{2} \frac{p^B_\mu p^B_{\mu'}}{(p^A p^B)^2} , \; N = \sqrt{q_1^2 q_2^2} \left( \frac{p^A}{kp^A} - \frac{p^B}{kp^B} \right) ,$$

$$C = -q_1^\perp - q_2^\perp + p_1 \left( \frac{q_1^2}{kp_1} + \frac{kp_2}{p_1 p_2} \right) - p_2 \left( \frac{q_2^2}{kp_1} + \frac{kp_1}{p_1 p_2} \right)$$

# 10 Graviton trajectory in supergravity

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,, \;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k,q-k)^2 \left( \frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2}(k,q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5 \gamma_\nu \partial_\rho \psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left( \ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{|\Lambda|^2}{|q|^2} \right)$$

# 11 Double-logarithms in (super) gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2 \ln \frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\ \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for supergravity (BLS (2012))

$$f_\omega = 1 + b\frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - b\,\frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\,,\ b = \alpha|q|^2\,,\ \alpha = \frac{\kappa^2}{8\pi^2}\,,\ \xi = \alpha\,|q|^2\,\ln^2\frac{s}{|q|^2}$$

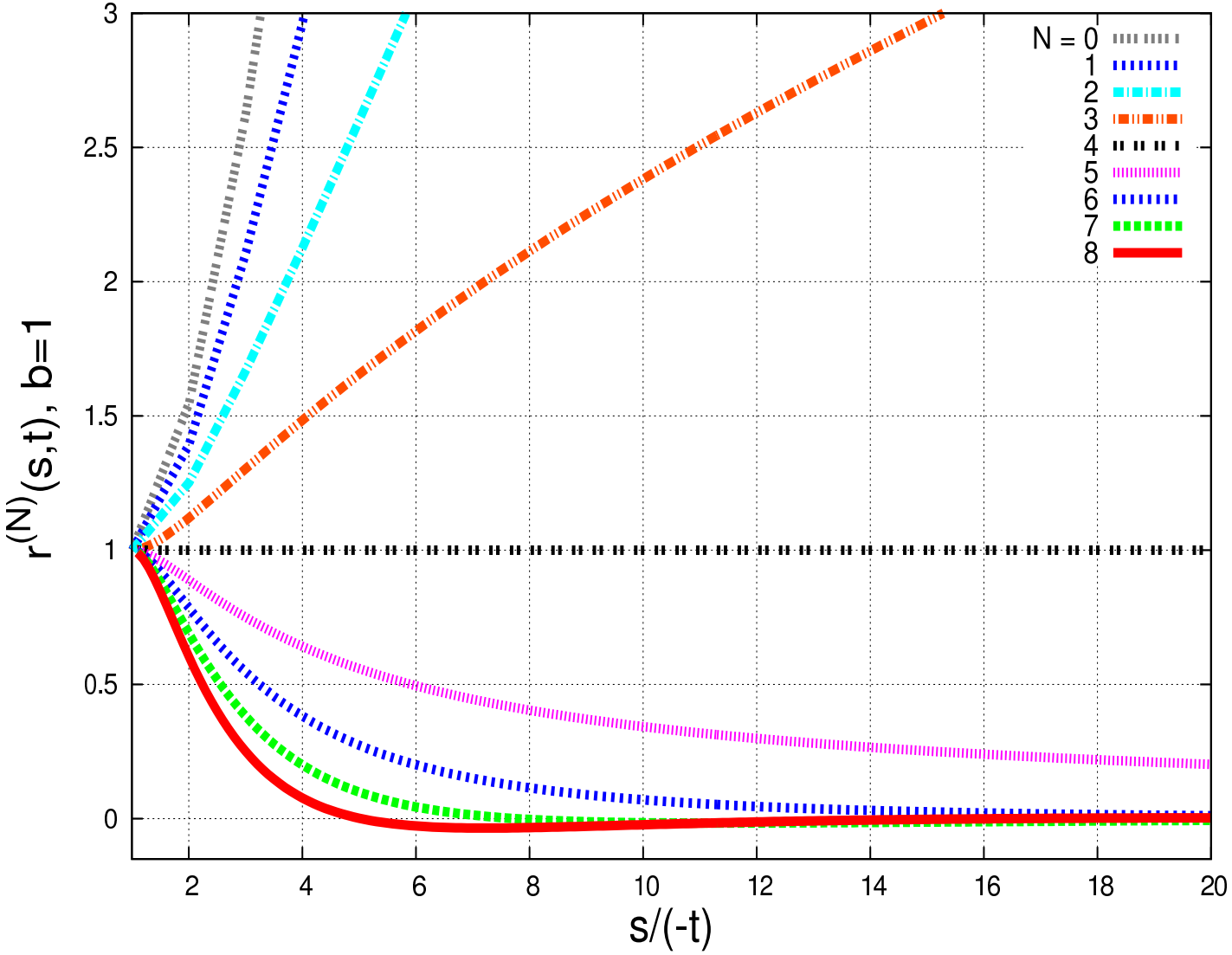
Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2 - 26N + 36)\,\frac{\xi^3}{6!} + \ldots$$

Parabolic cylinder function solution (BLS)

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln d^{(N)}(x)\,,\ d^{(N)}(x) = e^{\frac{x^2}{4}}\,D_{\frac{6-N}{2}}(x)\,,\ x = \frac{\omega}{\sqrt{b}}$$

# 12 Amplitudes in DL approximation



# 13 Discussion

1. Sudakov variables and double logarithms.
2. Electron form-factor in DLA.
3. Backward $\gamma e$ scattering.
4. Forward and backward $e^+e^-$ scattering.
5. Non-linear equations for amplitudes on mass shell.
6. Double-logarithm resummation in QCD.
7. Divergency of the graviton Regge trajectory.
8. Double logarithmic amplitudes in (super) gravity