Peculiarities of the BFKL approach

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Introduction

Gluon Reggeization

BFKL representation of scattering amplitudes

New problems in the NNLLA
  - Violation of the Regge factorization
  - Account of imaginary parts of amplitudes

Summary
Introduction

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Peculiarities of the BFKL approach
Asymptotic Freedom in Parton Language
Guido Altarelli (Ecole Normale Superieure), G. Parisi (IHES, Bures-sur-Yvette)

A novel derivation of the $Q^2$ dependence of quark and gluon densities as predicted by quantum chromodynamics is presented. The main body of predictions of the theory for deep-inelastic scattering on either unpolarized or polarized targets is re-obtained by a method which only makes use of the simplest tree diagrams and is entirely phrased in parton language with no reference to the conventional operator formalism.

5755 citations in INSPIRES
Cross sections of lepton($l$)-hadron $A$ collisions with a hard scale $Q^2$ are given by convolutions of parton distributions $F^a_A(z, Q^2)$ (aisapartonflavour, $x$ is a fraction of longitudinal momenta) and cross sections of lepton-parton interactions $\sigma_{la}(z, Q^2)$

$$\sigma_{IA} = F^a_A \bigotimes \sigma_{la}.$$ 

The sign $\bigotimes$ means convolution over $z$ and $a$. For hadron-hadron collisions

$$\sigma_{AB} = F^a_A \bigotimes \sigma_{ab} \bigotimes F^b_B.$$ 

$\sigma_{ab}(z_a, z_b; Q^2)$ – cross sections of parton-parton interactions.
Evolution of the parton distributions with $\tau = \ln \left( \frac{Q^2}{\Lambda_{QCD}^2} \right)$ is determined by the DGLAP equations

$$\frac{d}{d\tau} F^a_A = \frac{\alpha_s(Q^2)}{2\pi} P^a_b \otimes F^b_A,$$

$P^b_a(z)$ are the parton splitting functions,

$$\frac{\alpha_s}{2\pi} \int_0^1 dzz^{i-1} P^b_a(z) = \gamma^{ab}_j(\alpha_s),$$

$\gamma^{ab}_j(\alpha_s)$ – anomalous dimensions matrix for twist-2 operators of Lorenz spin $j$ in expansion of products of two currents.
Colour coherence

The splitting functions $P_{ab}^a(z)$ are determined by diagrams

The virtual (proportional to $\delta(z - 1)$) contributions to $P_{ab}^a(z)$ can be obtained using flavour conservation and longitudinal momentum conservation. The parton $a$ distribution $F_A^a(z, Q^2)$

with strong ordering of parton transverse momenta.
Another derivation, based on the results of direct summation in perturbation series
was performed in
for several field-theoretical models. For QCD
Introduction

The DGLAP sums terms enhanced by powers of $\ln Q^2$ (collinear logarithms), arising at integration over angles between parton momenta. There are another logarithms (soft, or Regge logarithms), arising at integration over ratios of parton energies. These logarithms are present both in parton distributions and in partonic cross sections.

At small $x = Q^2/s$ ($s$ is c.m.s. energy squared), it is necessary to sum the terms of the perturbation series enhanced by powers of $\log(1/x)$. The resummation of leading $\left(\alpha_s \ln(1/x)\right)^n$ and next-to-leading $\alpha_s \left(\alpha_s \ln(1/x)\right)^n \log(1/x)$-terms can be performed using the BFKL equation.
The BFKL equation usually is associated with the equation for unintegrated gluon density \( \mathcal{F}(x, \vec{k}) \) (\( \vec{k} \) is transverse momentum), connected with the gluon distribution \( F^g(x, Q^2) \) by the equation

\[
F^g(x, Q^2) = \int_{Q^2}^{\infty} \mathcal{F}(x, \vec{k}) d\vec{k}/(\pi \vec{k}^2).
\]

The equation describes evolution of \( \mathcal{F}(x, \vec{k}) \) with change of \( \ln(1/x) \) and has the structure

\[
\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{K} \otimes \mathcal{F},
\]

where \( \mathcal{K} \) is the BFKL kernel and \( \otimes \) means convolution with respect to transverse momenta.
The small x resummation:

"A fully small x resummed global PDF analysis is unavoidable in order to quantify the modifications in the PDFs due to small x resummation, and to propagate these modifications into relevant observables at the LHC. Such a programme will be necessary in order to achieve phenomenology at the percent level for many LHC signal, background and standard candle processes. It would be even more important for phenomenology at a future LHeC electron-proton collider, and mandatory for the treatment of extremely high-energy scattering processes, such as those induced by Ultra- High Energy cosmic neutrinos, which are currently under investigation".
The small $x$ resummation:
By different method
Gluon Reggeization

The basis of the BFKL approach
E. A. Kuraev, L. N. Lipatov, V.S. F., ZhETF 71 (1976) 840
E. A. Kuraev, L. N. Lipatov, V.S. F., ZhETF 72 (1977) 377
I.I. Balitsky, L.N. Lipatov, Yad. Fiz. 28 (1978) 1597

is the remarkable property of QCD – gluon Reggeization
L. N. Lipatov, Yad. Fiz. 23 (1976) 642.

The Reggeization allows to express an infinite number of amplitudes through several effective vertices and gluon trajectory.

Validity of the Reggeization is proved now in all orders of perturbation theory in the coupling constant $g$ both in the leading logarithmic approximation (LLA), where in each order of the perturbation theory only terms with the highest powers of $\ln s$ are kept, and in the next-to-leading one (NLLA), where terms with one power less are also kept.
For elastic scattering processes $A + B \rightarrow A' + B'$ in the Regge kinematical region: $s \simeq -u \rightarrow \infty$, $t$ fixed (i.e. not growing with $s$) the Reggeization means that scattering amplitudes with the gluon quantum numbers in the $t$-channel can be presented as

$$\mathcal{A}_{AB}^{A'B'} = \Gamma_{A'A}^c \left[ \left( \frac{-s}{-t} \right)^\omega(t) - \left( \frac{s}{-t} \right)^\omega(t) \right] \Gamma_{B'B}^c;$$
Gluon Reggeization

\( \Gamma^c_{P', P} \) – particle-particle-Reggeon (PPR) vertices or scattering vertices ("c" are colour indices); \( j(t) = 1 + \omega(t) \) – Reggeon trajectory.

The Reggeization means definite form not only of elastic amplitudes, but of inelastic amplitudes in the multi-Regge kinematics (MRK) as well. It can be presented by the picture
Gluon Reggeization

and written as

$$\mathcal{RA}_{AB}^{\tilde{A}\tilde{B}+n} = 2s \Gamma_{\tilde{A}A}^{c_1} \left( \prod_{i=1}^{n} \gamma_{c_i c_{i+1}}^{J_i}(q_i, q_{i+1}) \left( \frac{s_i}{s_0} \right)^{\omega(t_i)} \frac{1}{t_i} \right)$$

$$\frac{1}{t_{n+1}} \left( \frac{s_{n+1}}{s_0} \right)^{\omega(t_{n+1})} \Gamma_{\tilde{B}B}^{c_{n+1}}$$

Here $\gamma_{c_i c_{i+1}}^{J_i}(q_i, q_{i+1})$ – the Reggeon-Reggeon-particle (RRP) or production vertices.

MRK is the kinematics where all particles have limited (not growing with $s$) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with $s$) invariant masses of any pair of the jets.

The MRK gives dominant contributions to cross sections of QCD processes at high energy $\sqrt{s}$. In the LLA only a gluon can be produced. In the NLA one has to account production of $Q\bar{Q}$ and $GG$ jets.
Amplitudes of processes with all possible quantum numbers in the \( t \)–channel are calculated using \( s \)-channel unitarity and analyticity.

The \( s \)-channel discontinuity

\[
\sum_n \quad \rho^A \quad \rho^A' \quad \rho^B \quad \rho^B' \quad q_1 \quad q_i \quad q_{i+1} \quad q_{n+1} \quad q_1' \quad q_i' \quad q_{i+1}' \quad q_{n+1}'
\]
BFKL representation of scattering amplitudes

The amplitudes are presented in the form:

\[ \Phi_{A'A} \otimes G \otimes \Phi_{B'B}. \]
BFKL representation of scattering amplitudes

Impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \rightarrow A'$, $B \rightarrow B'$, $G$ – Green’s function for two interacting Reggeized gluons,

$$\hat{G} = e^{Y\hat{K}},$$

$\hat{K}$ – BFKL kernel, $Y = \ln(s/s_0)$,

$$\hat{K} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{K}_r$$

$$\hat{K}_r = \hat{K}_G + \hat{K}_{Q\bar{Q}} + \hat{K}_{GG}$$

Energy dependence of scattering amplitudes is determined by the BFKL kernel.

The BFKL kernel and the impact factors are expressed in terms of the Reggeon vertices and trajectory. The kernel is universal (process independent).
The first observation of the violation of the Regge factorization was made by
V. Del Duca, N. Glover, 2001
in the consideration of the high-energy limit of the two-loop amplitudes for parton-parton scattering.
They consider the interference of the Born amplitude with the two-loop amplitude for $gg$, $gq$ and $qq$ elastic scattering

$$M_{ij}M_{ij}^{(0)} = |M_{ij}^{(0)}|^2 \left( 1 + \tilde{g}_s^2 M_{ij}^{(1)} + \tilde{g}_s^4 M_{ij}^{(2)} + O(\tilde{g}_s^6) \right),$$

with $i, j = g, q$,

$$\tilde{g}_s^2 = g_s^2 c_\Gamma \left( \frac{\mu^2}{-t} \right)^\epsilon \quad c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon) \frac{\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}}$$

V. S. Fadin
Peculiarities of the BFKL approach
and compare the results of using directly calculated amplitudes and the Reggeized amplitudes

\[ M_{ij} = \left[ g_s T_i C^i (p_a, p_a') \right] \frac{s}{t} \left[ \left( \frac{-s}{-t} \right) \omega(t) + \left( \frac{s}{-t} \right) \omega(t) \right] \left[ g_s T_j C^j (p_b, p_b') \right] \]

\[ + \kappa \left[ g_s T_i C^i (p_a, p_a') \right] \frac{s}{t} \left[ \left( \frac{-s}{-t} \right) \omega(t) - \left( \frac{s}{-t} \right) \omega(t) \right] \left[ g_s T_j C^j (p_b, p_b') \right] , \]

where \( \kappa \) is zero for \( gg \) and \( gq \) scattering and

\[ \kappa = \frac{N_c^2 - 4}{N_c^2} \]

for quark-quark scattering.
Violation of the Regge factorization in the NNLLA

They use the perturbative expansion

\[ \omega(t) = \tilde{g}_s^2 \omega^{(1)}(t) + \tilde{g}_s^4 \omega^{(2)}(t) \]

\[ C^i = C^{i(0)}(1 + \tilde{g}_s^2 C^{i(1)} + \tilde{g}_s^4 C^{i(2)}) \]

Then the one-loop coefficient

\[ M_{ij}^{(1)} = \omega^{(1)}(t) \ln \left( \frac{s}{-t} \right) + C^{i(1)} + C^{j(1)} - i \frac{\pi}{2} \left( 1 + \kappa \frac{N^2 - 4}{N^2} \right) \omega^{(1)}(t) , \]

and two-loop
\[ M^{(2)}_{ij} = \frac{1}{2} \left( \omega^{(1)}(t) \right)^2 \ln^2 \left( \frac{s}{-t} \right) + \ln \left( \frac{s}{-t} \right) \]
\[ \times \left[ \omega^{(2)}(t) + \left( C^{i(1)} + C^{j(1)} \right) \omega^{(1)}(t) - i \frac{\pi}{2} \left( 1 + \kappa \frac{N^2 - 4}{N^2} \right) \left( \omega^{(1)}(t) \right)^2 \right] \]
\[ + \left[ C^{i(2)} + C^{j(2)} + C^{i(1)} C^{j(1)} - \frac{\pi^2}{4} \left( 1 + \kappa \frac{N^2 - 4}{N^2} \right) \left( \alpha^{(1)}(t) \right)^2 \right] \]
\[ - i \frac{\pi}{2} \left( 1 + \kappa \frac{N^2 - 4}{N^2} \right) \left[ \alpha^{(2)}(t) + \left( C^{i(1)} + C^{j(1)} \right) \alpha^{(1)}(t) \right]. \]
The interference of the tree- and two-loop amplitudes for each of the parton-parton scattering processes have been explicitly computed.


After expansion in $\epsilon$ and taking the high-energy limit, the interference between the $n$-loop and the tree amplitudes for $ij$ elastic scattering has the form

$$\operatorname{Re} \left( M^{(0)*} M^{(n)} \right)_{ij} = |M^{(0)}|_{ij}^2 \tilde{g}_s^{2n} \sum_{m=0}^{n} B_{ij}^{nm} \ln^m \left( -\frac{s}{t} \right),$$

$$B_{00}^{ij} = 1.$$
The discrepancy appears in non-logarithmic two-loop terms. If the Reggeization would be correct in the NNLLA, they should satisfy the equation

\[ B_{qg}^{20} - \frac{1}{2} \left( B_{qg}^{10} \right)^2 - \frac{1}{2} \left[ B_{g}^{20} - \frac{1}{2} \left( B_{g}^{10} \right)^2 \right] - \frac{\pi^2}{2\epsilon^2} \left( N^2 - 4 \right) = 0. \]
The explicit calculation of the $B_{ij}$ coefficients gives that this relation is violated by terms of $\mathcal{O}(\pi^2/\epsilon^2)$,

$$B_{20}^{gg} - \frac{1}{2} \left( B_{10}^{gg} \right)^2 - \frac{1}{2} \left[ B_{20}^{gg} - \frac{1}{2} \left( B_{10}^{gg} \right)^2 + B_{20}^{qq} - \frac{1}{2} \left( B_{10}^{qq} \right)^2 \right]$$

$$- \frac{\pi^2}{2\epsilon^2} \left( N^2 - 4 \right) = \frac{3\pi^2}{\epsilon^2} \left( \frac{N^2 + 1}{N^2} \right) + \mathcal{O}(\epsilon).$$
Detailed consideration of the terms responsible for the Regge factorization breaking in the case of two-loop and three-loop quark and gluon amplitudes in QCD was performed by V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, 2014. In particular, the non-logarithmic double-pole contribution at two-loops was recovered and all non-factorizing single-logarithmic singular contributions at three loops were found using the techniques of infrared factorization.
Violation of the Regge factorization in the NNLLA

For comparison of Regge and infrared factorizations the representation of scattering amplitudes

\[ \mathcal{M}^{[8]}_{rs} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) = 2\pi \alpha_s H^{(0)}_{rs}^{[8]} \]

\[
\times \left\{ C_r \left( \frac{t}{\mu^2}, \alpha_s \right) \left[ A_+ \left( \frac{s}{t}, \alpha_s \right) + \kappa_{rs} A_- \left( \frac{s}{t}, \alpha_s \right) \right] C_s \left( \frac{t}{\mu^2}, \alpha_s \right) \right. \\
+ \left. \mathcal{R}^{[8]}_{rs} \left( \frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s \right) \right\}, \quad \kappa_{gg} = \kappa_{qg} = 0, \quad \kappa_{qq} = \frac{4 - N_c^2}{N_c^2},
\]

was used. \( H^{(0)}_{rs}^{[8]} \) represents the tree-level amplitude. A non-factorizing remainder function \( \mathcal{R}_{rs} \) was introduced.
The results obtained:

\[ R_{qq}^{(2),0,[8]} = \frac{\pi^2}{4\epsilon^2} \left( 1 - \frac{3}{N_c^2} \right) \left( 1 - \epsilon^2 \zeta(2) \right), \]

\[ R_{gg}^{(2),0,[8]} = -\frac{3\pi^2}{2\epsilon^2} \left( 1 - \epsilon^2 \zeta(2) \right), \]

\[ R_{qg}^{(2),0,[8]} = -\frac{\pi^2}{4\epsilon^2} \left( 1 - \epsilon^2 \zeta(2) \right). \]

The factor \((1 - \epsilon^2 \zeta(2))\) can be absorbed in the constant \(c_\Gamma^2\) by performing the expansion in terms of \(\bar{\alpha}_s = \alpha_s c_\Gamma\), instead of using \(\alpha_s\).
Violation of the Regge factorization in the NNLLA

\[
R^{(3),1,[8]}_{qq} = \left( \frac{\alpha_s}{\pi} \right)^3 \frac{\pi^2}{\epsilon^3} \left( \frac{2N_c^2 - 5}{12N_c} \right) \left( 1 - \frac{3}{2} \epsilon^2 \zeta(2) \right) + \mathcal{O}(\epsilon^0),
\]

\[
R^{(3),1,[8]}_{gg} = -\left( \frac{\alpha_s}{\pi} \right)^3 \frac{\pi^2}{\epsilon^3} \left( \frac{2}{3} N_c \right) \left( 1 - \frac{3}{2} \epsilon^2 \zeta(2) \right) + \mathcal{O}(\epsilon^0),
\]

\[
R^{(3),1,[8]}_{qg} = -\left( \frac{\alpha_s}{\pi} \right)^3 \frac{\pi^2}{\epsilon^3} \left( \frac{N_c}{24} \right) \left( 1 - \frac{3}{2} \epsilon^2 \zeta(2) \right) + \mathcal{O}(\epsilon^0),
\]
Another difficulty in development of the BFKL approach in the NNLLA is necessity to account imaginary parts of amplitudes in the unitarity relations. In the simplest two-particle intermediate state:

\[ p_A p_A' \quad p_B p_B' \]

In imaginary parts, one \( \ln s \) is lost. Products of imaginary and real parts in the unitarity relations cancel due to summation of contributions complex conjugated to each other. Due to this, imaginary parts don’t play any role in the NLLA. But they become important in the NNLLA.
Attenuating circumstance:
For the amplitudes in the unitarity relation, LLA is sufficient.

Nevertheless, account of imaginary parts leads to great changes.
Remind that for two Reggeized gluons in the \( t \)-channel in QCD, that is for tree colours, there are 6 irreducible representations:

\[ 1, 8_a, 8_s, 10, 10, 27. \]

For \( N_c > 3 \) there are 7 possible representations. The representations \( 8_a, 10, 10 \) are anti-symmetric the representations \( 1, 8_s, 27 \) and the extra one are symmetric.
In real parts, with the NLLA accuracy, only the Reggeon channel, $8_a$, is important.

It provides universality of the NLLA:

$gg$, $qg$ and $qq$ scattering can be considered in an unique way.

But account imaginary parts violate the universality.

$gg$ scattering amplitudes can contain all the representations, while $qg$ and $qq$ scattering amplitudes only $1, 8_a, 8_s$. 
Consideration of many-particle states in the unitarity condition

\[ \sum_{n} \]

is even more complicated problem.
The basis of the BFKL approach is the remarkable property of QCD – gluon Reggeization.

The Reggeization hypothesis is extremely powerful: all scattering amplitudes are expressed in terms of the gluon trajectory and several Reggeon vertices.

In the LLA and NLLA the Reggeization means a simple factorized form for real parts of amplitudes with gluon exchanges, which is proved to be true.

This simple form is broken in the NNLLA.

In the NNLLA one has to account also imaginary parts of multiparticle amplitudes in the unitarity relations.