Multiloop running/evolution in QCD and beyond

Konstantin Chetyrkin

Guido Altarelli Memorial session

5-th International Conference on New Frontiers in Physics

AOK, Kolymbari, Crete, Tuesday, July 12, 2016

Guido Altarelli (from arXiv:0804.4147):

“Since α_s is not too small, $\alpha_s(M_Z) \approx 0.12$, the need of high oder perturbative calculations, resummation of logs at all orders etc. is particularly acute. Ingenious new computational techniques and software have been developed and many calculations have been realized that only a decade ago appeared as impossible.”

In fact, these words are equally well applicable today (8 years later). I will try to demonstrate you many significant advances of $\approx$ last ten years which have happened in a rather small subset of the vast field of pQCD.

Namely, I will discuss higher order (multiloop) analytical calculations of various RG-functions and 2-point (massless) correlators (the later are tigtly related to the former).

Many of them (but not all) were done by the Karlsruhe-Moscow group:

Pavel Baikov (Moscow State University)

Johan Kühn (KIT) and myself (KIT).

running $\equiv$ evolution

running: quark-gluon c.c. $\alpha_s(\mu)$, a quark mass, say, $m_b(\mu)$ or a /local/ operator $O(x)$/

It is governed by so-called RG-functions: $\beta(\alpha_s)$, $\gamma_m(\alpha_s)$ or $\gamma_O(\alpha_s)$ /for a /local/ operator $O(x)$/

In good (read: *minimal* renormalization schemes) RG-functions are just series (with constant coefficients) in $\alpha_s \rightarrow$ (relatively) simple to compute analytycally

The running of $\alpha_s(\mu)$ and $m_q(\mu)$ is extremely important in QCD:

the fact that $\beta_0 =$ is negative directly leads to the famous property of asymptotic freedom of the QCD c.c. and, as a direct consequence, to the very possibility (along with the factorization) to employ PT to the high-energy QCD processes

The status of the QCD β-function: until just **two weeks** ago it was as summarized by Guido Altarelli (in arXiv:0804.4147):

"The QCD beta function that fixes the running coupling is known in QCD up to 4 loops in the MS or $\overline{\text{MS}}$ definitions and the expansion is well behaved. The 4-loop calculation [$\star$] involving about 50.000 4-loop diagrams is a great piece of work."

The quark mass AD γ_m in 4 loops is also analytically known since 1997 [$\star\star$]

NOW: *Five-Loop Running of the QCD coupling constant*, by Baikov, K.Ch and Kühn, arXiv:1606.08659, Jun 28, 2016. (About one and a half million of 5-loop diagrams contribute!)

Main result (details will be discussed later):

The expansion is still *very* well behaved!. No drastic effects (in contrary some earlier predictions/expectations)

The 5-loop γ_m is also quite well behaved as known since recently from:

Quark Mass and Field Anomalous Dimensions to $\mathcal{O}(\alpha_s^5)$, by Baikov, K.Ch and Kühn, arXiv:1402.6611, Feb 26, 2014

$\star$ and $\star\star$ J.A.M. Vermaseren, S.A. Larin, T. van Ritbergen (1997); $\star\star$ K. Ch. (1997)

$\star$ was also checked by M. Czakon (2005);

evolution (from running!, see the end of the page): of a parton density via the famous

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation

$$\frac{dq^{\mathsf{NS}}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi}\int_o^1 \frac{dy}{y} q^{\mathsf{NS}}(y,t)\, P(\frac{x}{y})$$

The *splitting function* $P(z)$ is related with *anomalous dimensions* A_N^{NS} of the whole tower of (non-singlet) spin N, twist two operators O_N

$$\int dz z^{N-1} P(z) = A_N^{\mathsf{NS}}, \quad O_n = \overline{\psi}\gamma_{\mu_1}\mathcal{D}_{\mu_2}\mathcal{D}_{\mu_3}\ldots\mathcal{D}_{\mu_n}\psi, \quad P = P_1\alpha_s + P_2\,\alpha_s^2 + P_3\,\alpha_s^3 + \ldots$$

Untill very recently the status of the P splitting function was as expressed by Guido Altarelli (from his paper in scholarpedia):

"in recent years the next-next-to-leading results (that is those for P_3) have been first derived in analytic form for the first few moments and, then the full analytic calculation, a really monumental work, was completed in 2004 by Moch, Vermaseren and Vogt (2004) and Vogt, Moch and Vermaseren (2004)."

Technically, P_3 was constructed from the the AD's A_N^{NS} computed for a generic value of spin N (starting from 3 loops this the only currently available way). All techniques developed for anomalous dimensions should also work for the splitting function! But generic spin induces extra **(and quite significant!) complications**. But, we do do'nt know better way . . .

Less then 2 months ago:

First Forcer results on deep-inelastic scattering and related quantities, B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies, A. Vogt, arXiv:1605.08408, May 26, 2016.

here the 4-loop terms in P_4 have been computed for few fist values of spin N for the non-singlet case as well as the fully analytic N-dependence for of the n_f^3 and n_f^2 parts of P_4

Motivations:

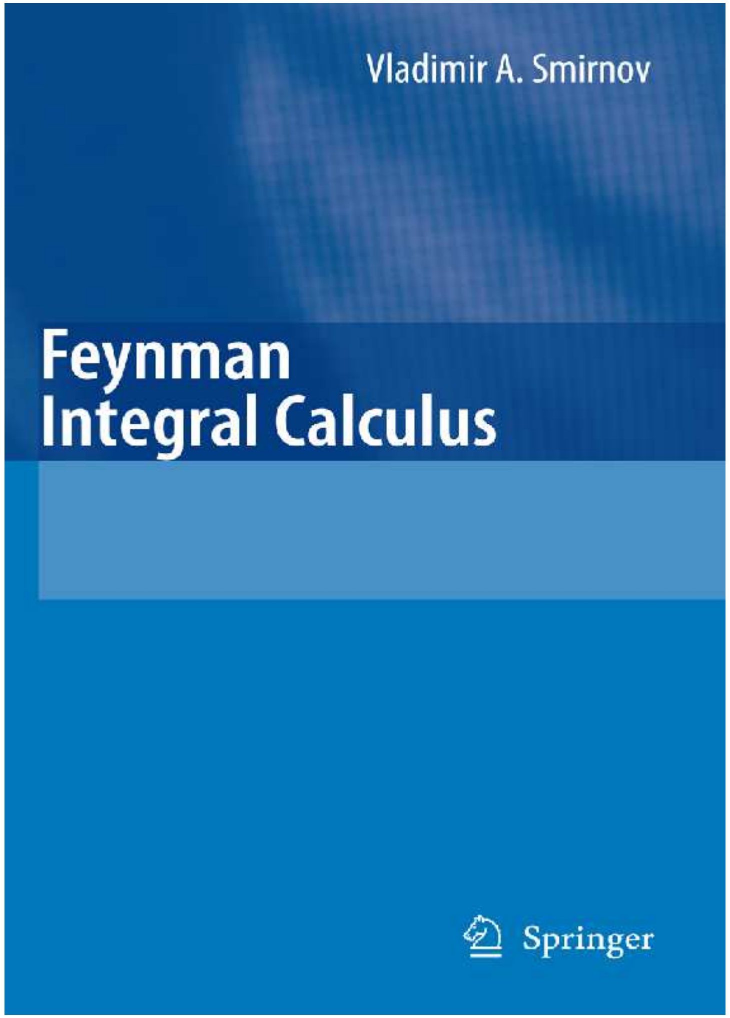
there are cases where the next order, NNNLO, is of interest due to (a) very high requirements on the theoretical accuracy, such as in the determination of the strong coupling constant α_s from deep-inelastic scattering (DIS) or (b) a slow convergence of the perturbation series, such as for Higgs production in proton-proton collisions . . .

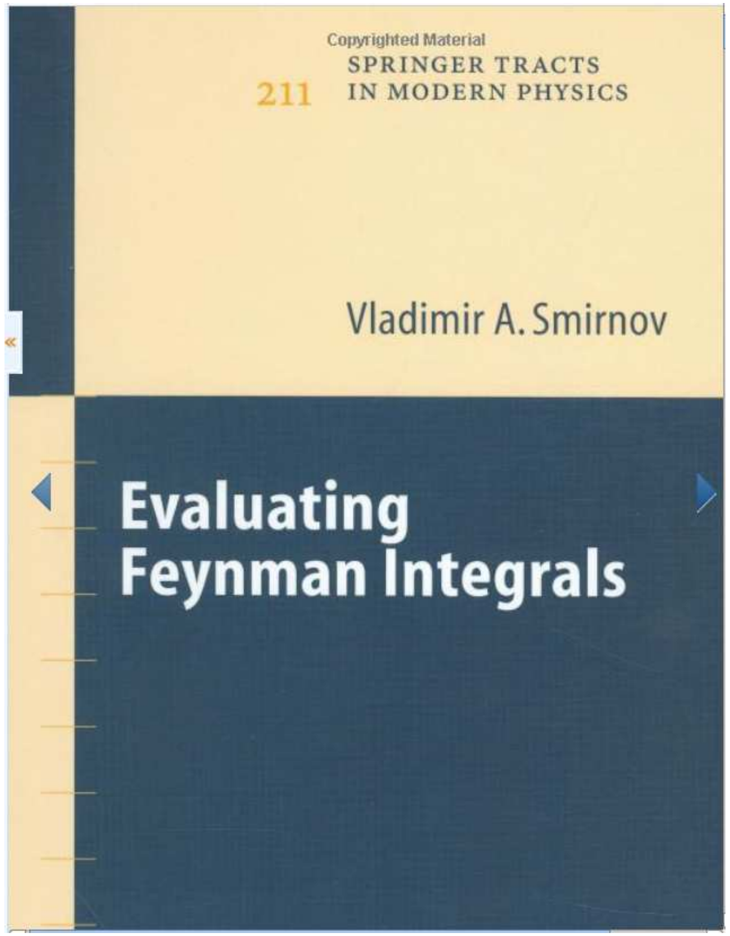
Last 30 years $\rightarrow$ revolution in our ability to deal with multiloop Feynman Integrals

Main (but not all) ingredients:

- Dim. Reg. /G. t' Hooft & M. Veltman (72)/
- IBP method (see below) + Laporta's way of solution of IBP identities
- effective theory, Eucliudean and non-Eucldean expansion of FI's, method of regions (M. Beneke & V. Smirnov (1998) + . . .)
- "IR-reduction" $\longrightarrow$ most useful trick to automatically reduce # of loops by one in computing Z-factors (read **any** β-function and anomalous dimension in **any** theory) /see below/
- Computer Algebra: from legendary SCHOONSCHIP (M. Veltman) to Mathematica and, **especially, FORM (J. Vermaseren, . . .) and its thread version /effective parallelization!/**

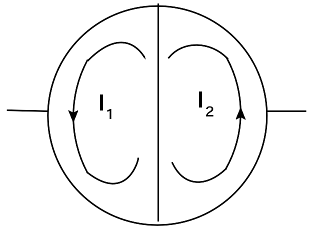
Now: Evaluation of FI = established, fastly developing part of math. physics





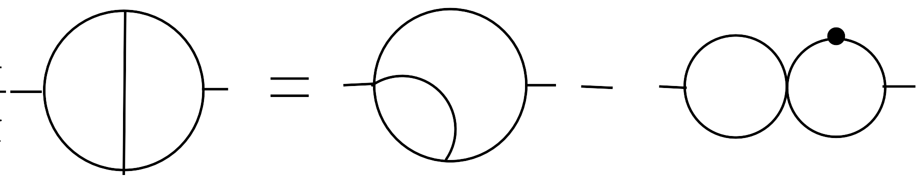
The main breakthrough happened 35 years ago with discovery of the method of integration by parts (IBP) of DR integrals. Here is (now a textbook) example:

$$\mathbf{0} = \int d^D \ell_1 \; d^D \ell_2 \; \frac{\partial}{\partial \ell_1^{\alpha}} (\ell_1 - \ell_2)_{\alpha}$$



which is equivalent to the **exact** D-dimensional equality:

$\frac{1}{\epsilon}$



At one loop, IBP (for DR integrals) was used in [*], a crucial step — an appropriate modification of the *integrand* before differentiation was undertaken first [**] (in position space, 2 loops) and in [***] (in momentum space, 2 and 3 loops)

[*] G. 't Hooft and M. Veltman (1979)
[**] A. Vasiliev, Yu. Pis'mak and Yu. Khonkonen (1981)
[***] F. Tkachov (1981); K. Ch. and F. Tkachov (1981)

COMMENTS on IBP relations

- **IBP identities are *exact ones* valid for general Feynman amplitudes,**
  (with the seagull identity discusssed in the talk of Joannis Papavassiliou being a simple example)
- **IBP identities relate complicated (for calculations) topologies to simpler ones**
- **For a given class of FI's there exist only *finite* very limited number of (further irreducible) so-called *master* integrals**
- **As a result: "Since then IBP relations evolved into a fantastically universal and efficient method for reducing all integrals of a given topology to a few *master integrals*"** ⋆

- **IBP reduction greatly helps to find masters, e.g. the differential equations method (A. Kotikov (1991) ... E. Remiddi ... J. Henn (2013) ...)**

⋆ A. Grozin, Published in Int.J.Mod.Phys. A27 (2012) 1230018

COMMENTS on “IR-reduction”

- “IR-reduction” is a method to maximally simplify evaluation of UV Z-factors (read RG-functions) by modifying the low energy behaviour of of FI’s (not essential for UV behaviour). It uses so-called R^*-operation which allows to subtract (recursively) *both* UV *and* from any (euclidian) FI. It reduces evaluation of an UV Z-factor at (L+1) loop level in any theory to calculation of a set of massless propagator-like L-loop FI’s (p-integrals) /K.Ch., V. Smirnov, (1984)

- In fact, the existence of such generalization was envisaged by Giorgio Parisi (in a quite diffrent context):
  ”We conjecture that there is an extension of the Bogoliubov-Parasiuk-Hepp theorem which copes also with inrared divergences” / *”On infrarred divergences”*, NPB 150 (1979) 163)/

- IR-reduction is a generalization of a method suggested by A. Vladimirov (JINR, 1979)

COMMENTS on “p-integrals”

- **current theoretical status: any four-loop p-integral is analytically computable.** (This is one of results of 15 years of research and development /2001 – 2016/ by K-M group).

- **Together with the IR reduction it leads to analytical calculability of RG-functions in any model at the 5-loop level.**

- The previous level: three loop p-integrals were “in making” for about a decade: 1979 -1991

- There are three (different!) algorithms with corresponding computer implementations (only one is public):

  BAICER (by Pavel Baikov, 2001 – , FORM) uses $1/D$-expansion for reduction to masters

  LiteRed (by Roman Lee, 2012, Mathematica) uses explicit solution of IBP relations for reduction to masters

  FORCER (by Takashiro Ueda, Ben Ruijl and Jos Vermaseren, 2016, FORM) uses explicit solution of IBP trelations for reduction to masters

Massless correlators from p-integrals: a technical note

Starting object: the correlator of two currents $\boldsymbol{j} = \bar{\boldsymbol{q}}\boldsymbol{\Gamma}\boldsymbol{q}$ and $\boldsymbol{j}^\dagger$

$$\boldsymbol{\Pi}(\boldsymbol{q}^2 = -\boldsymbol{Q}^2) = \boldsymbol{i}\int \mathrm{d}\boldsymbol{x}\boldsymbol{e}^{\boldsymbol{iqx}}\langle \mathbf{0}|\boldsymbol{T}[\ \boldsymbol{j}(\boldsymbol{x})\boldsymbol{j}^\dagger(\mathbf{0})\ ]|\mathbf{0}\rangle$$

related to the corresponding absorptive part (which we are interested in) through $R(s) \approx \Im\,\Pi(s)$. In massless limit of PT $\Pi(Q^2)$ is just a combination of $\ln^n(\mu^2/Q^2)$ with *constant coefficients* and the constant terms ($n = 0$) do not contribute to R(s).

$\Pi(Q)$ **is not completely physical due to a divergency of** $T(j(x)j^\dagger(0))$ **at** $x \to 0$**, as a result the corresponding evolution equation read**$\star$($a_s \equiv \alpha_s/\pi$)

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s)a_s\frac{\partial}{\partial a_s}\right)\Pi = \gamma^{jj}(a_s)$$

At first sight, it would be advantageous to avoid this by considering (obviously RG invariant!) Adler function defined as $D = Q^2\frac{\partial}{\partial Q^2}\frac{\Pi(Q)}{Q^2}$

BUT, this is not true!

$\star$we assume that j is scale-invariant, or made such by a proper factor

For massless $(L+1)$ loop Π RG equation amounts to

$$\frac{\partial}{\partial L}\Pi = \gamma^{jj}(a_s) - \left(\beta(a_s) a_s \frac{\partial}{\partial a_s}\right)\Pi$$

anom.dim. at a_s^L (L+1) loop integrals most complicated part of calculations

L-loop integrals only contribute due to the factor of $\beta(a_s)$

- **to find Log-dependent part of Π at (L+1)-loops one should only to know (L+1)-loop anomalous dimension γ^{jj} and only L-loop Π (BUT! including its constant part)**
- **(L+1) loop anom.dim. reducible to L-loop p-integrals (via IR-reduction)**

As a result, (L+1)-loop R(s) $\Longleftrightarrow$ L-loop $\Pi(Q)$ and /usually significantly more complicated/

(L+1)-loop AD γ^{jj}

However, final result is more convenient to present in terms of Adler function and R(s) (also for summing higher RG-logs due to their scale invariance)

Armed with BAICER and IR-reduction our group has carried out a number of NNNLO (5-loop) fully analytical calculations (since 2004 till now). Among them:

SS correlator describing the total rate of the Higgs into $q\bar{q}$ pairs:

$$\Gamma(H \to \bar{f}f) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_f^2(\mu)\, R^S(s = M_H^2, \mu)$$

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4 \\ &= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140 \end{aligned}$$

with $a_s = \alpha_s/\pi = 0.0360$**,** $M_H = 125$ **GeV and** $\alpha_s(M_Z) = 0.118$**)**

VV and AA correlators: relevant for the Adler function, $R(s)$ ratio, Z- and tau-decays into hadrons

Let me cite again Guido Altarelli (from arXiv:1303.2842):

" But by now the 4-th term (NNNLO!) has also been computed for inclusive hadronic Z and τ decays. This remarkable calculation of about 20.000 diagrams, for the inclusive hadronic Z width ...”

These results of the K-M group +some other NNNLO ones (Higgs decays into gluons, the QCD β-function, various DIS sum rules like the Bjorken one, etc. etc.) can be found in:

Phys.Rev.Lett. 88 (2002) 01200
Phys.Rev.Lett. 95 (2005) 012003
Phys.Rev.Lett. 96 (2006) 012003
Phys.Rev.Lett. 97 (2006) 061803
Phys.Rev.Lett.101:012002,2008
Phys.Rev.Lett. 102 (2009) 212002
Phys.Rev.Lett.104:132004,2010
Phys.Rev.Lett. 108 (2012) 222003 plus some more publications in other journals . . .

We are jumping now to the most last

(and most demanding in calculational sense)

"the cherry on top of the cake":

the QCD β-function in five loops

QCD β-function in FIVE loops: result

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s = \beta(a_s)\, a_s, \quad a_s \equiv \frac{\alpha_s}{\pi}, \quad \beta(a_s) = \sum_{i \geq 0} \beta_i\, a_s^{i+1}$$

$$\begin{aligned}
4^5\,\beta_4 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right]}
\end{aligned}$$

n_f^4 term is in FULL AGREEMENT with the 20 years old result by John Gracey (in the framework of the conformal bootstrap method of A. Vasiliev, Yu. Pis'mak and J. Honkonen (1981))

n_f^3 term is in FULL AGREEMENT with the the very recent (general gauge group!) calculation by Th. Luthe, A. Maier, P. Marquard and Y. Schröder (arXiv:1606.08662)

QCD β-function in FIVE loops: Zeta's

In general any 5-loop beta in any theory will have the following "transcendental structure" (an obvious outcome of our knowledge of the corresponding masters)

1 and 2 loops: rational

3 loops: rationals + ζ_3

4 loops: rationals + ζ_3 + ζ_4 + ζ_5

5 loops: rationals + ζ_3 + ζ_4 + ζ_5 + ζ_5 +ζ_7

$$\beta_0 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\},$$

$$\beta_3 = \frac{1}{4^4}\left\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right\}$$

The one-loop-delayed appearance of zeta's (well-known at 3 and 4 loops) shows itself also at 5 loops. Any explanation is missing, indeed!

$$
\begin{aligned}
4^5\,\beta_4 =& \ \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f\left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\
+ \quad & n_f^2\left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\
+ \quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]}
\end{aligned}
$$

QCD β-function in FIVE loops: Numerics

$$
\begin{aligned}
-\beta &= a_s\,(2.75 - 0.166667\,n_f) + a_s^2\,(6.375 - 0.791667\,n_f) \\
&+ a_s^3\,(22.3203 - 4.36892\,n_f + 0.0940394\,n_f^2) \\
&+ a_s^4\,(114.23 - 27.1339\,n_f + 1.58238\,n_f^2 + 0.0058567\,n_f^3) \\
&+ a_s^5\,\underbrace{(\mathbf{524.56 - 181.8\,n_f + 17.16\,n_f^2 - 0.22586\,n_f^3 - 0.0017993 n_f^4})}_{\beta_4}
\end{aligned}
\quad (1)
$$

It is instructive to compare β_4 with a (20 years old!) prediction by J. Ellis, I. Jack, D.R.T. Jones, M. Karliner, M.A. Samuel, "Asymptotic Pade approximant predictions: Up to five loops in QCD and SQCD", Phys. Rev. (1998).

$$
\beta_4^{\mathbf{APAP}} = \mathbf{740 - 213\,n_f + 20\,n_f^2} - \mathbf{0.0486\,n_f^3} - \boxed{0.0017993\,n_f^4}
$$

QCD β-function in FIVE loops: Numerics

Unfortunately, this strikingly good agreement does not always survive for fixed values of n_f:

n_f	0	1	2	3	4	5	6
$\beta_4^{\rm exact}$	525	360	228	127	57	15	0.27
$\beta_4^{\rm APAP}$	741	548	395	281	205	169	170

due to severe cancellations between different powers of n_f

Very similar picture for the quark mass anomalous dimension:

$$\gamma_4^{exact} = 559.71 - 143.6\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4$$

$$\gamma_4^{APAP} = 530 - 143\, n_f + 6.67\, n_f^2 + 0.037\, n_f^3 - \boxed{0.00008535\, n_f^4}$$

QCD β-function in FIVE loops: Apparent Convergence

It is instructive to consider the properly normalized $a_s \overline{\beta} \equiv \frac{\beta}{-\beta_0}$:

$$
\begin{aligned}
\overline{\beta}(n_f = 0) &= 1. + 2.32\, a_s + 8.12\, a_s^2 + 41.54\, a_s^3 + 190.75\, a_s^4 \\
\overline{\beta}(n_f = 1) &= 1 + 2.16\, a_s + 6.99\, a_s^2 + 34.33\, a_s^3 + 139.23\, a_s^4 \\
\overline{\beta}(n_f = 2) &= 1 + 1.983\, a_s + 5.776\, a_s^2 + 27.45\, a_s^3 + 94.24\, a_s^4 \\
\overline{\beta}(n_f = 3) &= 1 + 1.78\, a_s + 4.47\, a_s^2 + 20.99\, a_s^3 + 56.59\, a_s^4 \\
\overline{\beta}(n_f = 4) &= 1 + 1.54\, a_s + 3.05\, a_s^2 + 15.07\, a_s^3 + 27.33\, a_s^4 \\
\overline{\beta}(n_f = 5) &= 1 + 1.261\, a_s + 1.47\, a_s^2 + 9.83\, a_s^3 + 7.88\, a_s^4 \\
\overline{\beta}(n_f = 6) &= 1 + 0.93\, a_s - 0.29\, a_s^2 + 5.52\, a_s^3 + 0.15\, a_s^4
\end{aligned}
$$

We see very modest growth of the coefficients, that is (apparent) convergence is better than one would expect (from comparison with other examples)

Instead of conclusion: few words for "running beyond QCD"

- ϕ^4 model is significantly simpler than QCD $\Longrightarrow$ 5-loop β-function is known since about 30 years, the SIX-loop (!) one has been published less than 2 weaks ago by Mikhail Kompaniets and Erik Panzer, arXiv:1606.09210, Jun 29, 2016. (Interesting applications for critical indexes . . .).

- During last 3-4 years there were significant advances in computing β-functions for the SM (in the unbroken phase), especially for the Higgs self-coupling Λ. The motivation: investigation of the stability of the SM vacuum state at large scales in dependence on Higgs and top masses and other SM parameters following the lines of important earlier works Cabbibo, Maiani, Parisi, Petronzio (1979) and by Altarelli, Isidor (1994), . . . :

  Stability of SM vacuum $\Leftrightarrow \lambda(\Lambda) > 0$

  (Λ: scale up to which the SM is valid)

  Three loops are known completely, some leading terms also at four loops. See, the next /final/ slide.

- Theoretically, full four loop calculations should be certainly possible, but there is still not completly understood issue with the "γ_5-problem" (anomaly cancellation in $D \neq 4$)

SM Running: State of the art

2 loop

[M. Fischler, C. Hill (1981); D. Jones (1982); M. Machacek, M. Vaughn (1983,1984,1985); I. Jack, H. Osborn (1984,1985)] [M. Fischler, J. Oliensis (1982); M. Machacek, M. Vaughn (1984); C. Ford, I. Jack, D. Jones (1992); M. Luo, Y. Xiao (2003)]

3 loop

- for gauge couplings g_1, g_2, g_s [L. Mihaila, J. Salomon, M. Steinhauser (2012); A. Bednyakov, A. Pikelner, Velizhanin (2012)]

- for Yukawa couplings y_t, y_b, y_τ, etc. [K. Ch. , M.Zoller (2012); A. Bednyakov, A. Pikelner, Velizhanin (2013)]

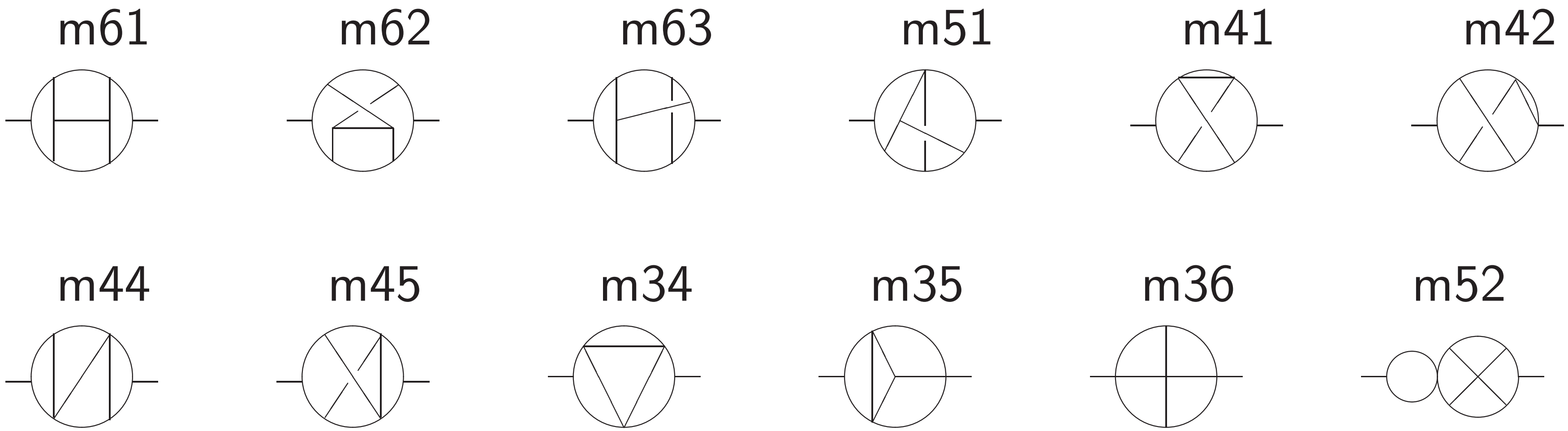
- for the Higgs self-coupling λ (and the mass parameter m^2) [K. Ch. , M.Zoller. (2012 and 2013); A. Bednyakov, A. Pikelner, Velizhanin (2013)]

4 loop

- $\beta_{g_s}(g_s, y_t, \lambda)$ [A. Bednyakov, A. Pikelner (2015); M. Zoller, (2015-2016)]

Important for RG-claculations:

for all non-trivial 4-loop massless propagators (masters)



about 10 terms of ϵ expansion (up to the transcendentality level 12!) have been computed first numerically (around **500 significant** digits!) and then their full analytical structure has been reconstructed (V. Smirnov & R. Lee, 2011).

Tools: IBP + recurrence relations in the space-time dimension D (O. Tarasov,1996) + a lot of ingenuity