Renormalization group analysis of a turbulent compressible fluid near $d = 4$: Crossover between local and non-local scaling regimes.

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Key objects

We will consider the **passive** advection of the **vector** (impurity) field by the **turbulent** flow.

In fact, we will apply **quantum field theory** techniques to the problem of **statistical physics**.

Key points of the work:

- Turbulence – we work in the inertial range;
- Stochastic differntial equations – we model a turbulence via random force, which brings energy to the system;
- Renormalization group – we use the quantum field theory techniques;
- Objects of interest – we study inertial range asymptotic behaviour of the correlation functions of composite operators.

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Stochastic differential equations and quantum field theory

Standart problem of stochastic dynamics:

$$\partial_t \phi(x) = U(x,\phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x'),$$

where $\phi(x) = \phi(t,\mathbf{x})$ is a random field, $U(x,\phi)$ is a given t-local functional, $f(x)$ is a random force – with Gaussian distribution, zero mean, and given pair correlator D.

Statement:

such stochastic differential equations are equivalent to the field theoretic models with double number of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and with actions functional

$$S(\widetilde{\phi}) = \frac{1}{2} \int \int dx dx' \phi'(x) D(x,x') \phi'(x') + \int dx \phi'(x) \left[-\partial_t \phi(x) + U(\phi(x)) \right].$$

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Quantum field theory: What does it mean?

The formulation of original stochastic problem via quantum field theory means:

- Statistical averages of random quantities in the stochastic problem $\rightarrow$ functional averages with weight $\exp S(\widetilde{\phi})$;
- Correlation functions, response (on the external force) functions $\rightarrow$ Green's functions of the quantum field theory;
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Fully developed turbulence

The turbulence is characterized by

- Cascades of energy;
- Scaling behaviour with universal "anomalous exponents"
- Intermittency.

The key parameters:

- W and L – power of the external source of energy and integral (external) scale;
- ν and l – viscosity coefficient and dissipation (internal) scale.

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Inertial range $l \ll r \ll L$

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Kolmogorov–Obukhov theory

The equal-time structure functions

$$S_n(\mathbf{r}) = \left\langle \left[v_r(t, \mathbf{x}) - v_r(t, \mathbf{x}') \right]^n \right\rangle ,$$

where v_r is the component of the velocity field along the direction $\mathbf{r} = \mathbf{x} - \mathbf{x}'$.

From the two Kolmogorov hypotesis (independence of L for $L \gg r$ and independence of l for $l \ll r$) it follows, that in the inertial range $l \ll r \ll L$

$$S_n(\mathbf{r}) = C_n \,(Wr)^{n/3}$$

with exact exponents and universal amplitudes C_n.

Anomalous scaling

Due to the imtermittency statistical properties of the velocity are dominated by rare spatiotemporal configurations – the main contributions are given by infrequent, but strong events.

This phenomenon is connected with the strong fluctuations of the energy flux and leads to the violation of the classical K41 theory:

$$S_n(\mathbf{r}) = (Wr)^{n/3}\,(r/L)^{\gamma_n}$$

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Outline

We will consider the stochastic Navier-Stokes equation for a compressible fluid. The task is divided into several steps:

- Definition of the model — stochastic differential equations;
- Field theoretic formulation, diagrammatic technique;
- Renormalization and fixed point, which defining the critical dimensions of the fields and parameters.

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Stochastic equation for a compressible fluid

The Navier-Stokes equation for a compressible fluid can be written in the following form:

$$\rho[\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] = \nu_0[\boldsymbol{\nabla}^2\mathbf{v} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})] + \mu_0\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla} p + \mathbf{f},$$

where ρ is the fluid density, $\mathbf{v}$ is the velocity field, ∂_t is a time derivative $\partial/\partial t$, $\boldsymbol{\nabla}^2$ is the Laplace operator, ν_0 and μ_0 are molecular viscosity coefficients, p is pressure field, and $\mathbf{f}$ is an external field per unit mass.

Taking into account continuity equation and an equation of state between deviations δp and $\delta\rho$ from the equilibrium values and introducing scalar field $\phi = c_0^2 \ln(\rho/\overline{\rho})$ we obtain

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Set of equations:

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \nu_0[\boldsymbol{\nabla}^2\mathbf{v} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})] + \mu_0\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla}\phi + \mathbf{f};$$
$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\phi = c_0^2 \ln(\rho/\overline{\rho}).$$

Condition to the random force:

$$\langle f_i(t,\mathbf{x}) f_j(t',\mathbf{x}') \rangle = \frac{\delta(t-t')}{(2\pi)^d}\int_{k>m} \mathrm{d}^d\mathbf{k}\; D_{ij}(\mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \quad \text{where}$$
$$D_{ij}(\mathbf{k}) = g_{10}\nu_0^3 k^{4-d-y}\Big\{P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})\Big\}.$$

Here $P_{ij}(\mathbf{k})$ and $Q_{ij}(\mathbf{k})$ are transverse and longitudinal projectors.

Action Functional

Our stochastic problem is equivalent to the field theoretic model of the extended set of four felds $\Phi = \{\phi, \phi', \boldsymbol{v}, \boldsymbol{v'}\}$ with action functional

$$\mathcal{S}(\varphi) = \frac{v_i' D^f_{ik} v_k'}{2} + v_i' \Big\{ -\partial_t v_i - v_j \partial_j v_i + \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + u_0 \nu_0 \partial_i \partial_k v_k$$
$$+ \phi' [-\partial_t \phi + v_j \partial_j \phi + v_0 \nu_0 \partial^2 \phi - c_o^2 (\partial_i v_i)].$$

At $d = 4$ there appears an additional divergence, in the Green's function $v'v'$ $\Rightarrow$ the kernel function has to be generalized:

$$D_{ij}(\mathbf{k}) \to g_{10} \nu_0^3 k^{4-d-y} \Big\{ P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k}) \Big\} + g_{20} \nu_0^3 \delta_{ij},$$

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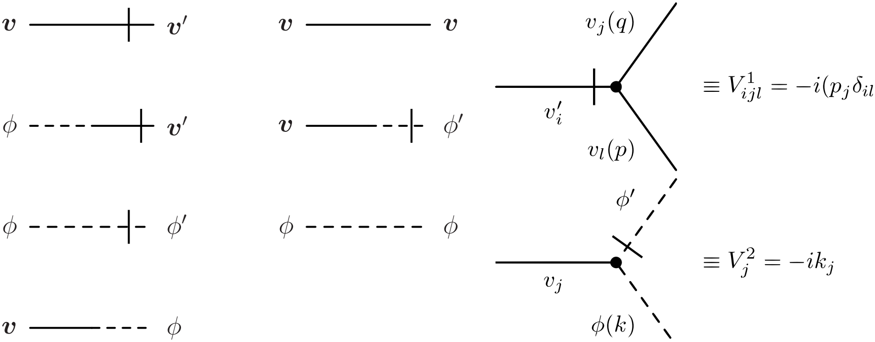
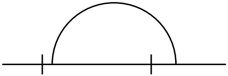
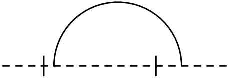
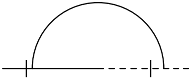
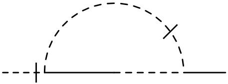
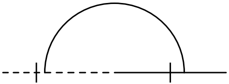


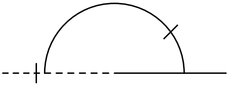
Renormalization constants

$$\Gamma_{v'v} = i\omega - (\delta_{ij}p^2 - p_i p_j)Z_1\nu - p_i p_j Z_2 u\nu +$$  ,

$$\Gamma_{\phi\phi'} = i\omega - p^2 Z_3 v\nu +$$  ,

$$\Gamma_{v'\phi} = -iZ_4 p_i +$$  ,

$$\Gamma_{\phi'v} = -iZ_5 p_i c^2 +$$  $+$  $+$

$+$  .

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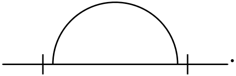

From the relations between renormalized parameters it follows that

$$Z_\nu = Z_1, \quad Z_{g_1} = Z_1^{-3}, \quad Z_u = Z_2 Z_1^{-1}, \quad Z_\phi = Z_4,$$
$$Z_{\phi'} = Z_4^{-1}, \quad Z_v = Z_3 Z_1^{-1}, \quad Z_c = (Z_4 Z_5)^{1/2}, \quad Z_{g_2} = Z_6 Z_1^{-3}. \tag{3}$$

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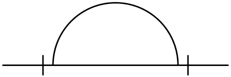
$$+ \frac{1}{2}$$  .

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Fixed points and asymptotic

From remormalization group (RG) it follows, that in the case of one charge the asymptotic behaviour of the invariant charge $\bar{g}$ is

$$\bar{g}\,(s) \cong g^* + const \cdot s^\omega,$$

where $s = 1/\mu r$, μ is the renormalization mass, g^* is fixed point:

$$\beta_g(g^*) = 0.$$

IR asymptotic behaviour $(s \to 0 \Leftrightarrow r \to \infty)$: $\omega = \beta'(g^*) > 0$.

In the case of many charges $\beta_i(g_i^*) = 0$ and $\Omega_{ik} = \partial\beta_i/\partial g_k$ at the point $g_j = g_j^*$ has to be positive.

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Fixed points and asymptotic

Depending of the exponents y and ε the model possesses 4 fixed points:

- Gaussian,

$$g_1^* = 0, \quad g_2^* = 0, \quad u^* \quad \text{and} \quad v^* \quad \text{are undetermined.}$$

  The corresponding eigenvalues of the matrix Ω are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\varepsilon, \quad \lambda_4 = -y.$$

- Local regime,

$$g_1^* = 0, \quad g_2^* = \frac{8\varepsilon}{3}, \quad u^* = 1, \quad v^* = 1.$$

  The eigenvalues of the matrix Ω are

$$\lambda_1 = \frac{7\varepsilon}{18}, \quad \lambda_2 = \frac{5\varepsilon}{6}, \quad \lambda_3 = \varepsilon, \quad \lambda_4 = \frac{3\varepsilon - 2y}{2}.$$

Fixed points and asymptotic

- Non-local regime,

$$g_1^* = \frac{16y(2y-3\varepsilon)}{9(y(2+\alpha)-3\varepsilon)}, \quad g_2^* = \frac{16\alpha y^2}{9(y(2+\alpha)-3\varepsilon)}, \quad u^* = v^* = 1;$$

the required eigenvalues are

$$\lambda_1 = \frac{y[2y(10\alpha+11)-3\varepsilon(3\alpha+11)]}{54[y(2+\alpha)-3\varepsilon]},$$

$$\lambda_2 = \frac{y[2y(2\alpha+3)-\varepsilon(\alpha+9)]}{6[y(\alpha+2)-3\varepsilon]}, \quad \lambda_{3,4} = \frac{A \pm \sqrt{B}}{C},$$

where A, B, C – some functions of ε, y and α.

This point is stable for $y > 0$ and $y > 3\varepsilon/2$.

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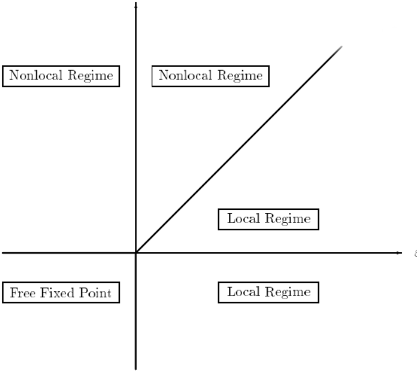
Picture of the IR–attractive fixed points (scaling regimes) on the $y - \varepsilon$ plane.



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- We applied *the field theoretic renormalization group* to the analysis of the stochastic Navier-Stokes equation of a *compressble* fluid;
- The one additional divergent function appears at special space dimension $d = 4$;
- Simple analysis near $d = 3$ shows us only two scaling regimes – Gaussian and non-local, whereas analysis near $d = 4$ providing three stable fixed points in the IR region – Gaussian, local and non-local.
- This means, that the simple analysis around $d = 3$ is *incomplete* in this case.

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Thank you for your attention!