# Analytical and numerical study of Weyl and Dirac Quantum Walks

5th International Conference on New Frontiers in Physics
Orthodox Academy of Crete, Kolymbari, Crete, Greece
July, 9th 2016

**Outline:**

1. Quantum Walk model
2. Dirac and Weyl QWs
3. Path-integral solution

**Authors:**
Giacomo Mauro D'Ariano
Nicola Mosco
Paolo Perinotti
Alessandro Tosini

# Quantum Walks

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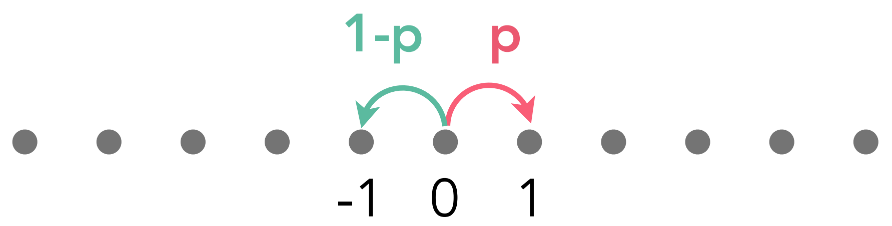
Random Walk:
go right or left with some probability



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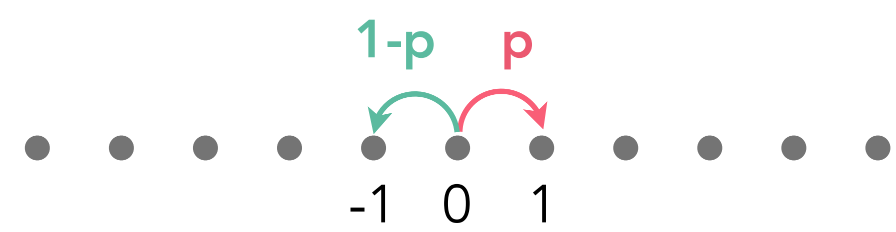
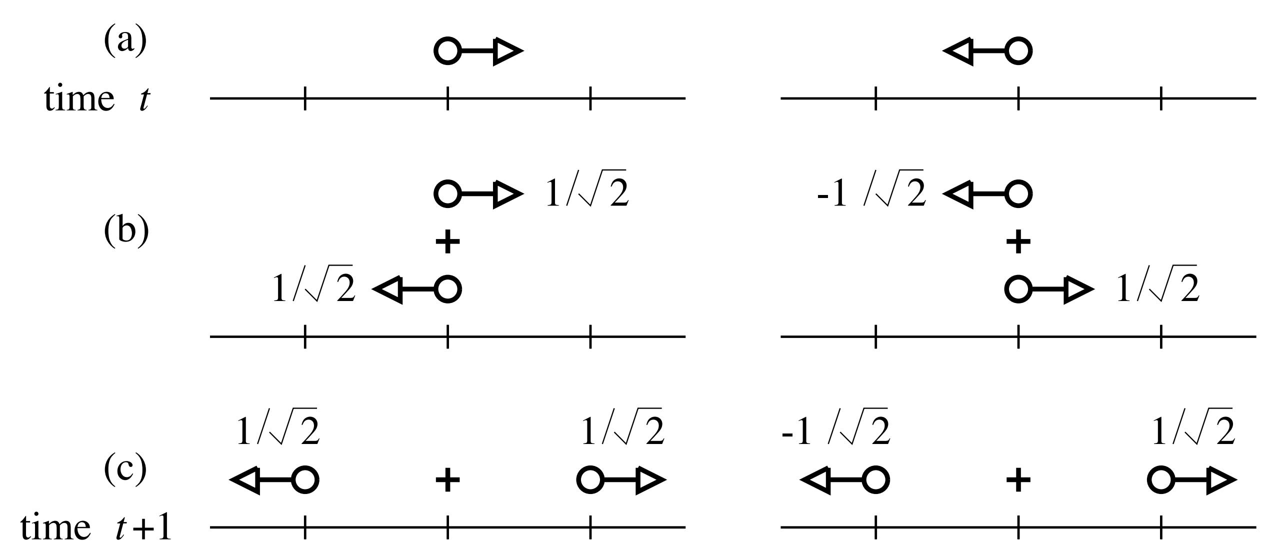



Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A 48(2):1687–1690 (1993)



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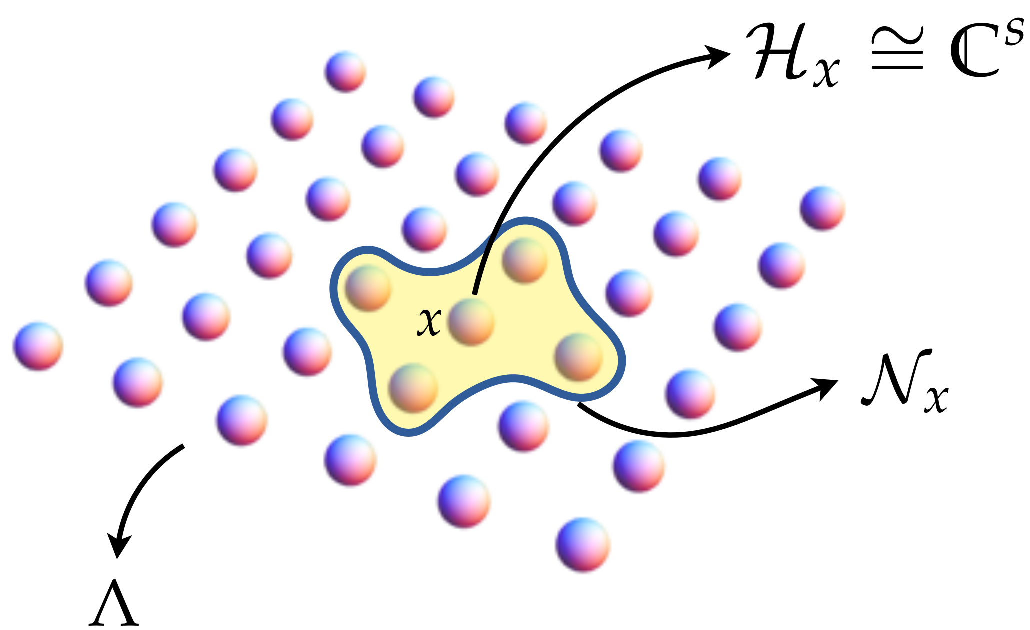


Lattice $\Lambda$

Locality

$\Downarrow$

Finite neighbourhood scheme $\mathcal{N}_x$

D. Aharonov, A. Ambainis, J. Kempe, and U. Vazirani. Quantum walks on graphs. In Proceedings of the thirty-third annual ACM symposium on Theory of computing - STOC '01 (2001)

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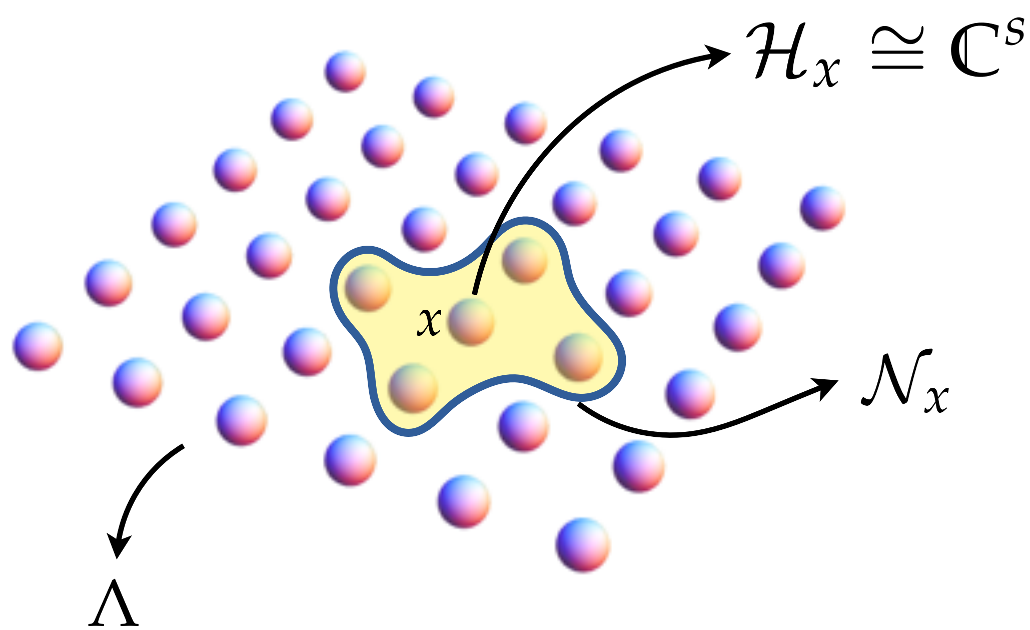


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$$\mathcal{H} = \bigoplus_{x \in \Lambda} \mathcal{H}_x \cong \ell^2(\Lambda) \otimes \mathbb{C}^s$$

$$U : \mathcal{H}_x \longrightarrow \bigoplus_{y \in \mathcal{N}_x} \mathcal{H}_y$$

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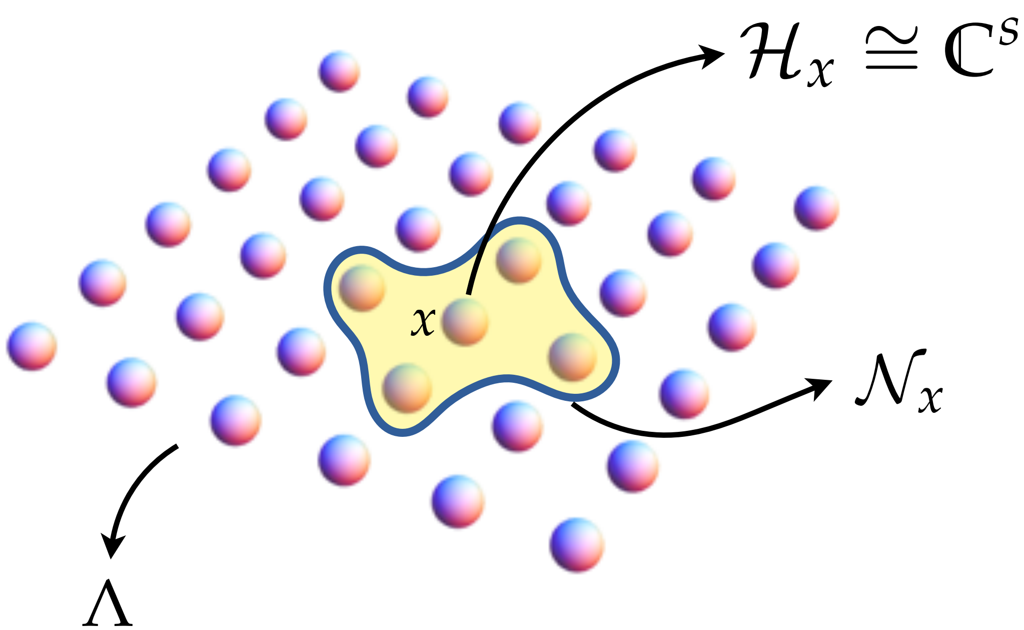


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Translational invariance:

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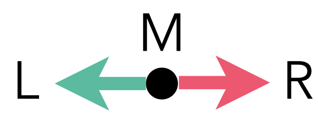
1+1 dimensions: 

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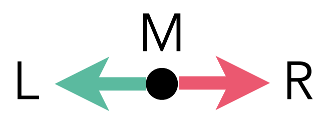
1+1 dimensions: 


Dirac QW

$$U_{\mathrm{R}} = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \quad U_{\mathrm{L}} = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}$$

$$U_{\mathrm{M}} = \begin{pmatrix} 0 & im \\ im & 0 \end{pmatrix} \quad n^2 + m^2 = 1$$

A. Bisio, G. M. D'Ariano, and
A. Tosini, Ann. Phys. 354(0):244
– 264 (2015)

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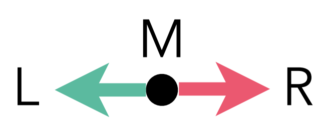
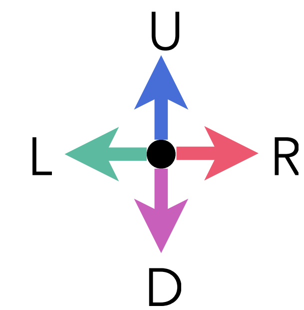
1+1 dimensions: 


2+1 dimensions: 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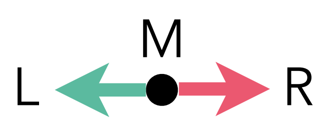


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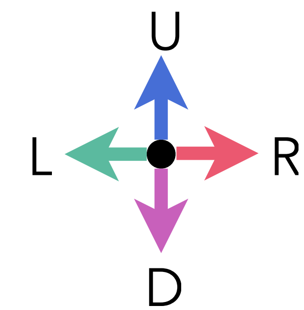


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## Weyl QW

$$U_{\mathrm{R}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -\nu & 0 \end{pmatrix} \quad U_{\mathrm{L}} = \frac{1}{2}\begin{pmatrix} 0 & \nu^* \\ 0 & 1 \end{pmatrix}$$

$$U_{\mathrm{U}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ \nu & 0 \end{pmatrix} \quad U_{\mathrm{D}} = \frac{1}{2}\begin{pmatrix} 0 & -\nu^* \\ 0 & 1 \end{pmatrix}$$

A. Bisio, G. M. D'Ariano, and A. Tosini, Ann. Phys. 354(0):244 – 264 (2015)

G. M. D'Ariano and P. Perinotti, Phys. Rev. A 90(6):062106 (2014)

# Dirac and Weyl Quantum Walks

$$\boldsymbol{k} \in \mathcal{B} \qquad U_{\boldsymbol{k}} = \sum_{\boldsymbol{h} \in S} e^{-i\boldsymbol{k}\cdot\boldsymbol{h}} U_{\boldsymbol{h}} = e^{-iH(\boldsymbol{k})} \qquad U_{\boldsymbol{k}} |u_r(\boldsymbol{k})\rangle = e^{-i\omega(\boldsymbol{k})} |u_r(\boldsymbol{k})\rangle$$

Walk Hamiltonian

Walk dispersion relation

Power expansion for small momenta $\|\boldsymbol{k}\| \ll 1 \implies \boxed{H(\boldsymbol{k}) = \gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{k} + m\gamma_0 + \mathcal{O}(m^2) + \mathcal{O}(\|\boldsymbol{k}\|^2)}$

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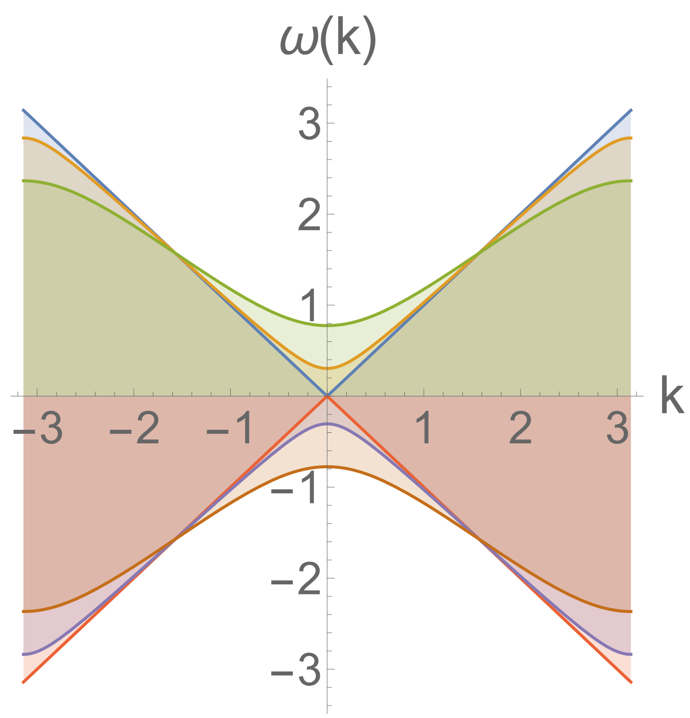
Dirac QW dispersion relation 1+1 dim



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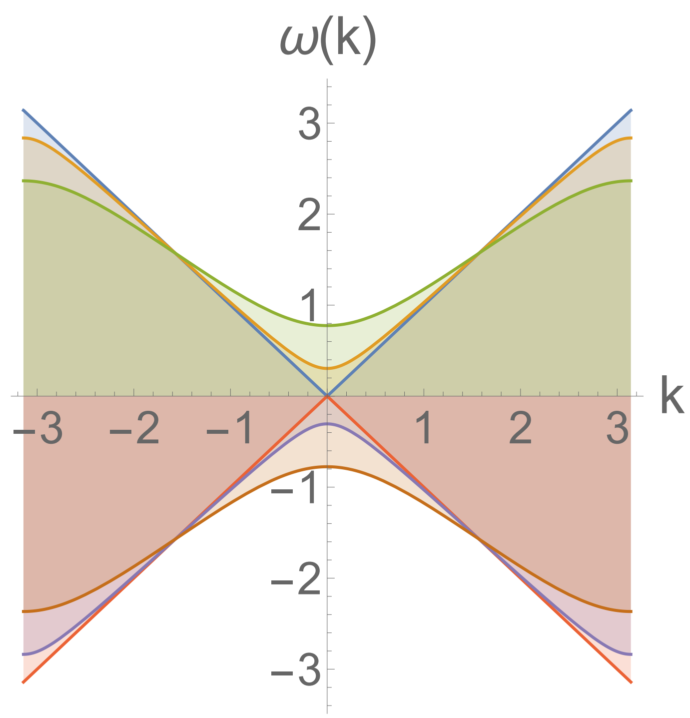
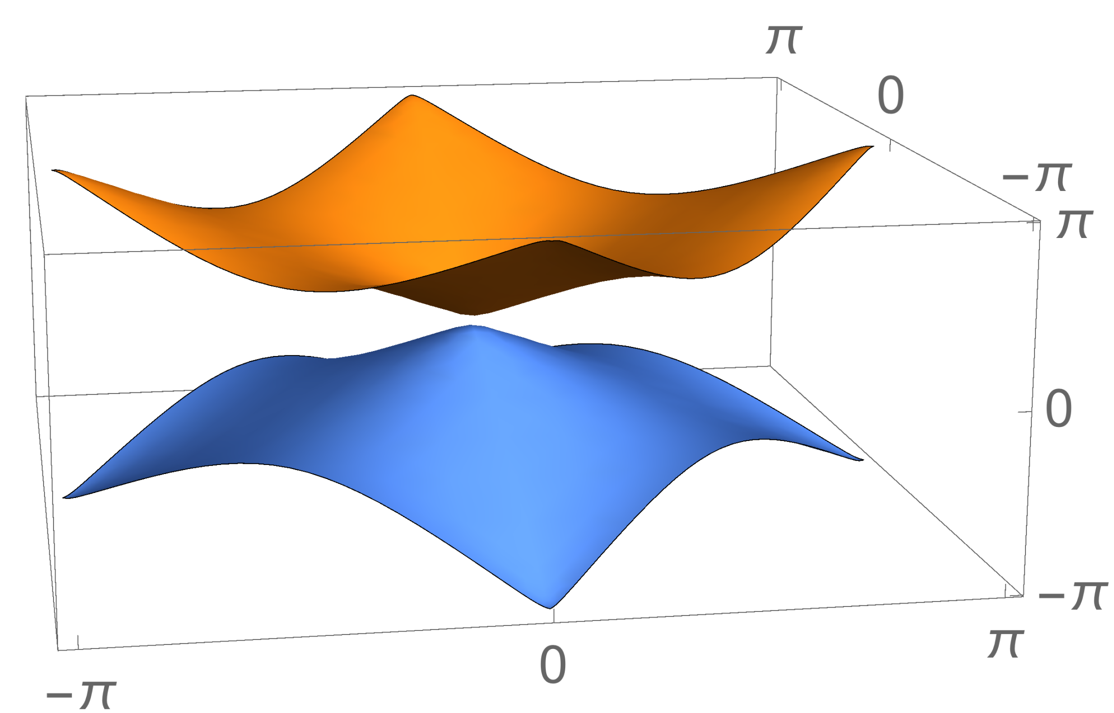



Dirac QW dispersion relation 2+1 dim



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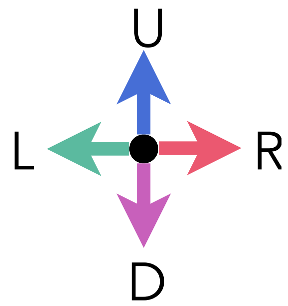


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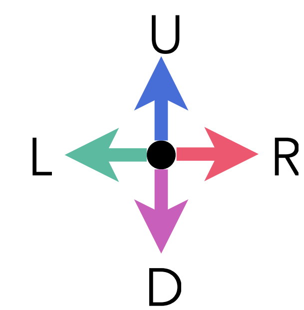


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$$\psi_x(t) = \sum_{x'} \sum_{\sigma} U_{h_t} \cdots U_{h_2} U_{h_1} \psi_{x'}(0)$$

Points in the past causal cone of $x$

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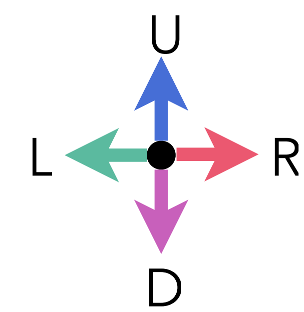


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$$U_{h_t} \cdots U_{h_2} U_{h_1} = \varphi(\sigma) U_{h(h_1,h_t)} \quad \Rightarrow \quad \psi_x(t) = \sum_{x'} \sum_{h} c_h U_h \psi_{x'}(0)$$

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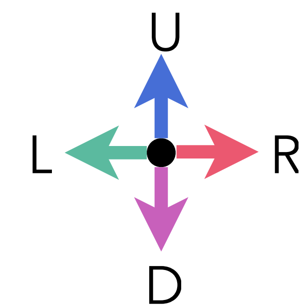


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Phase depends on the full path!

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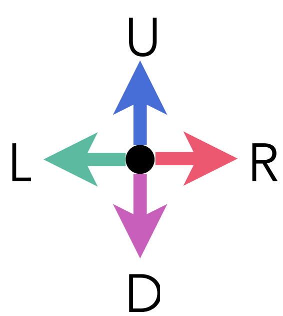


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U=10, D=01

$$h = ab$$

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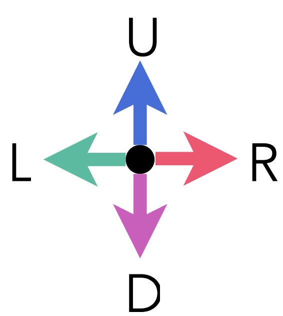


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Admissible paths:

$(x', y', 0) \longrightarrow (x, y, t)$

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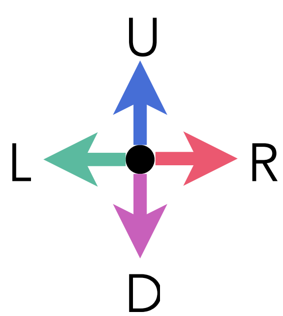


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Phase is then a binary function ⇒ Solve combinatorial problem

G. M. D'Ariano, N. Mosco, P. Perinotti, and A. Tosini, EPL 109(4):40012 (2015)

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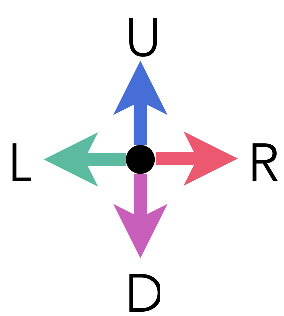
Idea: binary encoding of paths $\Rightarrow$ Topology of paths translated in properties of binary strings

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$$\boxed{\begin{aligned} c_{ab} = \sum_p \sum_{a'} \sum_k & (-1)^{k+(a\oplus a')b} \\ & \binom{K_1 - 1}{p-1} \binom{t - K_1 - 1}{p - a - a'} \binom{\mu}{k} \binom{t - \mu - 1}{K_2 - k - b} \\ \mu = 2p - a - a' & \end{aligned}}$$

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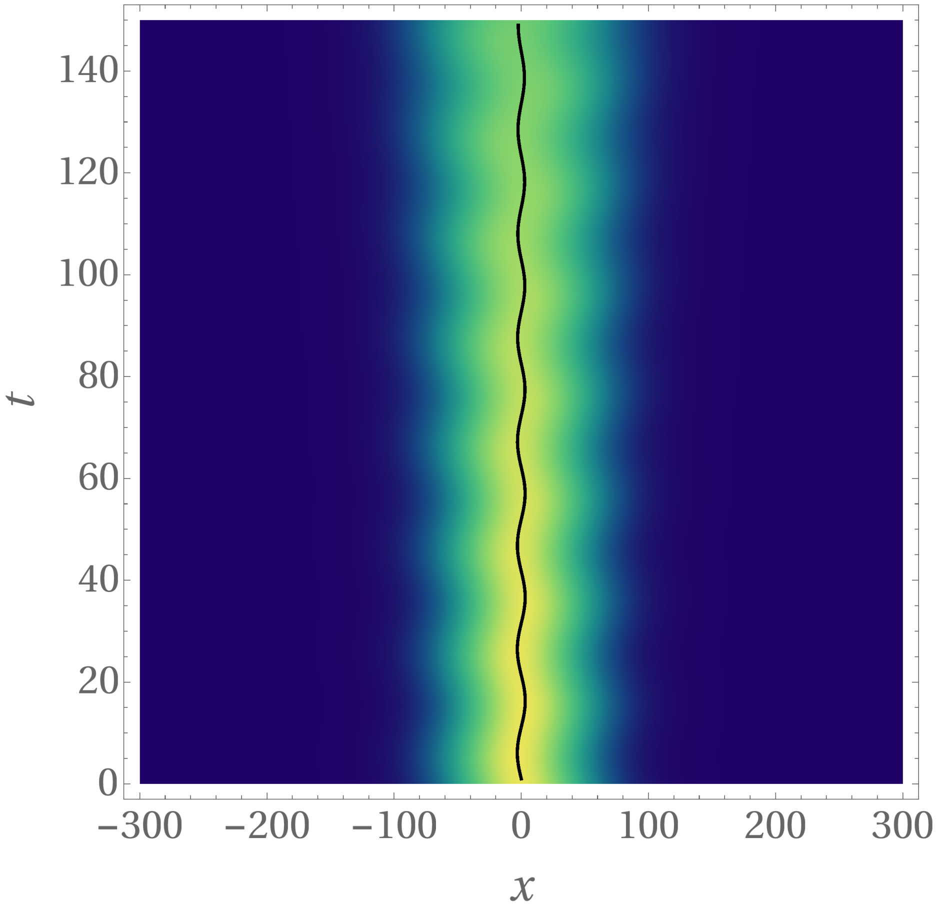
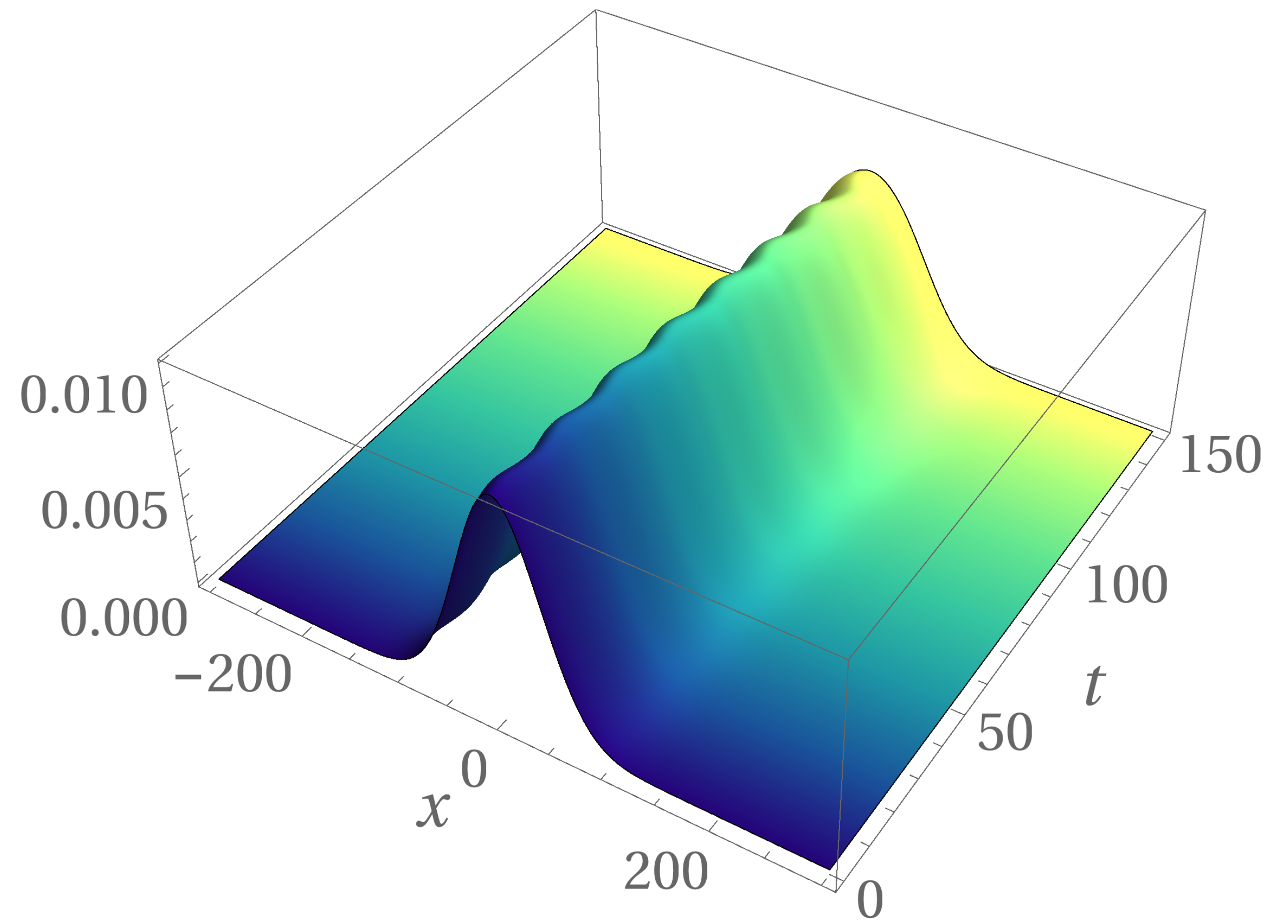
# Dirac QW in 1+1 dimensions



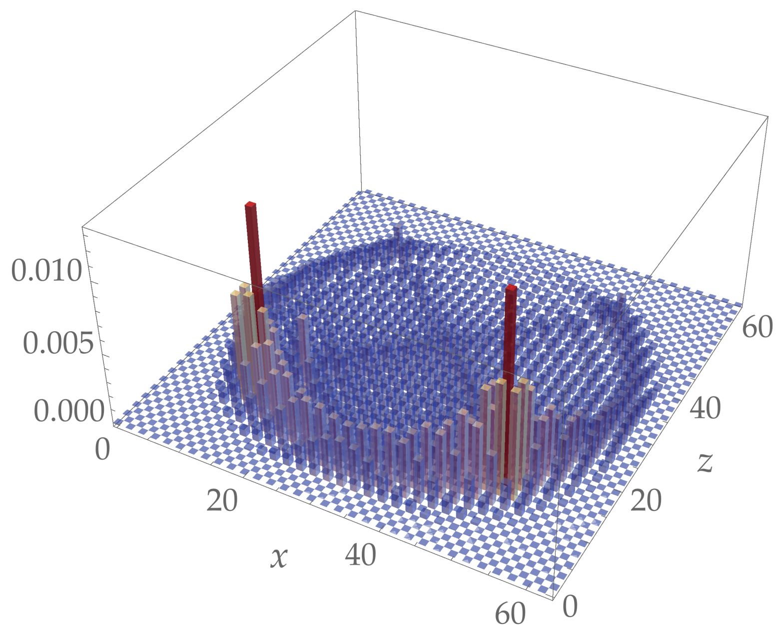



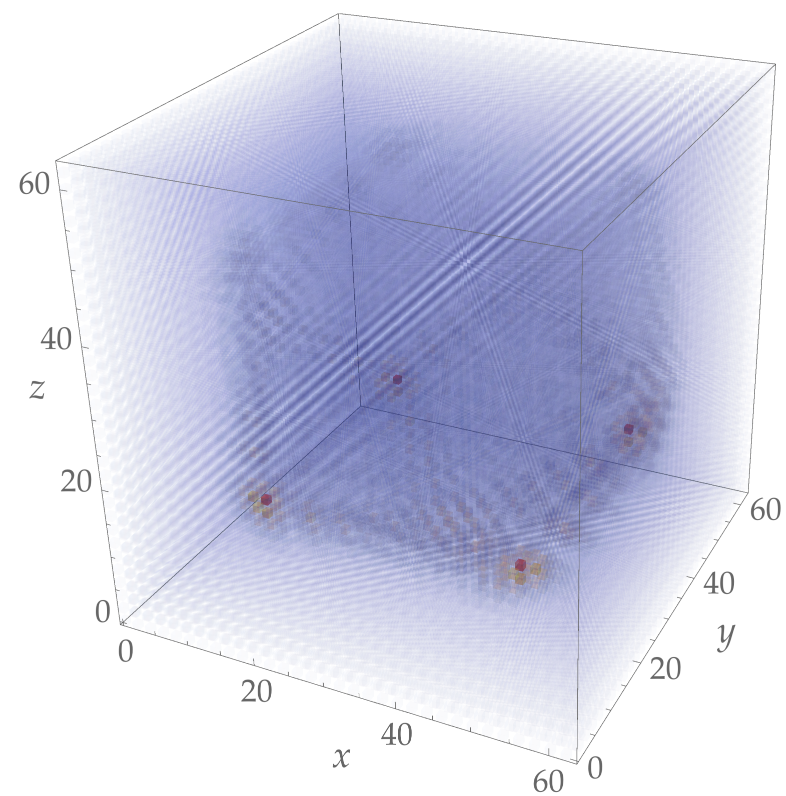

Particle state *Zitterbewegung*

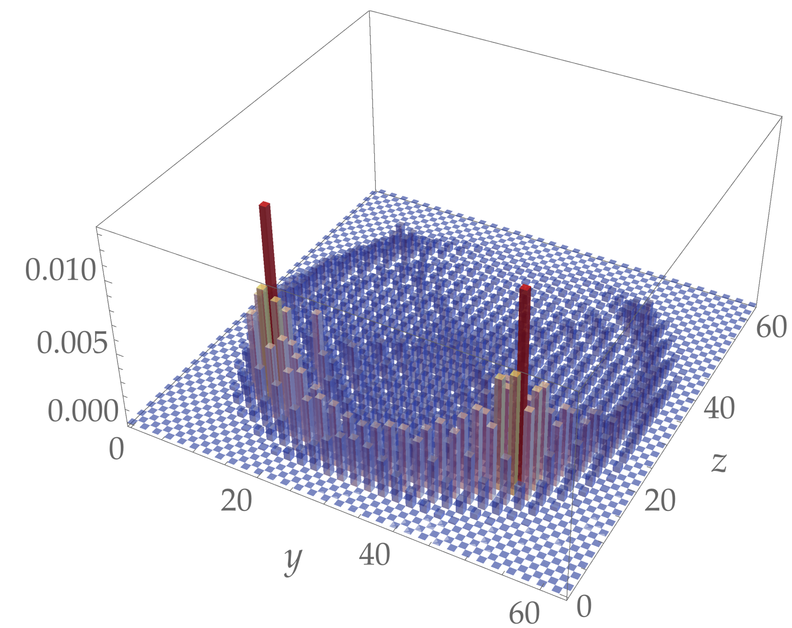
- mass $m = 0.15$
- width $\sigma = 40^{-1}$
- mean wave-vector $k_0 = 0.01\pi$
- spinor components $c_+ = c_- = 1/\sqrt{2}$

G. M. D'Ariano, N. Mosco, P. Perinotti, and A. Tosini, Entropy 18(6) (2016)

# Dirac QW in 3+1 dimensions

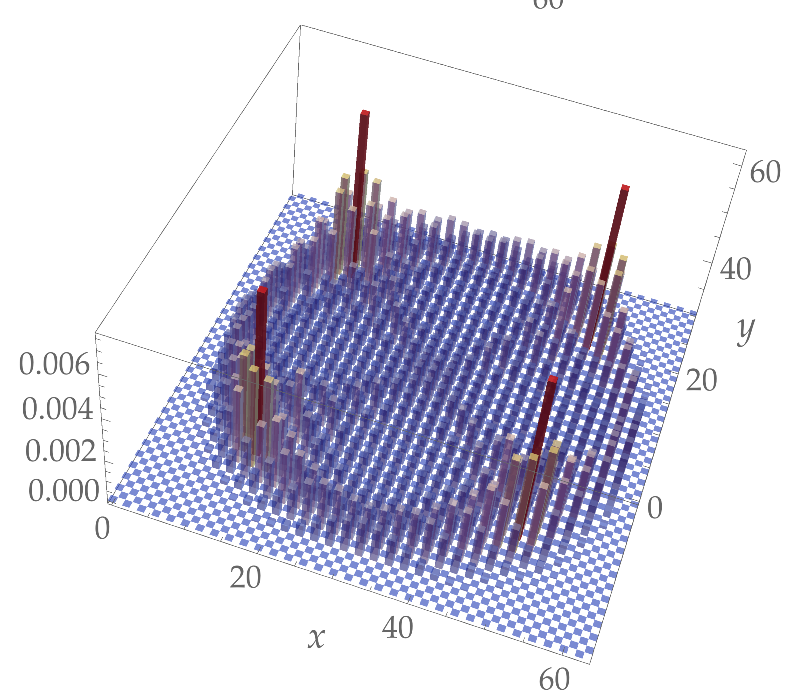



Perfectly localised state

mass $m = 0.03$

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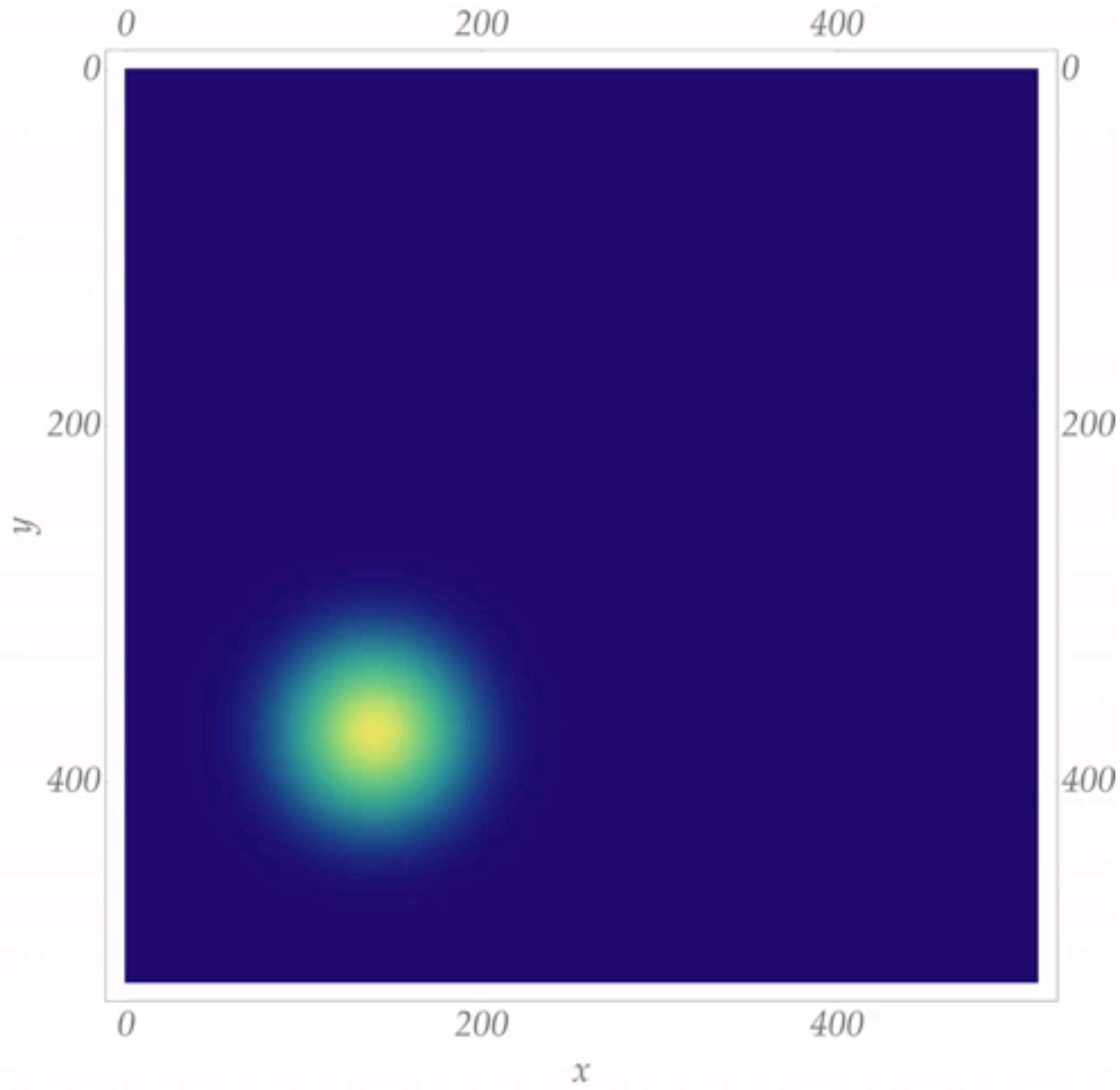


Particle state *Zitterbewegung*

- size: 512
- steps: 1200
- mass: 0.002
- width: 32

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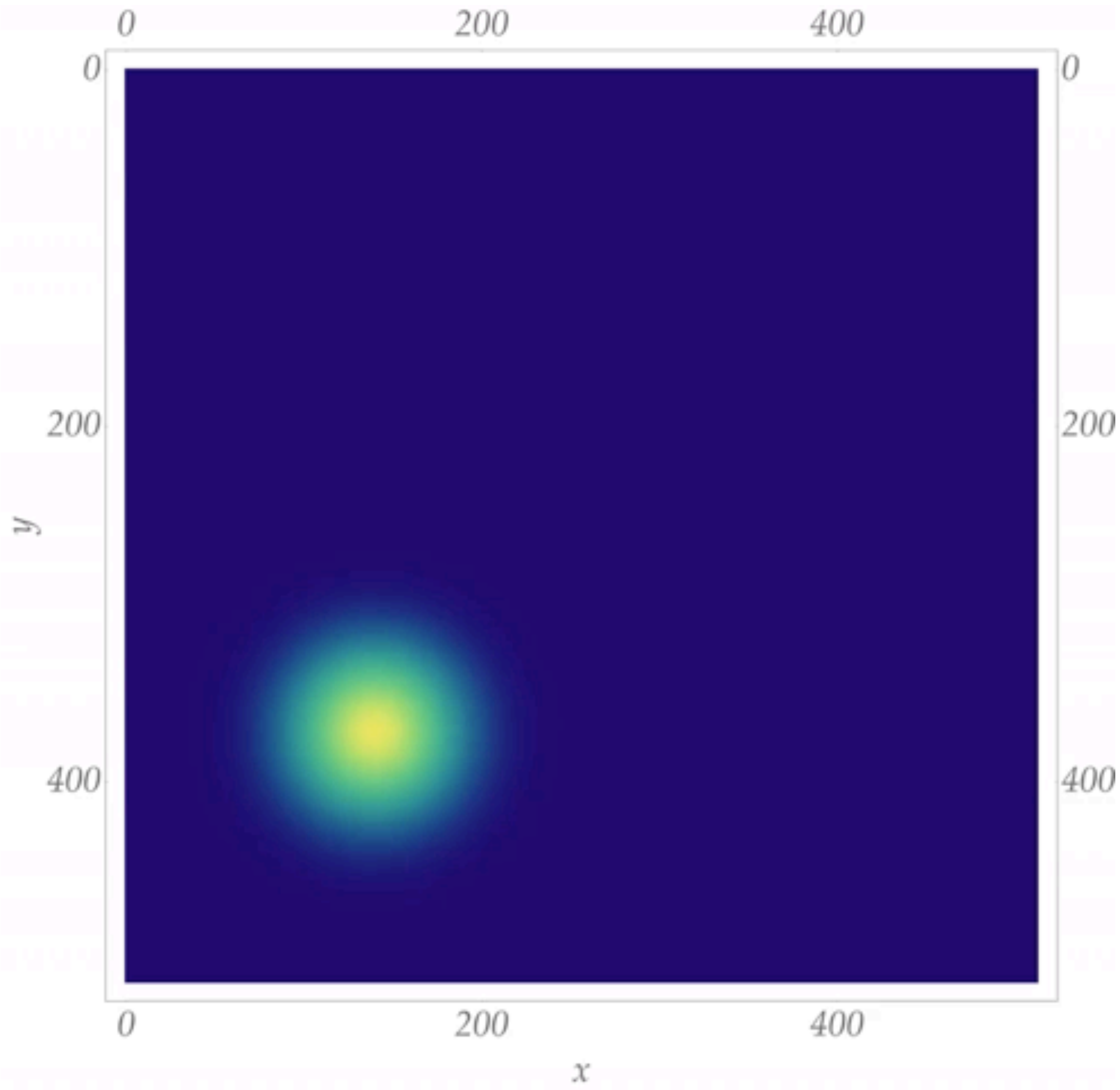


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- Solution in position representation in terms of a discrete path-integral
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quantum information theory group

Thank you!

Beto Collaboration Project