Intro to SUSY I: SUSY Basics

Archil Kobakhidze

CoEPP
ARC Centre of Excellence for Particle Physics at the Terascale

THE UNIVERSITY OF SYDNEY

PRE-SUSY 2016 SCHOOL
27 JUNE -1 JULY 2016, MELBOURNE

What are these lectures about?

This lectures are about arguably one of the most beautiful theoretical symmetry concepts (I hope you'll be convinced), with far reaching implications for fundamental physics,

which has no empirical evidence whatsoever in particle physics (hopefully) so far.

Supersymmetry has been already discovered in nuclear and condensed matter physics!

- Ground states of complex nuclei
- Disordered systems

We are not looking into this

Part I – SUSY Basics

Part II – SUSY QFT

Part III – MSSM

Suggested literature

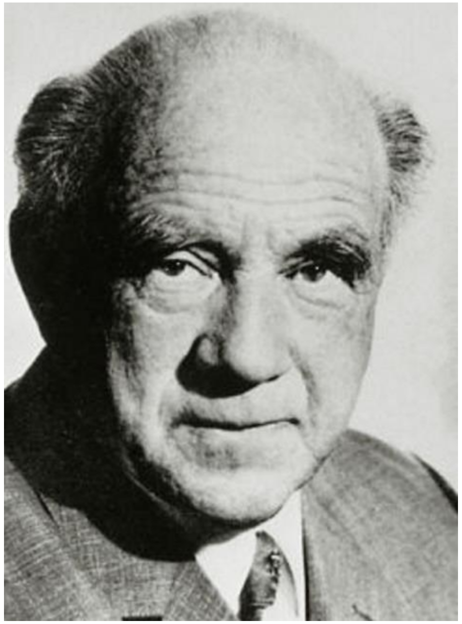
- J. Wess and J. Bagger
  Supersymmetry and supergravity
  Princeton, Univ. Press (1992)

- S. P. Martin
  A Supersymmetry primer
  hep-ph/9709356.

- J. D. Lykken
  Introduction to supersymmetry
  hep-th/9612114

QFT is an assumed knowledge; I follow Lykken's conventions.

SUSY history in brief



Since early 1950's W. Heisenberg was working on his "unified field theory of elementary particles" based on nonliner model of a 'fundamental' spinor field.

This work is largely forgotten…

Introduction to the Unified Field Theory of Elementary Particles

W. HEISENBERG
Max-Planck-Institut für Physik und Astrophysik

1966
INTERSCIENCE PUBLISHERS
a division of John Wiley & Sons London New York Sydney

677

W. Blum et al. (eds.), *Scientific Review Papers, Talks, and Books Wissenschaftliche Übersichtsartikel, Vorträge und Bücher* © Springer-Verlag Berlin Heidelberg 1984

SUSY history in brief

In the last chapter “Week Interactions” of his book he contemplates about possible reasons of neutrinos being massless and suggests:

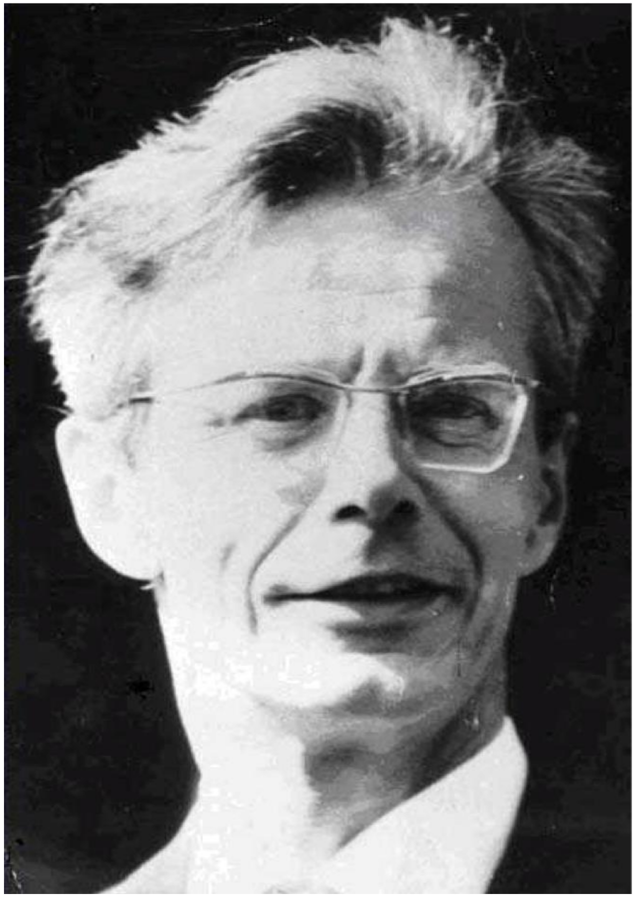
“… this makes to think that the neutrino might play a role of a Goldstone particle emerging due to asymmetry of a ground state… though here the usual Goldstone argumentation needs to be modified…”

A crazy idea – the broken generators must be spinorial

…and even wrong idea – neutrinos are massive

…and yet, a visionary idea!

SUSY history in brief



1972 – nonlinear realisation of SUSY; Goldstino

POSSIBLE UNIVERSAL NEUTRINO INTERACTION

D.V. Volkov and V.P. Akulov
Physico-technical Institute, Ukrainian Academy of Sciences
Submitted 13 October 1972
ZhETF Pis. Red. 16, No. 11, 621 - 624 (5 December 1972)

Much attention is being paid of late in elementary-particle physics to the possible degeneracy of vacuum, and the ensuing spontaneous breaking of one symmetry or another. The most direct consequence of vacuum degeneracy is the occurrence of the zero-mass particles called Goldstone particles [1].

Of all the presently known elementary particles, only the neutrino, photon, and graviton have zero mass. The last two, however, correspond to gauge fields and apparently do not require vacuum degeneracy for their description. The neutrino is therefore the only particle whose existence may be directly connected with vacuum degeneracy.

We wish to point out here that the hypothesis that the neutrino is a Goldstone particle leads to a definite type of neutrino interaction both with other neutrinos and with all other particles. The interaction is fully defined by a single phenomenological coupling constant and is in this sense universal.

To determine the type of symmetry whose spontaneous violation causes the degeneracy of vacuum and the corresponding properties of the neutrino as a Goldstone particle, let us consider the symmetry properties of the equation for the free neutrino

$$i\sigma_\mu \frac{\partial}{\partial x_\mu}\psi = 0. \qquad (1)$$

This equation is invariant both with respect to the Poincare group in chiral transformations, and with respect to shifts in spinor space, i.e., with respect to transformations of the type

$$\begin{aligned} \psi \to \psi' &= \psi + \zeta, \\ x \to x' &= x, \end{aligned} \qquad (2)$$

where ζ is a constant spinor that anticommutes with ψ.

We retain the character of the transformations of x_μ and ψ in the transformations of the Poincare group, and replace the transformations (2) by the transformations

$$\begin{aligned} \psi \to \psi' &= \psi + \zeta, \\ \psi^+ \to \psi^{+\prime} &= \psi^+ + \zeta^+, \\ x_\mu \to x'_\mu &= x_\mu - \frac{a}{2i}(\zeta^+\sigma_\mu\psi - \psi^+\sigma_\mu\zeta). \end{aligned} \qquad (3)$$

The resultant structure with ten commuting and four anticommuting parameters has the structure of a group[1] and is the only possible generalization of (2)

[1] Lie groups with commuting and anticommuting parameters were recently considered by Berezin and Kats [2].

438

SUSY history in brief

1971 - N=1 SUSY algebra;

Super-QED Lagrangian

EXTENSION OF THE ALGEBRA OF POINCARE GROUP GENERATORS AND VIOLATION OF P INVARIANCE

Yu.A. Gol'fand and E.P. Likhtman
Physics Institute, USSR Academy of Sciences
Submitted 10 March 1971
ZhETF Pis. Red. 13, No. 8, 452 - 455 (20 April 1971)

One of the main requirements imposed on quantum field theory is invariance of the theory to the Poincare group [1]. However, only a fraction of the interactions satisfying this requirement is realized in nature. It is possible that these interactions, unlike others, have a higher degree of symmetry. It is therefore of interest to study different algebras and groups, the invariance with respect to which imposes limitations on the form of the elementary particle interaction. In the present paper we propose, in constructing the Hamiltonian formulation of the quantum field theory, to use as the basis a special algebra $\mathcal{R}$, which is an extension of the algebra $\mathcal{P}$ of the Poincare group generators. The purpose of the paper is to find such a realization of the algebra $\mathcal{R}$, in which the Hamiltonian operator describes the interaction of quantized fields.

The extension of the algebra $\mathcal{P}$ is carried out in the following manner: we add to the generators P_μ and $M_{\mu\nu}$ the bispinor generators W_α and $\bar{W}_\beta$, which we shall call the generators of spinor translations. In order to obtain the algebra $\mathcal{R}$, it is necessary to find the Lorentz-invariant form of the permutation relations between the translation generators. In order not to violate subsequently the connection between the spin and statistics, we shall consider anticommutators of the operators W_α and $\bar{W}_\beta$. A generalization of the Jacobi identities imposes stringent limitations on the form of the possible commutation relations between the algebra operators. We confine ourselves to consideration of only those algebras $\mathcal{R}$, in which there are no subalgebras Q such that $\mathcal{P} \subset Q$ and $\mathcal{P} \neq Q$. This choice is governed by the fact that the remaining algebras $\mathcal{R}$ are obtained by further extending the algebras $\mathcal{R}$, and the field theories corresponding to them will have a still higher degree of symmetry.

An investigation of the albegras $\mathcal{R}$ has shown that upon spatial inversion they do not go over into themselves for any choice of the structure constants of the algebra. As a result, in a field theory that is invariant against such an algebra, the parity should not be conserved[1]), and the form of the nonconservation is completely determined by the algebra itself. We shall stop to discuss one of the algebras $\mathcal{R}$:

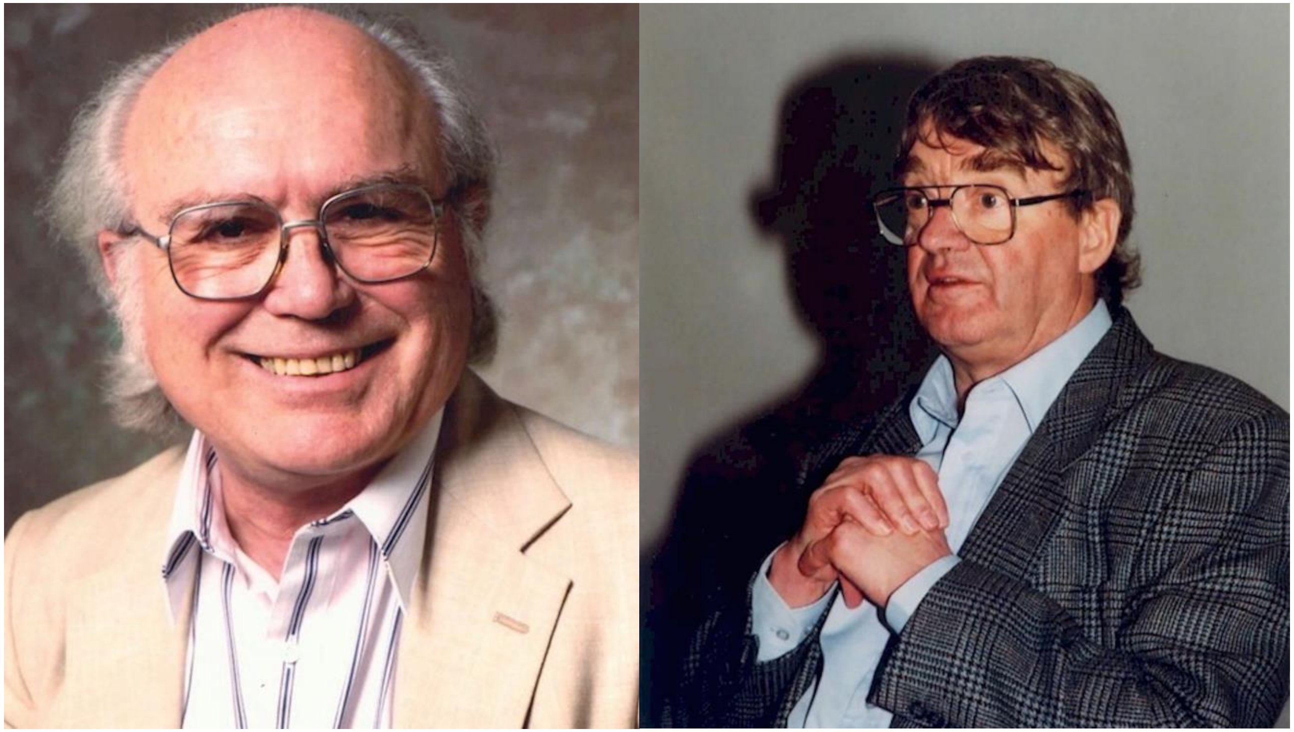
$$[M_{\mu\nu}, M_{\sigma\lambda}]_- = i(\delta_{\mu\sigma}M_{\nu\lambda} + \delta_{\nu\lambda}M_{\mu\sigma} - \delta_{\mu\lambda}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\lambda}); \quad [P_\mu, P_\nu]_- = 0;$$
$$[M_{\mu\nu}, P_\lambda]_- = i(\delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu); \quad [M_{\mu\nu}, W]_- = \frac{i}{4}[\gamma_\mu, \gamma_\nu]_- W; \quad \bar{W} = W^+\gamma_0. \qquad (1a)$$

$$[W_\pm, \bar{W}]_+ = \overset{+}{\gamma}_\mu P_\mu; \quad [W, W]_+ = 0; \quad [P_\mu, W]_- = 0, \qquad (1b)$$

1)A more detailed analysis of this question will be the subject of a separate paper.

323

SUSY history in brief

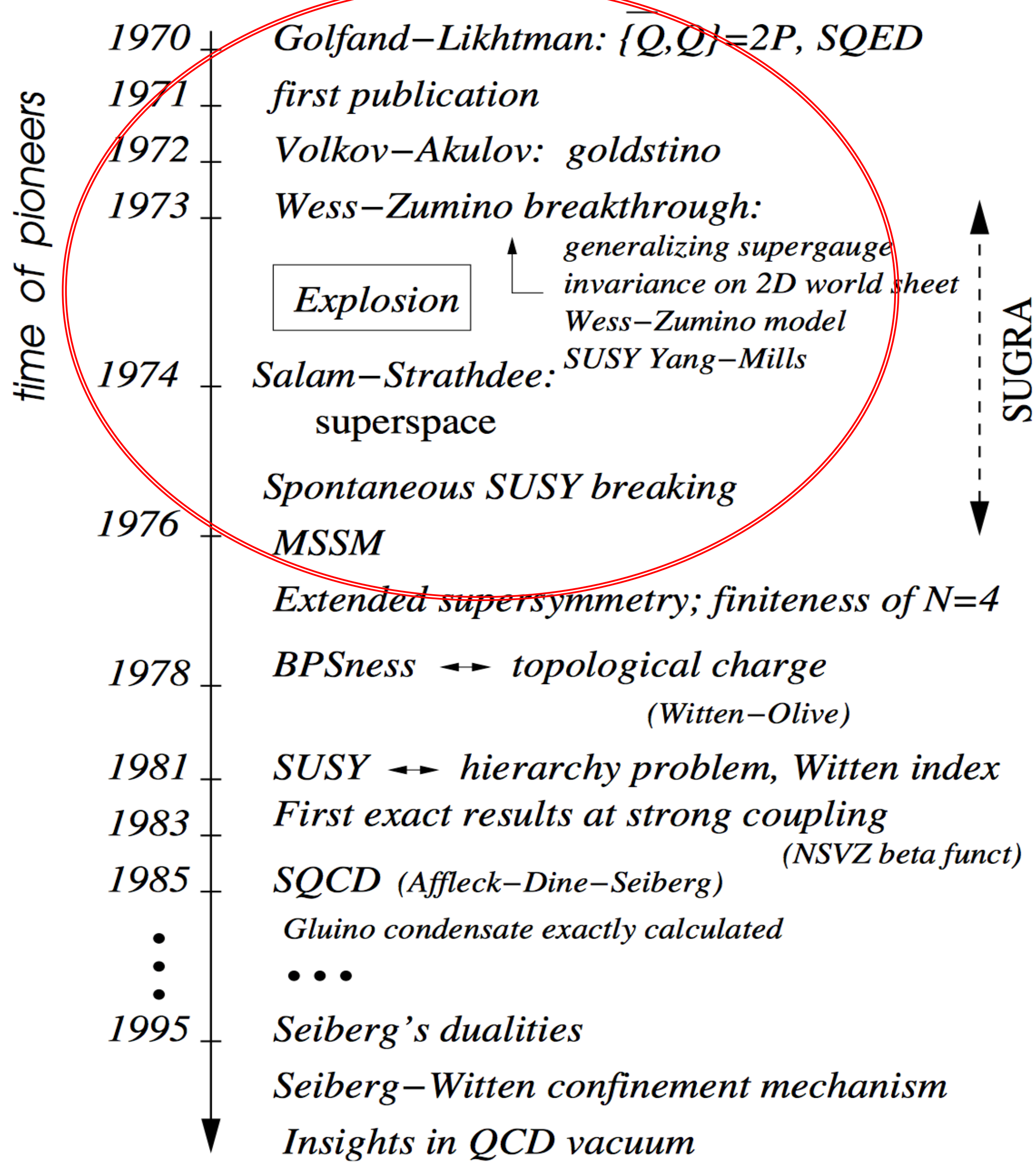


1974 - B. Zumino and J. Wes breakthrough paper:

- Linear realisation of SUSY, Wess-Zumino model, super-Yang-Mills – motivated by 2d world-sheet SUSY of Gervais and Sakita (1971)

SUSY time arrow

(taken from M. Shifman, Fortschr. Phys. 50 (2002), 552–561)



Other SUSY lectures at this school:

- Haber – SUSY Higgs
- Kuzenko – SUGRA
- White – SUSY searches

Outline of part I: Basics

- The road to supersymmetry
  - Symmetries in particle physics
  - Attempts at unification of spin and charge. Coleman-Mandula "no-go" theorem.

- Basics of supersymmetry
  - N=1 Superspace.
  - Gol'fand-Likhtman superalgebra

Symmetries in particle physics

'As far as I can see, all a priori statements in physics have their origin in symmetry.'
– Hermann Weyl – *Symmetry* (1980), p.126.

- Studies of elementary particles reveal an important role of symmetries:

(i) Kinematics of elementary particles is governed by the relativistic invariance (homogeneity and isotropy of space and time = physics is the same for all inertial observers)

(ii) Dynamics of elementary particles is governed by gauge symmetries (e.g., strong, weak and electromagnetic interactions in the Standard Model)

Symmetries in particle physics

- Mathematical description of these symmetries is provided by the Lie groups

$$\underbrace{G_{\mathsf{SM}}}_{SU(3)\times SU(2)\times U(1)} \quad \times \quad \underbrace{ISO(1,3)}_{T_4 \otimes SO(1,3)}$$

Symmetries in particle physics

- Lie group G is a set of elements which satisfy group axioms and is compatible with the smooth structure (differentiable manifold). Lie groups describe continuous transformations
- An element of the Lie group g can be represented as:

$$g = \exp\left\{ i \sum_{(A)} \xi^A T^A \right\}$$

$$= \left(\prod_{(A)} \{ i\xi^A T^A \} \right) \left(1 - \sum \sum_{A<B} \xi^A \xi^B \left[T^A, T^B \right] + ... \right)$$

Symmetries in particle physics

- Lie algebra of the group G:

$$[T^A, T^B] \equiv T^A T^B - T^B T^A = i \sum_{(C)} f^{ABC} T^C$$

for any representation of the generators T^A.

- G is an Abelian group if its algebra is commutative, $[T^A, T^B] = 0$, otherwise it is a non-Abelian group.

- Direct product $G_1 \times G_2$ is a group with $[T_1^A, T_2^B] = 0$;
- Semi-direct product $G_1 \otimes G_2$ is also a group, but $[T_1^A, T_2^B] \neq 0$

Symmetries in particle physics

- Poincaré group *ISO(1,3)* is a 10 parametric group describing relativistic invariance. An element of this group is given by:

$$g = \exp\{i\xi^{\mu}P_{\mu} + i\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}\}$$

- $P_{\mu} = -i\partial_{\mu}$ - generators of spacetime translations, T_4;
- $M_{\mu\nu} = x_{\mu}P_{\nu} - x_{\nu}P_{\mu}$ - generators of *SO(1,3)* rotations.

Symmetries in particle physics

- Lie algebra *iso(1,3):*

$$[P_\mu, P_\nu] = 0 \ ;$$
$$[P_\mu, M_{\rho\sigma}] = -i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) \ ;$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

Exercise: Verify explicitly these commutation relations

- Internal symmetries G: transformations of fields (quantum-mechanical states) that leaves observables (measured quantities) invariant.

$$[T^a, T^b] = i\sum_{(c)} f^{abc}T^c \ , \quad [T^a, P_\mu] = [T^a, M_{\mu\nu}] = 0$$

Symmetries in particle physics

- Nöether's theorem: n-parametric continuous symmetry ⇔ n conserved quantities (energy, momentum, angular momentum, electric charge, colour charges, etc.).

Exercise: Do the boost generators M_{0i} correspond to any conserved quantity?

- Can we describe internal and spacetime symmetries in unified manner, within a continuous group that covers ? $G \times ISO(1,3)$

$$\mathcal{G} \supset G \times ISO(1,3)$$

Symmetries in particle physics

- Despite considerable efforts in 1960's this idea of "spin-charge" unification turned out to be wrong. All the field theory models constructed were inconsistent for one or another reason.

- Coleman-Mandula "no-go" theorem: Every quantum field theory satisfying certain natural conditions that has non-trivial interactions can only have a symmetry Lie group which is always a direct product of the Poincaré group and internal group: no mixing between these two is possible.

S. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix". Physical Review 159 (1967) 1251.

The Coleman-Mandula theorem

- Let G be a symmetry group of a scattering matrix (S-matrix) of certain quantum field theory in more than (1+1)-dimensions, and let the following conditions hold:

i. G contains a group which locally isomorphic to *ISO(1,3)* (relativistic invariance);

i. All particle types correspond to a positive energy representations of *ISO(1,3).* For any finite mass M, there are only finite number of particles with mass less than M (particle finiteness);

The Coleman-Mandula theorem

iii. Elastic scattering amplitudes are analytic functions of center-of-mass energy s and invariant momentum transfer t in some neighborhood of physical region, except at normal thresholds (weak elastic analyticity);

iv. Let $|p_1>$ and $|p_2>$ be two one-particle momentum eigenstates, and $|p_1,p_2>$ is a two-particle eigenstate made out of these. Then,

$$\hat{T}|p_1, p_2\rangle \neq 0 \ , \qquad \hat{S} = 1 - i(2\pi)^4\delta(p_1^\mu - p_2^\mu)\hat{T} \ .$$

(occurrence of scattering);

Then,

$$\mathcal{G} = G \times ISO(1,3)$$

The Coleman-Mandula theorem

- Consider a theory of free scalar fields:

$$\mathcal{L} = -\phi_1 \Box \phi_1 - \phi_2 \Box \phi_2$$

- This theory contains infinite number of conserved currents:

$$J_\mu = (\partial_\mu \phi_2)\phi_1 - \phi_2(\partial_\mu \phi_1) \ ,$$
$$J_{\mu\nu} = (\partial_\mu \phi_2)(\partial_\nu \phi_1) - \phi_2(\partial_\mu \partial_\nu \phi_1) \ ,$$
$$J_{\mu\nu\rho} = (\partial_\mu \phi_2)(\partial_\nu \partial_\rho \phi_1) - \phi_2(\partial_\mu \partial_\nu \partial_\rho \phi_1) \ , \quad etc.$$

$$\partial^\mu J_\mu = \partial^\mu J_{\mu\nu} = \partial^\mu J_{\mu\nu\rho} = 0 \ \ (\text{use } \Box \phi_1 = \Box \phi_2 = 0).$$

and, hence, infinite number of conserved charges:

$$Q = \int d^3x J_0 \ ,$$
$$Q_\nu = \int d^3x J_{0\nu} \ ,$$
$$Q_{\nu\rho} = \int d^3x J_{0\nu\rho} \ , \quad etc.$$

The Coleman-Mandula theorem

- Suppose now we can introduce interactions that conserve, e.g., $Q_{\mu\nu}$

$$Q_{\mu\nu} \propto p_\mu p_\nu - \frac{1}{d}\eta_{\mu\nu}p^2$$

- Consider 2 → 2 elastic scattering:

$$p_{1\mu} + p_{2\mu} = q_{1\mu} + q_{2\mu}$$ 4-momentum conservation

$$p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} = q_{1\mu}q_{1\nu} + q_{2\mu}q_{2\nu}$$ $Q_{\mu\nu}$ conservation

$$\Downarrow$$

$$p_{1\mu} = q_{1\mu} \text{ or } p_{1\mu} = q_{2\mu}$$, i.e., no scattering!

The Coleman-Mandula theorem

- Non-trivial extension of the relativistic invariance is only possible if we abandon the Lie group framework by introducing spinorial generators Q_α which are anti-commuting!

Yu.A. Gol'fand and E.P. Likhtman, Extension of the Algebra of Poincaré Group Generators and Breakdown of P-invariance, JETP Lett. (1971) 323.

Uniqueness of SUSY - R. Haag, J.T. Łopuszański, M. Sohnius, ``All Possible Generators of Supersymmetries of the s Matrix," Nucl. Phys. B88 (1975) 257.

- Geometrically, this means that we must extend spacetime by introducing anti-commuting coordinates, i.e. to pass from space to superspace!

Superspace

A. Salam and J. Strathdee, Superfields and Fermi-Bose symmetry, Physical Review D 11 (1975) 1521.

Relativistic invariance

- The concept of space-time:

$$x^\mu = (t, x, y, z)$$

- 4 dimensions, coordinates are c-numbers,

$$x^\mu x^\nu - x^\nu x^\mu = 0$$

- 10 parameter Poncaré group:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = (1, -1, -1 - 1)$$

- Quantum field *F(x)*
  Particle – representation of the Poincare group

Supersymmetry

- The concept of superspace:

$$X^M = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$$

- 8 dimensions (N=1 case), θ' s are the Grassmann-numbers [$\bar{\theta}_{\dot{\alpha}} = (\theta^\alpha)^*$],

$$\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0, \quad e.g., \theta^\alpha \theta^\gamma \theta^\delta = 0$$

- 14 parameter super-Poincare group:

$$dS^2 = G_{MN} dX^M dX^N$$

$$G_{MN} = (\eta_{\mu\nu}, \epsilon^{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$$

- Superfield $S(x, \theta, \bar{\theta})$ – describes particles with different spins which form reps of super-Poincaré group

Superspace

- Some notations and properties of Grassmannian coordinates (I follow conventions of J.P. Lykken, Introduction to supersymmetry, hep-th/9612114; further reading: J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton Univ. Press (1992)):

$$\{\theta^\alpha, \theta^\beta\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0 \ (\{A, B\} = AB + BA)$$

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \ , \quad \theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta \ , \quad \bar\theta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta} \ , \quad \bar\theta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}$$

$$\theta^2 = \theta^\alpha\theta_\alpha = -\theta_\alpha\theta^\alpha \ , \quad \bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = -\bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha}$$

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Superspace

- Derivatives over Grassmannian coordinates:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha},\ \partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\partial_\beta,\ \bar{\partial}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}},\ \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}},$$

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta,\ \ \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}},$$

$$\partial^\alpha\theta^\beta = -\epsilon^{\alpha\beta},\ \ \partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta},$$

$$\bar{\partial}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}},\ \ \bar{\partial}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}},$$

$$\partial_\alpha(\theta^2) = 2\theta_\alpha,\ \ \bar{\partial}_{\dot{\alpha}}(\bar{\theta}^2) = -2\bar{\theta}_{\dot{\alpha}},$$

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \delta_\alpha^\gamma\theta^\beta,$$

$$\partial^2(\theta^2) = \bar{\partial}^2(\bar{\theta}^2) = 4\ .$$

Exercise: Verify the last two identities

Gol'fand-Likhtman (Poincaré) superalgebra

- Consider now an element of super-Poincaré group SP:

$$g(\xi^\mu, \omega^{\mu\nu}, \epsilon^\alpha, \bar{\epsilon}_{\dot{\alpha}}) = \exp i \left[\xi^\mu P_\mu + \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right]$$

- From the group closure property:

$$g_1(\xi_1^\mu, \omega_1^{\mu\nu}, \epsilon_1^\alpha, \bar{\epsilon}_{1\dot{\alpha}}) g_2(\xi_2^\mu, \omega_2^{\mu\nu}, \epsilon_2^\alpha, \bar{\epsilon}_{2\dot{\alpha}}) = g_3(\xi_3^\mu, \omega_3^{\mu\nu}, \epsilon_3^\alpha, \bar{\epsilon}_{3\dot{\alpha}}) \in \mathrm{SP}$$

$$\Downarrow$$

$$g_3 = \exp i \Big((\xi_1^\mu + \xi_2^\mu) P_\mu + \tfrac{1}{2}(\omega_1^{\mu\nu} + \omega_2^{\mu\nu}) M_{\mu\nu} + (\epsilon_1^\alpha + \epsilon_2^\alpha) Q_\alpha + (\bar{\epsilon}_{1\dot{\alpha}} + \bar{\epsilon}_{2\dot{\alpha}}) \bar{Q}^{\dot{\alpha}}$$

$$+ \tfrac{i}{2} \left[\xi_1^\mu P_\mu + \tfrac{1}{2}\omega_1^{\mu\nu} M_{\mu\nu} + \epsilon_1^\alpha Q_\alpha + \bar{\epsilon}_{1\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \,,\ \xi_2^\mu P_\mu + \tfrac{1}{2}\omega_2^{\mu\nu} M_{\mu\nu} + \epsilon_2^\alpha Q_\alpha + \bar{\epsilon}_{2\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right] + \dots \Big)$$

Gol'fand-Likhtman (Poincaré) superalgebra

- Consider,e.g.,

$$\left[\xi_1^\mu P_\mu, \epsilon_2^\alpha Q_\alpha\right] = \xi_1^\mu \epsilon_2^\alpha \left[P_\mu, Q_\alpha\right] \Longrightarrow \left[P_\mu, Q_\alpha\right] = 0.$$

- Similarly,

$$\left[\xi_1^\mu P_\mu, \bar{\epsilon}_{2\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right] = \xi_1^\mu \bar{\epsilon}_{2\dot{\alpha}} \left[P_\mu, \bar{Q}^{\dot{\alpha}}\right] \Longrightarrow \left[P_\mu, \bar{Q}^{\dot{\alpha}}\right] = 0$$

$$\left[\epsilon_1^\alpha Q_\alpha, \epsilon_2^\beta Q_\beta\right] = \epsilon_1^\alpha \epsilon_2^\beta \{Q_\alpha, Q_\beta\} \Longrightarrow \{Q_\alpha, Q_\beta\} = 0$$

$$\left[\bar{\epsilon}_1^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}, \bar{\epsilon}_2^{\dot{\beta}} \bar{Q}_{\dot{\beta}}\right] = \bar{\epsilon}_1^{\dot{\alpha}} \bar{\epsilon}_2^{\dot{\beta}} \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} \Longrightarrow \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0$$

Gol'fand-Likhtman (Poincaré) superalgebra

- Non-trivial commutators:

$$[M_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta \ , \quad \left[M_{\mu\nu}, \bar{Q}^{\dot{\alpha}}\right] = -(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}} \ .$$

- Non-trivial anti-commutator:

$$\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\} = 2(\sigma^\mu)_\alpha^{\ \dot{\alpha}} P_\mu$$

Notations: $\sigma^\mu = (I, \vec{\sigma}) = \bar{\sigma}_\mu \ , \quad \bar{\sigma}^\mu = (I, -\vec{\sigma}) = \sigma_\mu \ ,$

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}\left((\sigma^\mu)_{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - (\sigma^\nu)_{\alpha\dot{\gamma}}(\bar{\sigma}^\mu)^{\dot{\gamma}\beta}\right) \ ,$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Gol'fand-Likhtman (Poincaré) superalgebra

- Supercharges:

$$Q_\alpha = -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu$$

$$\bar{Q}^{\dot{\alpha}} = i\bar{\partial}^{\dot{\alpha}} + \theta^\beta(\sigma^\mu)_\beta{}^{\dot{\alpha}}\partial_\mu$$

Exercise: Check that supercharges indeed satisfy the (anti)commutation relations

Gol'fand-Likhtman (Poincaré) superalgebra

- N=1 SUSY algebra is unaffected by the U(1) chiral U(1) chiral phase transformations of supercharges:

$$Q_\alpha \to e^{i\phi} Q_\alpha,\ \bar{Q}^{\dot{\alpha}} \to e^{-i\phi} \bar{Q}^{\dot{\alpha}}$$

- An extra Abelian $U_R(1)$ isometry of superspace known as R-symmetry

$$e^{iR\phi} Q_\alpha e^{-iR\phi} = e^{i\phi} Q_\alpha,\ e^{iR\phi} \bar{Q}^{\dot{\alpha}} e^{-iR\phi} = e^{-i\phi} \bar{Q}^{\dot{\alpha}}$$

$$[Q_\alpha, R] = +Q_\alpha,\ \left[\bar{Q}^{\dot{\alpha}}, R\right] = -\bar{Q}^{\dot{\alpha}}\ .$$

- Z_2 discrete subgroup of $U_R(1)$ (R-parity) is an important symmetry in phenomenology – stability of matter, dark matter candidate!

Gol'fand-Likhtman (Poincaré) superalgebra

These (anti)commutation relations together with the commutation relations of Poincaré algebra defines the simplest N=1 supersymmetric (super-Poincaré) algebra

More on superspace: Covariant derivatives

- Supersymmetric transformation of superspace coordinates:

$$\theta'^{\alpha} = \left(1 + i\epsilon^{\beta}Q_{\beta} + i\bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right)\theta^{\alpha} = \theta^{\alpha} + \epsilon^{\alpha},$$
$$\bar{\theta}'^{\dot{\alpha}} = \left(1 + i\epsilon^{\beta}Q_{\beta} + i\bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right)\bar{\theta}^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}},$$

$$x'^{\mu} = \left(1 + i\epsilon^{\beta}Q_{\beta} + i\bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right)x^{\mu} = x^{\mu} - i\epsilon\sigma^{\mu}\bar{\theta} + i\theta\sigma^{\mu}\bar{\epsilon}\ .$$

- Local supersymmetry with $\epsilon(x)$, $\bar{\epsilon}(x)$ implies (super)gravity!

More on superspace: Covariant derivatives

- The set of derivatives $\partial_M = \left(\partial_\mu,\ \partial_\alpha,\ \bar{\partial}_{\dot{\alpha}}\right)$ is not covariant under SUSY transformations, e.g.,

$$(\partial_\alpha x^\mu)'_{SUSY} \neq \partial_\alpha \left(x^\mu\right)'_{SUSY}$$

- SUSY covariant derivatives $D_M = (\partial_\mu,\ D_\alpha,\ \bar{D}_{\dot{\alpha}})$:

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \ ,$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu \ ,$$

- Flat superspace is a space with torsion

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma)_{\alpha\dot{\alpha}}\partial_\mu$$

Exercise: Show that covariant derivatives anticommute with supercharges

More on superspace: Integration

- The Berezin integral over a single Grassmannian variable θis defined as:

$$\int d\theta\theta = 1 \text{ and } \int d\theta = 0.$$

- For an arbitrary function $f(\theta)=f_0 + \theta f_1$:

$$\int d\theta \frac{df}{d\theta} = 0; \quad \delta(\theta) = \theta \text{ - Grassmann delta-function;}$$

$$\int d\theta f(\theta) = f_1 = \frac{d}{d\theta} f(\theta)$$

Grassmann integration is equivalent to differentiation

More on superspace: Integration

- The integration rules are straightforwardly generalized to superspace coordinates with the following notational conventions:

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \; d^2\bar{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \; d^4\theta = d^2\theta d^2\bar{\theta}$$

$$\int d^2\theta \; \theta^2 = \int d^2\bar{\theta} \; \bar{\theta}^2 = \int d^4\theta \; \bar{\theta}^2\theta^2 = 1$$

Exercise: Verify the last 3 integrations

Summary of Part I

- SUSY is a unique non-trivial continuous extension of the relativistic invariance.
- Provides unified description of fields of different spin and statistics.
- Superspace. Gol'fand-Likhtman (super-Poincaré) algebra.
- Differentiation and integration. Covariant derivative.
- Local SUSY implies (super)gravity! (Kuzenko's lectures).