Two introductory lectures on supergravity

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Outline

Introductory comments

- November 1915 — Einstein's general theory of relativity (GR)
- February 2016 — Discovery of gravitational waves
  Final confirmation of GR (100 years later)
- Since the creation of GR, Einstein was confident in the correctness of his theory. However he was not completely satisfied with it. Why?
- The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \ .$$

Here the left-hand side is purely geometric. The right-hand side is proportional to the energy-momentum tensor of matter, which is not geometric. While the geometry of spacetime is determined by the Einstein equations, the theory does not predict the structure of its matter sector.

Introductory comments

Action functional describing the dynamics of the gravitational field coupled to matter fields φ^i:

$$S = S_{\rm GR} + S_{\rm M},$$
$$S_{\rm GR} = \frac{1}{2\kappa^2}\int {\rm d}^4x\sqrt{-g}\,R~, \quad S_{\rm M} = \int {\rm d}^4x\sqrt{-g}\,L_{\rm M}(\varphi^i, \nabla_\mu\varphi^j, \dots)~,$$

with $\kappa^2 = 8\pi G c^{-4}$.

$S_{\rm GR}$ and $S_{\rm M}$ are the gravitational and matter actions, respectively.

The dynamical equations are:

(i) the matter equations of motion, $\delta S/\delta\varphi^i = 0$; and

(ii) the Einstein field equations with

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm M}}{\delta g^{\mu\nu}}~.$$

Einstein was concerned with the fact that the matter Lagrangian, $L_{\rm M}$, is essentially arbitrary!

Introductory comments

- Typical reasoning by Einstein:
  "Ever since the formulation of the general relativity theory in 1915, it has been the persistent effort of theoreticians to reduce the laws of the gravitational and electromagnetic fields to a single basis. It could not be believed that these fields correspond to two spatial structures which have no conceptual relation to each other."
  Science **74**, 438 (1931)
- Supergravity is the gauge theory of supersymmetry.
  Local supersymmetry is a unique symmetry principle to bind together the gravitational field (spin 2) and matter fields of spin $s < 2$.
- It is known that the Kaluza-Klein approach also makes it possible to unify the gravitational field with Yang-Mills and scalar fields. However, it is local supersymmetry which allows one to unite the gravitational field with fermionic fields into a single multiplet.
- If we believe in unity of forces in the universe, local supersymmetry should play a fundamental role, as a spontaneously broken symmetry.

Brief history of $\mathcal{N} = 1$ supergravity in four dimensions

- On-shell supergravity

  D. Freedman, P. van Nieuwenhuizen & S. Ferrara (1976)

  S. Deser & B. Zumino (1976)

- Super-Higgs effect (spontaneously broken supergravity)

  D. Volkov & V. Soroka (1973)

  S. Deser & B. Zumino (1977)

- Non-minimal off-shell supergravity

  P. Breitenlohner (1977)

  unpublished — W. Siegel (1977)

- Old minimal off-shell supergravity

  unpublished — W. Siegel (1977)

  Phys. Lett. B **74**, 51 — J. Wess & B. Zumino (1978)

  Phys. Lett. B **74**, 330 — K. Stelle & P. West (1978)

  Phys. Lett. B **74**, 333 — S. Ferrara & P. van Nieuwenhuizen (1978)

- New minimal off-shell supergravity

  M. Sohnius & P. West (1981)

Textbooks on $\mathcal{N} = 1$ supergravity in four dimensions

Superspace and component approaches:

- J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, Princeton, 1983 (Second Edition: 1992).
- S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry*, Benjamin/Cummings (Reading, MA), 1983, hep-th/0108200.
- I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, Bristol, 1995 (Revised Edition: 1998).

Purely component approach:

- D. Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge, 2012.

Weyl-invariant formulation for gravity

There exist three formulations for gravity in d dimensions:

- Metric formulation;
- Vielbein formulation;
- Weyl-invariant formulation.

I briefly recall the metric and vielbein approaches and then concentrate in more detail of the Weyl-invariant formulation.

S. Deser (1970)
P. Dirac (1973)

The latter formulation is a natural starting point to introduce supergravity as a generalisation of gravity.

Metric and vielbein formulations for gravity

Metric formulation

Gauge field: metric $g_{mn}(x)$

Gauge transformation: $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$

$\xi = \xi^m(x)\partial_m$ vector field generating an infinitesimal diffeomorphism.

Vielbein formulation

Gauge field: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

Metric is a composite field $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$

Gauge transformation: $\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a]$

Gauge parameters: $\xi^a(x) = \xi^m e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$

Covariant derivatives (M_{bc} Lorentz generators $M_{bc} V^a = \delta^a_b V_c - \delta^a_c V_b$)

$$\nabla_a = e_a + \omega_a \equiv e_a{}^m \partial_m + \frac{1}{2}\omega_a{}^{bc} M_{bc} , \qquad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}{}^{cd} M_{cd}$$

$e_a{}^m$ inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$

$\omega_a{}^{bc}$ torsion-free Lorentz connection $\omega_{abc} = \frac{1}{2}(\mathcal{C}_{bca} + \mathcal{C}_{acb} - \mathcal{C}_{abc})$

$[e_a, e_b] = \mathcal{C}_{ab}{}^c e_c$ $\mathcal{C}_{ab}{}^c$ anholonomy coefficients

Weyl transformations

Weyl transformations

The torsion-free constraint

$$T_{ab}{}^{c} = 0 \quad \Longleftrightarrow \quad [\nabla_a, \nabla_b] \equiv T_{ab}{}^{c}\nabla_c + \frac{1}{2}R_{ab}{}^{cd}M_{cd} = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

is invariant under Weyl (local scale) transformations

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma}\Big(\nabla_a + (\nabla^b\sigma)M_{ba}\Big) ~,$$

with the parameter $\sigma(x)$ being completely arbitrary.

$$e_a{}^m \to \mathrm{e}^{\sigma}e_a{}^m ~, \qquad e_m{}^a \to \mathrm{e}^{-\sigma}e_m{}^a ~, \qquad g_{mn} \to \mathrm{e}^{-2\sigma}g_{mn}$$

Weyl transformations are gauge symmetries of conformal gravity, which in the $d = 4$ case is described by action (C_{abcd} is the Weyl tensor)

$$S_{\rm conf} = \int \mathrm{d}^4x\, e\, C^{abcd}C_{abcd} ~, \quad C_{abcd} \to \mathrm{e}^{2\sigma}C_{abcd}$$

Einstein gravity possesses no Weyl invariance.

Weyl-invariant formulation for Einstein's gravity

Gauge fields: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$
& conformal compensator $\varphi(x)$, $\varphi \neq 0$

Gauge transformations ($\mathcal{K} := \xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}$)

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] + \sigma \nabla_a + (\nabla^b \sigma) M_{ba} \equiv (\delta_{\mathcal{K}} + \delta_\sigma) \nabla_a ,$$

$$\delta \varphi = \xi^b \nabla_b \varphi + \frac{1}{2}(d-2)\sigma\varphi \equiv (\delta_{\mathcal{K}} + \delta_\sigma)\varphi$$

Gauge-invariant gravity action

$$S = \frac{1}{2} \int \mathrm{d}^d x \, e \left(\nabla^a \varphi \nabla_a \varphi + \frac{1}{4} \frac{d-2}{d-1} R \varphi^2 - \lambda \varphi^{2d/(d-2)} \right)$$

Imposing a Weyl gauge condition $\varphi = \frac{2}{\kappa}\sqrt{\frac{d-1}{d-2}} = \mathrm{const}$ reduces S to the Einstein-Hilbert action with a cosmological term

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \, e \, R - \Lambda \int \mathrm{d}^d x \, e$$

Conformal isometries

Conformal Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is conformal Killing if there exist local Lorentz, $K^{bc}[\xi]$, and Weyl, $\sigma[\xi]$, parameters such that

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0$$

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}\big(\nabla^b \xi^c - \nabla^c \xi^b\big) , \qquad \sigma[\xi] = \frac{1}{d} \nabla_b \xi^b$$

Conformal Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 2\eta^{ab} \sigma[\xi]$$

Conformal Killing vector fields for Minkowski space:

$$\xi^a = b^a + K^a{}_b x^b + \Delta x^a + f^a x^2 - 2 f_b x^b x^a , \quad K_{ab} = -K_{ba}$$

Conformal isometries

- Lie algebra of conformal Killing vector fields
- Conformally related spacetimes (∇_a, φ) and $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi$$

have the same conformal Killing vector fields $\xi = \xi^a e_a = \tilde{\xi}^a \tilde{e}_a$.

The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\mathcal{K}[\tilde{\xi}] := \tilde{\xi}^b \widetilde{\nabla}_b + \frac{1}{2} K^{cd}[\tilde{\xi}] M_{cd} = \mathcal{K}[\xi] ,$$
$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi \rho$$

- Conformal field theories

Isometries

Killing vector fields

Let $\xi = \xi^a e_a$ be a conformal Killing vector,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0 \ .$$

It is called Killing if it leaves the compensator invariant,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\varphi = \xi\varphi + \frac{1}{2}(d-2)\sigma[\xi]\varphi = 0 \ .$$

These Killing equations are Weyl invariant in the following sense: Given a conformally related spacetime $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^\rho \Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) \ , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi \ ,$$

the above Killing equations have the same functional form when rewritten in terms of $(\widetilde{\nabla}_a, \widetilde{\varphi})$, in particular

$$\xi\widetilde{\varphi} + \frac{1}{2}(d-2)\sigma[\tilde{\xi}]\widetilde{\varphi} = 0 \ .$$

Isometries

Because of Weyl invariance, we can work with a conformally related spacetime such that

$$\varphi = 1$$

Then the Killing equations turn into

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] = 0 , \qquad \sigma[\xi] = 0$$

Standard Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 0$$

Killing vector fields for Minkowski space:

$$\xi^a = b^a + K^a{}_b x^b , \quad K_{ab} = -K_{ba}$$

- Lie algebra of Killing vector fields
- Field theories in curved space with symmetry group including the spacetime isometry group.

Two-component spinor notation and conventions

The Minkowski metric is chosen to be $\eta_{ab} = \mathrm{diag}\,(-1, +1, +1, +1)$.
For two-component undotted spinors, such as ψ_α and ψ^α, their indices are raised and lowered by the rule:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\,\psi_\beta\ , \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\,\psi^\beta\ ,$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are 2×2 antisymmetric matrices normalised as

$$\varepsilon^{12} = \varepsilon_{21} = 1\ .$$

The same conventions are used for dotted spinors ($\bar\psi_{\dot\alpha}$ and $\bar\psi^{\dot\alpha}$).

$$\psi\lambda := \psi^\alpha\lambda_\alpha\ , \quad \psi^2 = \psi\psi\ , \qquad \bar\psi\bar\lambda := \bar\psi_{\dot\alpha}\bar\lambda^{\dot\alpha}\ , \quad \bar\psi^2 = \bar\psi\bar\psi\ .$$

Relativistic Pauli matrices $\sigma_a = \Big((\sigma_a)_{\alpha\dot\beta}\Big)$ and $\tilde\sigma_a = \Big((\tilde\sigma_a)^{\dot\alpha\beta}\Big)$

$$\sigma_a = (\mathbb{1}_2, \vec\sigma)\ , \qquad \tilde\sigma_a = (\mathbb{1}_2, -\vec\sigma)$$

Lorentz spinor generators $\sigma_{ab} = \Big((\sigma_{ab})_\alpha{}^\beta\Big)$ and $\tilde\sigma_{ab} = \Big((\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\Big)$:

$$\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde\sigma_b - \sigma_b\tilde\sigma_a)\ , \quad \tilde\sigma_{ab} = -\frac{1}{4}(\tilde\sigma_a\sigma_b - \tilde\sigma_b\sigma_a)$$

Minkowski superspace and chiral superspace

Minkowski superspace $\mathbb{M}^{4|4}$ is parametrised by 'real' coordinates

$$z^A = (x^a, \theta^\alpha, \bar\theta_{\dot\alpha}) , \qquad \overline{x^a} = x^a , \qquad \overline{\theta^\alpha} = \bar\theta^{\dot\alpha} .$$

It may be embedded in complex superspace $\mathbb{C}^{4|2}$ (chiral superspace)

$$\zeta^{\underline{A}} = (y^a, \theta^\alpha)$$

as real surface

$$y^a - \bar y^a = 2\mathrm{i}\theta\sigma^a\bar\theta \equiv 2\mathrm{i}\mathcal{H}_0^a(\theta,\bar\theta) \quad \Longleftrightarrow \quad y^a = x^a + \mathrm{i}\theta\sigma^a\bar\theta .$$

Supersymmetry transformation on $\mathbb{M}^{4|4}$

$$x'^a = x^a - \mathrm{i}\epsilon\sigma^a\bar\theta + \mathrm{i}\theta\sigma^a\bar\epsilon , \qquad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha , \qquad \bar\theta'_{\dot\alpha} = \bar\theta_{\dot\alpha} + \bar\epsilon_{\dot\alpha}$$

is equivalent to a holomorphic transformation on $\mathbb{C}^{4|2}$

$$y'^a = y^a + 2\mathrm{i}\theta\sigma^a\bar\epsilon + \mathrm{i}\epsilon\sigma^a\bar\epsilon , \qquad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha .$$

Every Poincaré transformation (translation and Lorentz one) on $\mathbb{M}^{4|4}$ is also equivalent to a holomorphic transformation on $\mathbb{C}^{4|2}$.

Family of superspaces $\mathcal{M}^{4|4}(\mathcal{H})$

Curved superspace $\mathcal{M}^{4|4}(\mathcal{H})$,
parametrised by real coordinates $z^A = (x^a, \theta^\alpha, \bar\theta_{\dot\alpha})$,
is defined by its embedding in $\mathbb{C}^{4|2}$:

$$y^a - \bar y^a = 2\mathrm{i}\theta\sigma^a\bar\theta \equiv 2\mathrm{i}\mathcal{H}^a(x,\theta,\bar\theta)\ , \qquad x^a = \frac{1}{2}(y^a + \bar y^a)\ ,$$

for some real vector superfield $\mathcal{H}^a(x,\theta,\bar\theta)$.

What is special about Minkowski superspace $\mathbb{M}^{4|4} = \mathcal{M}^{4|4}(\mathcal{H}_0)$?
It is the only super-Poincaré invariant superspace in the family of all supermanifolds $\mathcal{M}^{4|4}(\mathcal{H})$.

Family of superspaces $\{\mathcal{M}^{4|4}(\mathcal{H})\}$ and uniqueness of $\mathbb{M}^{4|4}$

- Spacetime translations

$$y'^a = y^a + b^a \ , \qquad \theta'^\alpha = \theta^\alpha$$

Condition of invariance

$$\begin{aligned} & y'^a - \bar{y}'^a = 2\mathrm{i}\mathcal{H}^a(x', \theta', \bar{\theta}') = y^a - \bar{y}^a = 2\mathrm{i}\mathcal{H}^a(x, \theta, \bar{\theta}) \\ \Longrightarrow \quad & \mathcal{H}^a(x + b, \theta, \bar{\theta}) = \mathcal{H}^a(x, \theta, \bar{\theta}) \quad \Longrightarrow \quad \mathcal{H}^a = \mathcal{H}^a(\theta, \bar{\theta}) \end{aligned}$$

- Lorentz transformations

$$\mathcal{H}^a(\theta, \bar{\theta}) = \kappa \theta \sigma^a \bar{\theta}$$

for some constant κ.

- Supersymmetry transformations

$$\mathcal{H}^a(\theta, \bar{\theta}) = \theta \sigma^a \bar{\theta} = \mathcal{H}_0^a$$

Superconformal transformations

Consider an infinitesimal holomorphic transformation on $\mathbb{C}^{4|2}$

$$y'^a = y^a + \lambda^a(y,\theta) , \qquad \theta'^\alpha = \theta^\alpha + \lambda^\alpha(y,\theta)$$

What are the most general infinitesimal holomorphic transformations on $\mathbb{C}^{4|2}$ which leave Minkowski superspace, $\mathcal{M}^{4|4}(\mathcal{H}_0)$, invariant ?

$$y'^a - \bar{y}'^a = 2\mathrm{i}\theta'\sigma^a\bar{\theta}'$$

The answer is: superconformal transformations

$$\lambda^a = b^a + K^a{}_b y^b + \Delta y^a + f^a y^2 - 2f_b y^b y^a + 2\mathrm{i}\theta\sigma^a\bar{\epsilon} - 2\theta\sigma^a\tilde{\sigma}_b\eta y^b ,$$
$$\lambda^\alpha = \epsilon^\alpha - K^\alpha{}_\beta\theta^\beta + \frac{1}{2}(\Delta - \mathrm{i}\Omega)\theta^\alpha + f^b y^c(\theta\sigma_b\tilde{\sigma}_c)^\alpha + 2\eta^\alpha\theta^2 - \mathrm{i}(\bar{\eta}\tilde{\sigma}_b)^\alpha y^b$$

$K_{ab} = -K_{ba} \longleftrightarrow K_{\alpha\beta} = K_{\beta\alpha}$ Lorentz transformation;
Δ dilatation; Ω R-symmetry transformation (or U(1) chiral rotation);
f^a special conformal transformation; η^α S-supersymmetry transformation.

Converting vector indices into spinor ones and vice versa

$$V_a \to V_{\alpha\dot\alpha} := (\sigma^a)_{\alpha\dot\alpha} V_a \ , \qquad V_a = -\frac{1}{2}(\tilde\sigma_a)^{\dot\alpha\alpha} V_{\alpha\dot\alpha}$$

Second-rank antisymmetric tensor $K_{ab} = -K_{ba}$ is equivalent to

$$K_{\alpha\dot\alpha\beta\dot\beta} := (\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta} K_{ab} = 2\varepsilon_{\alpha\beta} K_{\dot\alpha\dot\beta} + 2\varepsilon_{\alpha\delta\dot\beta} K_{\alpha\beta} \ ,$$

where

$$K_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} K_{ab} = K_{\beta\alpha} \ , \qquad K_{\dot\alpha\dot\beta} = \frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta} K_{ab} = K_{\dot\beta\dot\alpha}$$

If K_{ab} is real, then

$$K_{\dot\alpha\dot\beta} = \overline{K_{\alpha\beta}} \equiv \bar K_{\dot\alpha\dot\beta}$$

Gravitational superfield and conformal supergravity

We turn to introducing a supersymmetric generalisation of
(i) the gravitational field described by $e_a = e_a{}^m(x)\partial_m$; and
(ii) its gauge transformation

$$\delta e_a = \delta e_a{}^m(x)\partial_m = [\xi, e_a] + K_a{}^b(x)e_b + \sigma(x)e_a \ , \qquad \xi = \xi^m(x)\partial_m$$

which corresponds to conformal gravity.
Such a geometric formalism was developed by

V. Ogievetsky & E. Sokatchev (1978)

Equivalent, but less geometric approach, was developed by

unpublished W. Siegel (1977)

Gravitational superfield and conformal supergravity

Group of holomorphic coordinate transformations on $\mathbb{C}^{4|2}$

$$y^m \to y'^m = f^m(y,\theta) \ , \quad \theta^\alpha \to \theta'^\alpha = f^\alpha(y,\theta) \ , \quad \mathrm{Ber}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right) \neq 0$$

Every holomorphic transformation on $\mathbb{C}^{4|2}$ acts on the space of supermanifolds $\{\mathcal{M}^{4|4}(\mathcal{H})\}$

$$\mathcal{M}^{4|4}(\mathcal{H}) \to \mathcal{M}^{4|4}(\mathcal{H}')$$

In other words, the superfield $\mathcal{H}^m$, which defines the curved superspace $\mathcal{M}^{4|4}(\mathcal{H})$, transforms under the action of the group.

Superdeterminant = Berezinian

Nonsingular ever $(p,q)\times(p,q)$ supermatrix

$$F=\begin{pmatrix} A & B \\ C & D \end{pmatrix}~, \qquad \det A\neq 0~, \quad \det D \neq 0$$

Here A and D are bosonic $p\times p$ and $q\times q$ matrices, respectively; B and C are fermionic $p\times q$ and $q\times p$ matrices, respectively.

$$\begin{aligned} \mathrm{sdet} F \equiv \mathrm{Ber} F &:= \det(A-BD^{-1}C)\det{}^{-1}D \\ &= \det A \det{}^{-1}(D-CA^{-1}B) \end{aligned}$$

Change of variables on superspace $\mathbb{R}^{p|q}$ parametrised by coordinates $z^A=(x^a,\theta^\alpha)$, where x^a are bosonic and θ^α fermionic coordinates

$$\begin{aligned} z^A \to z'^A &= f^A(z) \\ \int \mathrm{d}^p x' \mathrm{d}^q\theta' \, L(z') &= \int \mathrm{d}^p x \mathrm{d}^q \theta \, \mathrm{Ber}\left(\frac{\partial(x',\theta')}{\partial(x,\theta)}\right) L\Big(z'(z)\Big) \end{aligned}$$

Gravitational superfield and conformal supergravity

Infinitesimal holomorphic coordinate transformation on $\mathbb{C}^{4|2}$

$$y^m \to y'^m = y^m - \lambda^m(y,\theta) , \qquad \theta^\alpha \to \theta'^\alpha = \theta^\alpha - \lambda^\alpha(y,\theta)$$

leads to the following coordinate transformation on $\mathcal{M}^{4|4}(\mathcal{H})$:

$$\begin{aligned} x^m \to x'^m &= x^m - \frac{1}{2}\lambda^m(x+\mathrm{i}\mathcal{H},\theta) - \frac{1}{2}\bar\lambda^m(x-\mathrm{i}\mathcal{H},\bar\theta) \\ \theta^\alpha \to \theta'^\alpha &= \theta^\alpha - \lambda^\alpha(x+\mathrm{i}\mathcal{H},\theta) . \end{aligned}$$

For $\mathcal{H}'^m(x',\theta',\bar\theta') = -\frac{\mathrm{i}}{2}(y'^m - \bar y'^m)$ we get

$$\mathcal{H}'^\mu(x',\theta',\bar\theta') = \mathcal{H}^m(x,\theta,\bar\theta) + \frac{\mathrm{i}}{2}\Big\{\lambda^m(x+\mathrm{i}\mathcal{H},\theta) - \bar\lambda^m(x-\mathrm{i}\mathcal{H},\bar\theta)\Big\}$$

Now we can read off the gauge transformation

$$\delta H^m := H'^m(x,\theta,\bar\theta) - H^m(x,\theta,\bar\theta)$$

Gravitational superfield and conformal supergravity

Nonlinear gauge transformation law of $\mathcal{H}^m$

$$\delta H^m = \frac{\mathrm{i}}{2}(\lambda^m - \bar{\lambda}^m) + \Big(\frac{1}{2}(\lambda^n + \bar{\lambda}^n)\partial_n + \lambda^\alpha \partial_\alpha + \bar{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}}\Big) H^m(x, \theta, \bar{\theta}) ,$$

$$\lambda^m(x, \theta, \bar{\theta}) = \lambda^m(x + \mathrm{i}\mathcal{H}, \theta) \text{ and } \bar{\lambda}^m(x, \theta, \bar{\theta}) = \bar{\lambda}^m(x - \mathrm{i}\mathcal{H}, \bar{\theta}).$$

Some component fields of $\mathcal{H}^m$ can be gauged away. Indeed

$$\begin{aligned}\mathcal{H}^m(x, \theta, \bar{\theta}) = &\ h^m(x) + \theta^\alpha \chi^m_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{m\dot{\alpha}}(x) + \theta^2 S^m(x) + \bar{\theta}^2 \bar{S}^m(x) \\ &+ \theta\sigma^a\bar{\theta} e_a{}^m(x) + \mathrm{i}\bar{\theta}^2\theta^\alpha \Psi^m_\alpha(x) - \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\Psi}^{m\dot{\alpha}}(x) + \theta^2\bar{\theta}^2 A^m(x) .\end{aligned}$$

The gauge transformation law of $\mathcal{H}^m$ can be written as

$$\delta\mathcal{H}^m(x, \theta, \bar{\theta}) = \frac{\mathrm{i}}{2}\lambda^m(x, \theta) - \frac{\mathrm{i}}{2}\bar{\lambda}^m(x, \bar{\theta}) + O(\mathcal{H})$$

The superfield gauge parameters are

$$\begin{aligned}\lambda^m(x, \theta) &= a^m(x) + \theta^\alpha \varphi^m_\alpha(x) + \theta^2 s^m(x) , \\ \lambda^\alpha(x, \theta) &= \epsilon^\alpha(x) + \omega^\alpha{}_\beta(x)\theta^\beta + \theta^2 \eta^\alpha(x) ,\end{aligned}$$

where all bosonic gauge parameters a^m, s^m and $\omega^\alpha{}_\beta$ are complex.

Gravitational superfield and conformal supergravity

Wess-Zumino gauge

$$\mathcal{H}^m(x,\theta,\bar\theta) = \theta\sigma^a\bar\theta e_a{}^m(x) + \mathrm{i}\bar\theta^2\theta^\alpha\Psi^m_\alpha(x) - \mathrm{i}\theta^2\bar\theta_{\dot\alpha}\bar\Psi^{m\dot\alpha}(x) + \theta^2\bar\theta^2 A^m(x)$$

Residual gauge freedom

$$\lambda^m(x,\theta) = \xi^m(x) + 2\mathrm{i}\theta\sigma^a\bar\epsilon(x)e_a{}^m(x) - 2\theta^2\bar\epsilon(x)\bar\Psi^m(x) \,,$$

$$\lambda^\alpha(x,\theta) = \epsilon^\alpha(x) + \frac{1}{2}[\sigma(x)+\mathrm{i}\Omega(x)]\theta^\alpha + K^\alpha{}_\beta(x)\theta^b + \theta^2\eta^\alpha(x) \,,$$

$$K_{\alpha\beta} = K_{\beta\alpha}$$

The bosonic parameters ξ^m, σ and Ω are real.
ξ^m general coordinate transformation;
$K_{\alpha\beta}$ local Lorentz transformation;
σ and Ω Weyl and local R-symmetry transformations, respectively;
ϵ^α local supersymmetry transformation;
η^α local S-supersymmetry transformation.

Gravitational superfield and conformal supergravity

General coordinate transformation

$$\delta_\xi e_a{}^m = \xi^n \partial_n e_a{}^m - e_a{}^n \partial_n \xi^m \quad \Longleftrightarrow \quad \delta_\xi e_a = \delta e_a{}^m(x)\partial_m = [\xi, e_a] ~,$$
$$\delta_\xi \Psi^m_\alpha = \xi^n \partial_n \Psi^m_\alpha - \Psi_\alpha{}^n \partial_n \xi^m ~, \qquad \delta_\xi A^m = \xi^n \partial_n A^m - A^n \partial_n \xi^m ~.$$

Each of the fields $e_a{}^m$, Ψ^m_α and A^m transforms as a world vector, with respect to the index m, under the general coordinate transformations. The index 'm' is to be interpreted as a curved-space index.

Local Lorentz transformation

$$\delta_K e_a{}^m = K_a{}^b e_b{}^m ~, \qquad \delta_K \Psi_\alpha{}^m = K_\alpha{}^\beta \Psi^m_\beta ~.$$

We see that $\delta_\xi e_a{}^m$ and $\delta_K e_a{}^m$ coincide with the transformation laws of the inverse vielbein. Since in Minkowski superspace

$$\mathcal{H}^m_0(x, \theta, \bar\theta) = \theta\sigma^a\bar\theta\delta_a{}^m \quad \Longrightarrow \quad {}_{(0)}e_a{}^m = \delta_a{}^m ~,$$

we interpret the field $e_a{}^m(x)$ as the inverse vielbein, $\det(e_a{}^m) \neq 0$.

Gravitational superfield and conformal supergravity

Field redefinition $A^m = \tilde{A}^m + \frac{1}{4} e_a{}^m \varepsilon^{abcd} \omega_{bcd}$
where ω_{bcd} is the torsion-free Lorentz connection.

Local Lorentz transformation

$$\delta_K \tilde{A}^m = 0 \ .$$

Weyl transformation

$$\delta_\sigma e_a{}^m = \sigma e_a{}^m \ , \qquad \delta_\sigma \Psi^m_\alpha = \frac{3}{2} \sigma \Psi^m_\alpha \ , \qquad \delta_\sigma \tilde{A}_m = 0 \ ,$$

with $\tilde{A}_m = g_{mn} \tilde{A}^n$.

Local chiral transformation

$$\delta_\Omega e_a{}^m = 0 \ , \qquad \delta_\sigma \Psi^m_\alpha = -\frac{\mathrm{i}}{2} \Omega \Psi^m_\alpha \ , \qquad \delta_\sigma \tilde{A}_m = \frac{1}{2} \partial_m \Omega \ .$$

$\tilde{A}_m$ is the R-symmetry gauge field.

Gravitational superfield and conformal supergravity

Local supersymmetry transformation

$$\begin{aligned} \delta_\epsilon e_a{}^m &= \mathrm{i}\epsilon\sigma_a\bar{\Psi}^m - \mathrm{i}\Psi^m\sigma_a\bar{\epsilon} \ , \\ \delta_\epsilon \Psi^m_\alpha &= (\sigma^a\tilde{\sigma}^b\nabla_a\epsilon)_\alpha e_b{}^m - 2\mathrm{i}\tilde{A}^m\epsilon_\alpha \ , \\ \delta_\epsilon \tilde{A}_m &= \dots \end{aligned}$$

Local S-supersymmetry transformation
see section 5.1 of I. Buchbinder & SMK, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*

Multiplet of conformal supergravity

$$(e_a{}^m, \Psi^m_\alpha, \bar{\Psi}^m_{\dot{\alpha}}, \tilde{A}_m)$$

$e_a{}^m$	graviton
$\Psi^m_\alpha, \bar{\Psi}^m_{\dot{\alpha}}$	gravitino
$\tilde{A}_m$	$\mathrm{U}(1)_R$ gauge field

Superconformal compensators and supergravity

- In order to obtain a supersymmetric extension of Einstein's gravity, a superconformal compensator $\Upsilon(x,\theta,\bar\theta)$ is required, in addition to the gravitational superfield $\mathcal{H}^m$. Unlike Einstein's gravity, there are several different supermultiplets
  ($\varphi(x)$ in gravity and $\Upsilon(x,\theta,\bar\theta)$ in supergravity)
  that may be chosen to play the role of superconformal compensator.
- One option is a chiral superfield $\phi(y,\theta)$ which is defined on $\mathbb{C}^{4|2}$ and transforms as follows

$$\phi'(y',\theta') = \left[\mathrm{Ber}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right)\right]^{-1/3}\phi(y,\theta)$$

under the holomorphic reparametrisation

$$y^m \to y'^m = f^m(y,\theta)~, \quad \theta^\alpha \to \theta'^\alpha = f^\alpha(y,\theta)~.$$

W. Siegel (1977); W. Siegel & J. Gates (1979)

Gauge-invariant chiral integration measure

$$\mathrm{d}^4y'\mathrm{d}^2\theta'\big(\phi'(y',\theta')\big)^3 = \mathrm{d}^4y\mathrm{d}^2\theta\big(\phi(y,\theta)\big)^3$$

Old minimal supergravity

- In the Wess-Zumino gauge, the residual gauge freedom includes the Weyl and local $\mathrm{U}(1)_R$ transformations (described by the parameters $\sigma(x)$ and $\Omega(x)$, respectively), as well as the local S-supersymmetry transformation (described by $\eta_\alpha(x)$ and its conjugate).
- These gauge symmetries may be used to bring

$$\phi^3(x,\theta) = e^{-1}(x)\Big\{F(x) + \theta^\alpha \chi_\alpha(x) + \theta^2 B(x)\Big\}$$

to the form:

$$\phi^3(x,\theta) = e^{-1}(x)\Big\{1 - 2\mathrm{i}\theta\sigma_a\bar{\Psi}^a(x) + \theta^2 B(x)\Big\}$$

Multiplet of old minimal supergravity

$$(e_a{}^m, \Psi^m_\alpha, \bar{\Psi}^m_{\dot{\alpha}}, \tilde{A}_m, B, \bar{B})$$

Differential geometry for supergravity

- Gravitational superfield $\mathcal{H}^m$ transforms in a nonlinear (non-tensorial) way. Its direct use for constructing supergravity matter actions is not very practical.
- In order to obtain powerful tools to generate supergravity-matter actions, we have to extend to curved superspace the formalism of differential geometry which we use for QFT in curved space.
- In curved spacetime $\mathcal{M}^4$ parametrised by coordinates x^m, the gravitational field is described in terms of covariant derivatives

$$\nabla_a = e_a + \omega_a \ , \qquad e_a = e_a{}^m(x) \ , \qquad \omega_a = \frac{1}{2}\omega_a{}^{bc}(x)M_{bc}$$

  In Einstein's gravity, the structure group is the Lorentz group, more precisely its universal covering $\mathsf{SL}(2, \mathbb{C})$.
- Gravity gauge transformation

$$\delta\nabla_a = [\mathcal{K}, \nabla_a] \ , \quad \delta U = \mathcal{K} U \ , \quad \mathcal{K} = \xi^b(x)\nabla_b + \frac{1}{2}K^{bc}(x)M_{bc} \ ,$$

  where $U(x)$ is a (matter) tensor field (with Lorentz indices only).

Differential geometry for supergravity

- In $\mathcal{N}=1$ supergravity, there are two ways to choose structure group:
  $\mathsf{SL}(2,\mathbb{C})$ R. Grimm, J. Wess & B. Zumino (1978)
  $\mathsf{SL}(2,\mathbb{C})\times \mathsf{U}(1)_R$ P. Howe (1982)
  Howe's approach is suitable for all off-shell formulations for $\mathcal{N}=1$ supergravity, while the Grimm-Wess-Zumino approach is ideal for the so-called old minimal formulation for supergravity.
  The two approaches prove to be equivalent, so here we follow the Grimm-Wess-Zumino approach, which is simpler.
- In curved superspace $\mathcal{M}^{4|4}$ parametrised by local coordinates $z^M=(x^m,\theta^\mu,\bar\theta_{\dot\mu})$, supergravity multiplet is described in terms of covariant derivatives

$$\mathcal{D}_A=(\mathcal{D}_a,\mathcal{D}_\alpha,\bar{\mathcal{D}}^{\dot\alpha})=E_A{}^M(z)\partial_M+\frac{1}{2}\Omega_A{}^{bc}(z)M_{bc}$$

- Supergravity gauge transformation

$$\delta_{\mathcal{K}}\mathcal{D}_A=[\mathcal{K},\mathcal{D}_A]~,\quad \delta_{\mathcal{K}}U=\mathcal{K}U~,\quad \mathcal{K}=\xi^B(z)\mathcal{D}_B+\frac{1}{2}K^{bc}(z)M_{bc}~,$$

for any tensor $U(z)$ a tensor superfield.

Constraints

- In curved space, the covariant derivatives ∇_a obey the algebra

$$[\nabla_a, \nabla_b] \equiv T_{ab}{}^c \nabla_c + \frac{1}{2} R_{ab}{}^{cd} M_{cd} = \frac{1}{2} R_{ab}{}^{cd} M_{cd} ,$$

where $T_{ab}{}^c(x)$ and $R_{ab}{}^{cd}(x)$ are the torsion tensor and the curvature tensor, respectively. To express the Lorentz connection in terms of the gravitational field, one imposes the torsion-free constraint

$$T_{ab}{}^c = 0$$

- In curved superspace, the covariant derivatives obey the algebra

$$[\mathcal{D}_A, \mathcal{D}_B\} = \mathcal{T}_{AB}{}^C \mathcal{D}_C + \frac{1}{2} \mathcal{R}_{AB}{}^{cd} M_{cd} .$$

We have to impose certain constraints on $\mathcal{T}_{AB}{}^C$ in order for superspace geometry to describe conformal supergravity.

Choosing right superspace constraints

- In Minkowski superspace, the vector derivative is given by an anti-commutator of spinor covariant derivatives,

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2\mathrm{i}\partial_{\alpha\dot\alpha}$$

  In curved superspace, we postulate

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha}$$

- In Minkowski superspace, there exist chiral superfields constrained by

$$\bar{D}_{\dot\alpha}\Phi = 0$$

  In curved superspace, we also want to have covariantly chiral scalar superfields constrained by

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0 \quad \Longrightarrow \quad 0 = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\}\Phi = \mathcal{T}_{\dot\alpha\dot\beta}{}^{c}\mathcal{D}_c\Phi + \mathcal{T}_{\dot\alpha\dot\beta}{}^{\gamma}\mathcal{D}_\gamma\Phi$$

  We are forced to require $\mathcal{T}_{\dot\alpha\dot\beta}{}^{c} = 0$ and $\mathcal{T}_{\dot\alpha\dot\beta}{}^{\gamma} = 0$.

- Similar to GR, we need constraints that would allow us to express the Lorentz connection in terms of the (inverse) vielbein.

Curved superspace covariant derivatives

Algebra of the superspace covariant derivatives

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha} ,$$
$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta} , \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4R\bar{M}_{\dot\alpha\dot\beta} ,$$
$$\Big[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}\Big] = \mathrm{i}\varepsilon_{\alpha\beta}\Big(\bar{R}\,\bar{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot\beta})M_{\gamma\delta} + 2\bar{W}_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\Big)$$
$$+\mathrm{i}(\bar{\mathcal{D}}_{\dot\beta}\bar{R})M_{\alpha\beta} .$$

The superfields R, $G_{\alpha\dot\alpha}$ and $W_{\alpha\beta\gamma}$ obey the Bianchi identities:

$$\bar{\mathcal{D}}_{\dot\alpha}R = 0 , \qquad \bar{\mathcal{D}}_{\dot\alpha}W_{\alpha\beta\gamma} = 0 ;$$
$$\bar{\mathcal{D}}^{\dot\alpha}G_{\alpha\dot\alpha} = \mathcal{D}_\alpha R , \qquad \mathcal{D}^\gamma W_{\alpha\beta\gamma} = \mathrm{i}\,\mathcal{D}_{(\alpha}{}^{\dot\gamma}G_{\beta)\dot\gamma} .$$

R is the supersymmetric extension of the scalar curvature.
G_a is the supersymmetric extension of the Ricci tensor.
$W_{\alpha\beta\gamma}$ is the supersymmetric extension of the Weyl tensor.

It may be shown that the gravitational superfield $\mathcal{H}^m$ originates by solving the supergravity constraints in terms of unconstrained prepotentials.

Super-Weyl invariance and conformal supergravity

The algebra of covariant derivatives is invariant under super-Weyl transformations P. Howe & R. Tucker (1978)

$$\delta_\Sigma \mathcal{D}_\alpha = (\bar\Sigma - \frac{1}{2}\Sigma)\mathcal{D}_\alpha + (\mathcal{D}^\beta \Sigma)\, M_{\alpha\beta} , \qquad \bar{\mathcal{D}}_{\dot\alpha}\Sigma = 0$$

$$\delta_\Sigma \bar{\mathcal{D}}_{\dot\alpha} = (\Sigma - \frac{1}{2}\bar\Sigma)\bar{\mathcal{D}}_{\dot\alpha} + (\bar{\mathcal{D}}^{\dot\beta}\bar\Sigma)\bar{M}_{\dot\alpha\dot\beta} ,$$

$$\delta_\Sigma \mathcal{D}_{\alpha\dot\alpha} = \{\delta_\Sigma \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} + \{\mathcal{D}_\alpha, \delta_\Sigma \bar{\mathcal{D}}_{\dot\alpha}\} = \frac{1}{2}(\Sigma + \bar\Sigma)\mathcal{D}_{\alpha\dot\alpha} + \dots ,$$

provided the torsion tensor superfields transform as follows:

$$\delta_\Sigma R = 2\Sigma R + \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar\Sigma ,$$

$$\delta_\Sigma G_{\alpha\dot\alpha} = \frac{1}{2}(\Sigma + \bar\Sigma)G_{\alpha\dot\alpha} + \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}(\Sigma - \bar\Sigma) , \qquad \delta_\Sigma W_{\alpha\beta\gamma} = \frac{3}{2}\Sigma W_{\alpha\beta\gamma} .$$

Gauge freedom of conformal supergravity:

$$\delta \mathcal{D}_A = [\delta_{\mathcal{K}}, \mathcal{D}_A] + \delta_\Sigma \mathcal{D}_A , \qquad \mathcal{K} = \xi^B(z)\mathcal{D}_B + \frac{1}{2}K^{bc}(z)M_{bc}$$

Reduction to component fields

Given a superfield $U(z)$, its bar-projection $U|$ is defined to be the $\theta, \bar\theta$-independent component of $U(x, \theta, \bar\theta)$ in powers of θ's and $\bar\theta$'s,

$$U| := U(x, \theta, \bar\theta)|_{\theta = \bar\theta = 0} \ .$$

$U|$ is a field on curved spacetime $\mathcal{M}^4$ which is the bosonic body of $\mathcal{M}^{4|4}$. In a similar way we define the bar-projection of the covariant derivatives:

$$\mathcal{D}_A| := E_A{}^M|\partial_M + \frac{1}{2}\Omega_A{}^{bc}|\mathcal{M}_{bc} \ .$$

Of special importance is the bar-projection of a vector covariant derivative

$$\mathcal{D}_a| = \hat\nabla_a + \frac{1}{2}\Psi_a{}^\beta \mathcal{D}_\beta| + \frac{1}{2}\bar\Psi_{a\dot\beta}\bar{\mathcal{D}}^{\dot\beta}| \ ,$$

where $\Psi_a{}^\beta$ is gravitino, and

$$\hat\nabla_a = e_a + \omega_a \equiv e_a{}^m(x)\partial_m + \frac{1}{2}\omega_a{}^{bc}(x)M_{bc},$$

is a spacetime covariant derivative with torsion.

Reduction to component fields

Wess-Zumino (WZ) gauge

$$\mathcal{D}_\alpha| = \delta_\alpha{}^\mu \frac{\partial}{\partial\theta^\mu} , \qquad \bar{\mathcal{D}}^{\dot\alpha}| = \delta^{\dot\alpha}{}_{\dot\mu} \frac{\partial}{\partial\bar\theta_{\dot\mu}} .$$

In this gauge, one obtains

$$E_a{}^m| = e_a{}^m , \qquad E_a{}^\mu| = \frac{1}{2}\Psi_a{}^\beta\delta_\beta{}^\mu , \qquad \Omega_a{}^{bc}| = \omega_a{}^{bc}$$

In the WZ gauge, we still have a tail of component fields which originates at higher orders in the $\theta, \bar\theta$-expansion of $E_A{}^M$, $\Omega_A{}^{bc}$ and which are pure gauge (that is, they may be completely gauged away). A way to get rid of such a tail of redundant fields is to impose a normal gauge around the bosonic body $\mathcal{M}^4$ of the curved superspace $\mathcal{M}^{4|4}$.

Vielbein (E^A) and connection (Ω^{cd}) super one-forms:

$$E^A = \mathrm{d}z^M E_M{}^A(z) , \quad \Omega^{cd} = \mathrm{d}z^M \Omega_M{}^{cd}(z) = E^A \Omega_A{}^{bc}$$

$$E_M{}^A E_A{}^B = \delta_M{}^N$$

Reduction to component fields

Normal gauge in superspace

$$\Theta^M E_M{}^A(x,\Theta) = \Theta^M \delta_M{}^A \ ,$$
$$\Theta^M \Omega_M{}^{cd}(x,\Theta) = 0 \ ,$$

where $\Theta^M \equiv (\Theta^m, \Theta^\mu, \bar\Theta_{\dot\mu}) := (0, \theta^\mu, \bar\theta_{\dot\mu})$.
In this gauge, $E_M{}^A(x,\Theta)$ and $\Omega_M{}^{cd}(x,\Theta)$ and $\Phi_M(x,\Theta)$ are given by Taylor series in Θ, in which all the coefficients (except for the leading Θ-independent terms given on previous page) are tensor functions of the torsion, the curvature and their covariant derivatives evaluated at $\Theta = 0$.

I. McArthur (1983)
SMK & G. Tartaglino-Mazzucchelli, arXiv:0812.3464

Analogue of the Fock-Schwinger gauge in Yang-Mills theories

$$x^m A_m{}^I(x) = 0 \ ,$$

where $A_m{}^I$ is the Yang-Mills gauge fields, with 'I' the gauge group index.

Reduction to component fields

The supergravity auxiliary fields occur as follows

$$R| = \frac{1}{3}\bar{B} , \qquad G_a| = \frac{4}{3}A_a .$$

The bar-projection of the vector covariant derivatives are

$$\mathcal{D}_a| = \nabla_a - \frac{1}{3}\varepsilon_{abcd}\, A^d M^{bc} + \frac{1}{2}\Psi_a{}^\beta\, \mathcal{D}_\beta| + \frac{1}{2}\bar{\Psi}_{a\dot\beta}\, \bar{\mathcal{D}}^{\dot\beta}| ,$$

where we have introduced the spacetime covariant derivatives, $\nabla_a = e_a + \frac{1}{2}\omega_{abc}M^{bc}$, with $\omega_{abc} = \omega_{abc}(e, \Psi)$ the Lorentz connection.

$$[\nabla_a, \nabla_b] = T_{ab}{}^c\, \nabla_c + \frac{1}{2}R_{abcd}M^{cd} ,$$
$$T_{abc} = -\frac{\mathrm{i}}{2}(\Psi_a\sigma_c\bar{\Psi}_b - \Psi_b\sigma_c\bar{\Psi}_a) .$$

The Lorentz connection is

$$\omega_{abc} = \omega_{abc}(e) - \frac{1}{2}(T_{bca} + T_{acb} - T_{abc}) ,$$

where $\omega_{abc}(e)$ is the torsion-free connection

Reduction to component fields

Some component results

$$\mathcal{D}_\alpha R| = -\frac{2}{3}(\sigma^{bc}\Psi_{bc})_\alpha - \frac{2\mathrm{i}}{3}A^b\Psi_{b\alpha} + \frac{\mathrm{i}}{3}\bar{B}(\sigma^b\bar{\Psi}_b)_\alpha ~,$$

$$\bar{\mathcal{D}}_{(\dot\alpha}G^\beta{}_{\dot\beta)}| = -2\Psi_{\dot\alpha\dot\beta,}{}^\beta + \frac{\mathrm{i}}{3}\bar{B}\bar{\Psi}^\beta{}_{(\dot\alpha,\dot\beta)} - 2\mathrm{i}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}\Psi_a{}^\beta A_b + \frac{2\mathrm{i}}{3}\Psi_{\alpha(\dot\alpha,}{}^\alpha A_{\dot\beta)}{}^\beta ~,$$

$$W_{\alpha\beta\gamma}| = \Psi_{(\alpha\beta,\gamma)} - \mathrm{i}(\sigma_{ab})_{(\alpha\beta}\Psi^a{}_{\gamma)}A^b ~,$$

where the gravitino filed strength $\Psi_{ab}{}^\gamma$ is defined by

$$\Psi_{ab}{}^\gamma = \nabla_a\Psi_b{}^\gamma - \nabla_b\Psi_a{}^\gamma - \mathcal{T}_{ab}{}^c\Psi_c{}^\gamma ~,$$

$$\Psi_{\alpha\beta,}{}^\gamma = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}\Psi_{ab}{}^\gamma ~, \qquad \Psi_{\dot\alpha\dot\beta,}{}^\gamma = -\frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}\Psi_{ab}{}^\gamma ~.$$

Reduction to component fields

Locally supersymmetric action principle

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, E\, \mathcal{L}~, \qquad E^{-1} = \mathrm{Ber}(E_A{}^M)$$

where the Lagrangian $\mathcal{L}$ is a scalar superfield.

Locally supersymmetric chiral action

$$S_c = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, \frac{E}{R}\, \mathcal{L}_c = \int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{E}\, \mathcal{L}_c~, \qquad \bar{\mathcal{D}}_{\dot\alpha}\mathcal{L}_c = 0$$

Chiral integration rule:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, E\, \mathcal{L} = -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{E}\, (\bar{\mathcal{D}}^2 - 4R)\mathcal{L}$$

Component action:

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{E}\, \mathcal{L}_c = \int \mathrm{d}^4x\, e\Big\{ -\frac{1}{4}\mathcal{D}^2\mathcal{L}_{\rm c}| - \frac{\mathrm{i}}{2}(\bar\Psi^b\tilde\sigma_b)^\alpha\, \mathcal{D}_\alpha\mathcal{L}_{\rm c}| + (B + \bar\Psi^a\tilde\sigma_{ab}\bar\Psi^b)\, \mathcal{L}_{\rm c}|\Big\}$$

WZ gauge and super-Weyl invariance

To choose the WZ + normal gauges, we make use of the general coordinate and local Lorentz transformations,

$$\delta\mathcal{D}_A = [\delta_{\mathcal{K}}, \mathcal{D}_A] \ , \qquad \mathcal{K} = \xi^B(z)\mathcal{D}_B + \frac{1}{2}K^{bc}(z)M_{bc}$$

The super-Weyl invariance remains intact. This local symmetry may be gauge-fixed by choosing useful conditions on the superconformal compensator(s) upon reducing the supergravity-matter system under consideration to components.

To see how this works in practice, see

SMK & S. McCarthy, arXiv:hep-th/0501172

Compensators and off-shell formulations for supergravity

Old minimal formulation for $\mathcal{N}=1$ supergravity
Its conformal compensators are Φ and $\bar{\Phi}$. Here Φ is a covariantly chiral, nowhere vanishing scalar Φ,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \ , \qquad \Phi \neq 0 \ ,$$

with the gauge transformation

$$\delta\Phi = \mathcal{K}\Phi + \Sigma\Phi \ .$$

The supergravity action is

$$S = -\frac{3}{\kappa^2}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, E\,\Phi\bar{\Phi} + \left\{\frac{\mu}{\kappa^2}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\Phi^3 + \mathrm{c.c.}\right\} \ ,$$

where κ is the gravitational coupling constant and μ a cosmological parameter.

Old minimal supergravity: Component action

$$
\begin{aligned}
S_{\rm SG} &= -\frac{3}{\kappa^2}\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\, E\,\Phi\bar\Phi \\
&= \frac{1}{\kappa^2}\int {\rm d}^4x\, e^{-1}\Big\{\frac{1}{2}\mathcal{R} + \frac{1}{4}\varepsilon^{abcd}(\bar\Psi_a\tilde\sigma_b\Psi_{cd} - \Psi_a\sigma_b\bar\Psi_{cd}) \\
&\qquad + \frac{4}{3}A^aA_a - \frac{1}{3}\bar B B\Big\}
\end{aligned}
$$

Supersymmetric cosmological term

$$
\begin{aligned}
S_{\rm cosm} &= \frac{\mu}{\kappa^2}\int {\rm d}^4x{\rm d}^2\theta\,\mathcal{E}\Phi^3 + {\rm c.c.} \\
&= \frac{\mu}{\kappa^2}\int {\rm d}^4x\, e\Big\{B - \frac{1}{2}\bar\Psi^a\tilde\sigma_a\sigma_b\bar\Psi^b - \frac{1}{2}\bar\Psi^a\bar\Psi_b\Big\} + {\rm c.c.}
\end{aligned}
$$

Eliminating the auxiliary fields B and $\bar B$ leads to the cosmological term

$$
3\frac{|\mu|^2}{\kappa^2}\int {\rm d}^4x\, e = -\Lambda\int {\rm d}^4x\, e \quad\Longrightarrow\quad \Lambda = -3\frac{|\mu|^2}{\kappa^2}
$$

Compensators and off-shell formulations for supergravity

New minimal formulation for $\mathcal{N} = 1$ supergravity

Its conformal compensator is a real covariantly linear, nowhere vanishing scalar $\mathbb{L}$,

$$(\bar{\mathcal{D}}^2 - 4R)\mathbb{L} = 0 , \qquad \bar{\mathbb{L}} = \mathbb{L} ,$$

with the gauge transformation

$$\delta\mathbb{L} = \mathcal{K}\mathbb{L} + (\Sigma + \bar{\Sigma})\mathbb{L} .$$

Supergravity action

$$S_{\rm SG} = \frac{3}{\kappa^2} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathbb{L} \ln \mathbb{L}$$

The action is super-Weyl invariant due to the identity

$$\int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathbb{L}\Sigma = 0 \quad \Longleftarrow \quad \bar{\mathcal{D}}_{\dot{\alpha}}\Sigma = 0$$

No cosmological term in new minimal supergravity

New minimal supergravity: The auxiliary field sector

The auxiliary fields of new minimal supergravity are gauge one- and two-forms, $A_1 = A_m(x)\mathrm{d}x^m$ and $B_2 = \frac{1}{2}B_{mn}(x)\mathrm{d}x^m \wedge \mathrm{d}x^n$. Here A_m is the $\mathrm{U}(1)_R$ gauge field, which belongs to the gravitational superfield, while the two-form B appears only via its gauge-invariant field strength

$$H_3 = \mathrm{d}B_2 = \frac{1}{3!}H_{mnr}\mathrm{d}x^m \wedge \mathrm{d}x^n \wedge \mathrm{d}x^r$$

The Hodge-dual H^m of H_{mnr} is a component field of the compensator. In the flat-superspace limit,

$$H_{\alpha\dot\alpha} \propto [D_\alpha, \bar D_{\dot\alpha}]\mathbb{L}| \ , \qquad \partial_a H^a = 0$$

The auxiliary fields contribute to the supergravity action as follows:

$$\int \mathrm{d}^4x \Big\{ c_1 (H^\star)_1 \wedge H_3 + c_2 A_1 \wedge H_3 \Big\} \ ,$$

with c_1 and c_2 numerical coefficients. Both fields are non-dynamical, $H_3 = 0$ and $F_2 = \mathrm{d}A_1 = 0$ on the mass shell.

New minimal supergravity coupled to σ-model matter

Consider a Kähler manifold parametrized by n complex coordinates ϕ^i and their conjugates $\bar{\phi}^{\underline{i}}$, with $K(\phi, \bar{\phi})$ the Kähler potential.
Supergravity-matter system:

$$S_{\rm new} = \frac{3}{\kappa^2} \int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar{\theta}\, E\, \mathbb{L}\, {\rm ln}\mathbb{L} + \int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar{\theta}\, E\, \mathbb{L}\, K(\phi, \bar{\phi}) \ .$$

The σ-model variables ϕ^i are covariantly chiral scalar superfields, $\bar{\mathcal{D}}_{\dot{\alpha}}\phi^i = 0$, being inert under the super-Weyl transformations.
The action is invariant under the Kähler transformations

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + \lambda(\phi) + \bar{\lambda}(\bar{\phi}) \ ,$$

with $\lambda(\phi)$ an arbitrary holomorphic function.

Classical equivalence of new & old minimal supergravities

First-order reformulation of the new minimal supergravity-matter system

$$S_{\rm new} = 3\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\, E\,\left(U\,\mathbb{L} - \Upsilon\right)~, \qquad \Upsilon = \exp\Big(U - \frac{1}{3}K(\phi,\bar\phi)\Big)~,$$

and U is an unconstrained real scalar superfield. Super-Weyl invariance:

$$\delta_\Sigma U = \Sigma + \bar\Sigma~.$$

To preserve Kähler invariance, the Kähler transformation of U should be

$$U ~\to~ U + \frac{1}{3}\left(\lambda(\phi) + \bar\lambda(\bar\phi)\right)~.$$

The equation of motion for $\mathbb{L}$ is $(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha U = 0$, and is solved by

$$U = \ln\Phi + \ln\bar\Phi~, \qquad \bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$$

The action turns into (restore κ^2)

$$S_{\rm old} = -\frac{3}{\kappa^2}\int {\rm d}^4x{\rm d}^2\theta{\rm d}^2\bar\theta\, E\,\bar\Phi\,\Phi\,\exp\Big(-\frac{\kappa^2}{3}K(\phi,\bar\phi)\Big)$$

Kähler invariance: $\Phi ~\to~ {\rm e}^{\kappa^2\lambda(\phi)/3}\,\Phi$

Chiral Goldstino superfield

Action for a free massless chiral superfield ($\bar{D}_{\dot{\alpha}} X = 0$)

$$S[X,\bar{X}] = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, X\bar{X} = \int \mathrm{d}^4x\, (\phi\Box\bar{\phi} - \mathrm{i}\rho\sigma^a\partial_a\bar{\rho} + F\bar{F})\,,$$

where the component fields are defined by

$$X| = \phi\,, \qquad D_\alpha X| = \sqrt{2}\rho_\alpha\,, \qquad -\frac{1}{4}D^2X| = F\,.$$

Goldstino superfield

M. Roček (1978)

$$X^2 = 0\,, \qquad -\frac{1}{4}X\bar{D}^2\bar{X} = fX\,,$$

where f is a non-zero parameter of dimension $(\mathrm{mass})^2$. The auxiliary field acquires a non-zero expectation value, $\langle F\rangle = f$.

Chiral Goldstino superfield

Solution to the constraint $X^2 = 0$ is

$$\phi = \frac{\rho^2}{2F} .$$

Solution to the second constraint, $-\frac{1}{4} X \bar{D}^2 \bar{X} = fX$, is

$$\begin{aligned} F &= f + \bar{F}^{-1} \langle \bar{u} \rangle - \frac{1}{4} \bar{F}^{-2} \bar{\rho}^2 \Box (F^{-1} \rho^2) \\ &= f \Big\{ 1 + f^{-2} \langle \bar{u} \rangle - f^{-4} (\langle u \rangle \langle \bar{u} \rangle + \frac{1}{4} \bar{\rho}^2 \Box \rho^2) + f^{-6} (\langle u \rangle^2 \langle \bar{u} \rangle + \text{c.c.}) \\ &\quad + \frac{f^{-6}}{4} \Big(\langle \bar{u} \rangle \rho^2 \Box \bar{\rho}^2 + 2 \langle u \rangle \bar{\rho}^2 \Box \rho^2 + \bar{\rho}^2 \Box (\rho^2 \langle \bar{u} \rangle) \Big) \\ &\quad - 3 f^{-8} \Big(\langle u \rangle^2 \langle \bar{u} \rangle^2 + \frac{1}{4} \rho^2 \bar{\rho}^2 \Box (\langle u \rangle^2 - \langle u \rangle \langle \bar{u} \rangle + \langle \bar{u} \rangle^2) + \frac{1}{16} \rho^2 \bar{\rho}^2 \Box \bar{\rho}^2 \Box \rho^2 \Big) \Big) \end{aligned}$$

where $\langle M \rangle = \text{tr} M = M_a{}^a$, and

$$u = (u_a{}^b), \quad u_a{}^b := \mathrm{i} \rho \sigma^b \partial_a \bar{\rho} \; ; \qquad \bar{u} = (\bar{u}_a{}^b), \quad \bar{u}_a{}^b := -\mathrm{i} \partial_a \rho \sigma^b \bar{\rho}$$

Chiral Goldstino superfield

Supersymmetry transformation

$$\delta_\epsilon \phi = \sqrt{2}\epsilon\rho\,, \quad \delta_\epsilon \rho_\alpha = \sqrt{2}\big(\epsilon_\alpha F + \mathrm{i}(\sigma^a\bar\epsilon)_\alpha \partial_a \phi\big)\,, \qquad \delta_\epsilon F = -\sqrt{2}\mathrm{i}(\partial_a \psi \sigma^a \bar\epsilon)\,.$$

Since F acquires the non-zero expectation value $\langle F\rangle = f$, supersymmetry becomes nonlinearly realised,

$$\delta_\epsilon \rho_\alpha = \sqrt{2} f \epsilon_\alpha + \dots$$

Goldstino action

$$S_{\rm Goldstino} = -\int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, X\bar X = -f\int \mathrm{d}^4x \mathrm{d}^2\theta\, X$$

$$= -\frac{1}{2}\int \mathrm{d}^4x \Big\{ 2f^2 + \langle u + \bar u\rangle + \frac{1}{2}f^{-2}(\partial^a\rho^2\partial_a\bar\rho^2 - 4\langle u\rangle\langle\bar u\rangle)$$

$$+f^{-4}\Big(\langle u\rangle(\bar\rho^2\Box\rho^2 + 2\langle u\rangle\langle\bar u\rangle) + \mathrm{c.c.}\Big)$$

$$+3f^{-6}\Big(\langle u^2\rangle\langle\bar u^2\rangle - 3\langle u\rangle^2\langle\bar u\rangle^2 - 2\langle u\rangle\langle\bar u\rangle\langle u\bar u\rangle - \frac{3}{8}\rho^2\bar\rho^2\Box\rho^2\Box\bar\rho^2\Big)\Big\}$$

Volkov-Akulov model for Goldstino

D. Volkov & V. Akulov (1972)

$$S_{\rm VA}[\lambda, \bar{\lambda}] = \frac{1}{2\kappa^2} \int {\rm d}^4x \Big(1 - \det \Xi\Big) ,$$

where κ denotes the coupling constant of dimension $(\rm length)^2$ and

$$\Xi_a{}^b = \delta_a{}^b + \kappa^2 \, (v + \bar{v})_a{}^b , \quad v_a{}^b := {\rm i} \lambda \sigma^b \partial_a \bar{\lambda} , \qquad \bar{v}_a{}^b := -{\rm i} \partial_a \lambda \sigma^b \bar{\lambda} .$$

$S_{\rm VA}$ is invariant under the nonlinear supersymmetry transformations

$$\delta_\xi \lambda_\alpha = \frac{1}{\kappa} \epsilon_\alpha - {\rm i} \kappa \Big(\lambda \sigma^b \bar{\epsilon} - \epsilon \sigma^b \bar{\lambda} \Big) \partial_b \lambda_\alpha .$$

Roček's Goldstino action is related to the Volkov-Akulov model by a nonlinear filed redefinition ($f^{-2} = 2\kappa^2$)

SMK and S. Tyler (2011)

Spontaneously broken (de Sitter) supergravity

Chiral Goldstino superfield coupled to supergravity

U. Lindström & M. Roček (1979)

$$S = -\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\mathrm{d}^2\bar{\theta}\, E\left(\frac{3}{\kappa^2}\bar{\Phi}\Phi + \bar{X}X\right) + \left\{\frac{\mu}{\kappa^2}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\Phi^3 + \mathrm{c.c.}\right\} ,$$

where X is covariantly chiral, $\bar{\mathcal{D}}_{\dot{\alpha}}X = 0$, and obeys the super-Weyl invariant constraints

$$X^2 = 0 , \qquad -\frac{1}{4}X(\bar{\mathcal{D}}^2 - 4R)\bar{X} = f\Phi^2 X .$$

Upon reducing the action to components and eliminating the supergravity auxiliary fields, for the cosmological constant one gets

$$\Lambda = f^2 - 3\frac{|\mu|^2}{\kappa^2} .$$

Cosmological constant is positive, $\Lambda > 0$, for $f^2 > 3\frac{|\mu|^2}{\kappa^2}$.

Alternative approaches to de Sitter supergravity

- E. A. Bergshoeff, D. Z. Freedman, R. Kallosh and A. Van Proeyen, arXiv:1507.08264.
- F. Hasegawa and Y. Yamada, arXiv:1507.08619.
- SMK and S. J. Tyler, arXiv:1102.3042;
  SMK, arXiv:1508.03190.
- I. Bandos, L. Martucci, D. Sorokin and M. Tonin, arXiv:1511.03024.

The first two groups used a nilpotent chiral Goldstino superfield

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, \bar X X + f\int \mathrm{d}^4x\mathrm{d}^2\theta\, X + f\int \mathrm{d}^4x\mathrm{d}^2\bar\theta\, \bar X~, \quad \bar D_{\dot\alpha} X = 0~,$$

where X is constrained by $X^2 = 0$.

R. Casalbuoni, S. De Curtis, D. Dominici, F. Feruglio & R. Gatto (1989)

Z. Komargodski & N. Seiberg (2009)

Complex linear Goldstino superfield

SMK & S. Tyler (2011)

$$-\frac{1}{4}\bar{D}^2\Sigma = f\,, \qquad f = \text{const}\,,$$

$$\Sigma^2 = 0\,, \qquad -\frac{1}{4}\Sigma\bar{D}^2 D_\alpha\Sigma = f\,D_\alpha\Sigma\,.$$

The constraints imply that all component fields of Σ are constructed in terms of a single spinor field $\bar{\rho}^{\dot\alpha}$.

$$\Sigma(\theta,\bar\theta) = \mathrm{e}^{\mathrm{i}\theta\sigma^a\bar\theta\partial_a}\Big(\phi + \theta\psi + \sqrt{2}\bar\theta\bar\rho + \theta^2 F + \bar\theta^2 f + \theta^\alpha\bar\theta^{\dot\alpha}U_{\alpha\dot\alpha} + \theta^2\bar\theta\bar\chi\Big)\,.$$

Goldstino superfield action

$$S[\Sigma,\bar\Sigma] = -\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Sigma\bar\Sigma\,.$$

Roček's Goldstino superfield is a composite object

$$f\,X = -\frac{1}{4}\bar{D}^2(\bar\Sigma\Sigma)$$

Bryce DeWitt, *Dynamical Theory of Groups and Fields* (1965), about the status of Yang-Mills theories in the 1960s:

"So far not a shred of experimental evidence exists that fields possessing non-Abelian infinite dimensional invariance groups play any role in physics at the quantum level. And yet motivation for studying such fields in a quantum context is not entirely lacking."

That situation completely changed in the early 1970s.

Nowadays, no one doubts that the Yang-Mills theories play a crucial role in physics.

What does the future hold for supersymmetry and supergravity?

Steven Weinberg, The Quantum Theory of Fields: Volume III: Supersymmetry (2000)

"I and many physicists are reasonably confident that supersymmetry will be found to be relevant to the real world, and perhaps soon."