

Cyclic Leibniz Rule, Cohomology and Non-renormalization Theorem in Lattice Supersymmetry

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It would be of great importance to reveal *non-perturbative aspects* of SUSY theories.

Great success of *Lattice Gauge theory* suggests that *Lattice SUSY Theory*, if possible, could provide a powerful tool to analyze non-perturvative properties of the theories.

But • • •

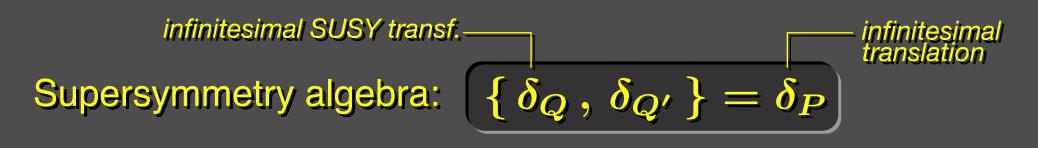


For more than 30 years, no one has succeeded satisfactorily to realize *supersymmetry algebra* on lattice!

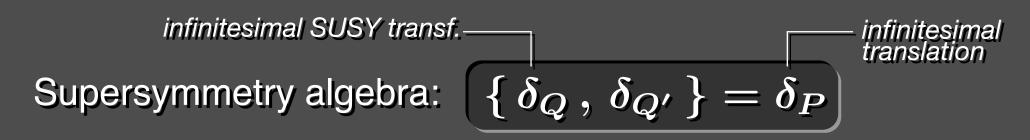
In my talk, I would like to discuss

What are obstacles to construct Lattice SUSY?
How can we circumvent them?



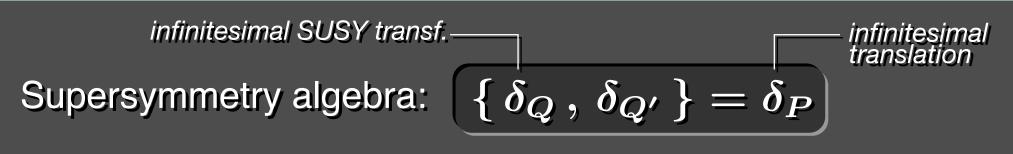






 $\delta_P(\phi\psi) = (\delta_P\phi)\psi + \phi(\delta_P\psi)$ - Leibniz rule





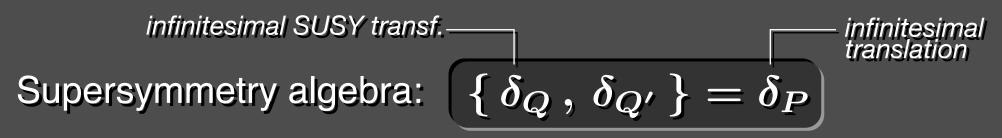
$$\delta_{P}(\phi\psi) = (\delta_{P}\phi)\psi + \phi(\delta_{P}\psi) \longleftarrow \text{Leibniz rule}$$

$$\lim_{\phi \to \phi} \delta_{P}\phi = \nabla\phi$$

$$= \nabla\phi$$

$$= quivalent \text{ to a certain difference operator on lattice}$$





$$\delta_{P}(\phi\psi) = (\delta_{P}\phi)\psi + \phi(\delta_{P}\psi) \longleftarrow \text{Leibniz rule}$$

linear realization of δ_{P} on lattice
 $\delta_{P}\phi = \nabla\phi$
equivalent to a certain
difference operator
on lattice
on lattice

SUSY algebra on lattice would require the Leibniz rule for ∇ .



No-Go Theorem*M.Kato, M.S. & H.So, JHEP 05(2008)057*There is *no* difference operator ∇ on lattice
satisfying the following three properties:i) discrete translation invariance
ii) locality
iii) Leibniz rule $\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$



No-Go Theorem*M.Kato, M.S. & H.So, JHEP 05(2008)057*There is **no** difference operator ∇ on lattice
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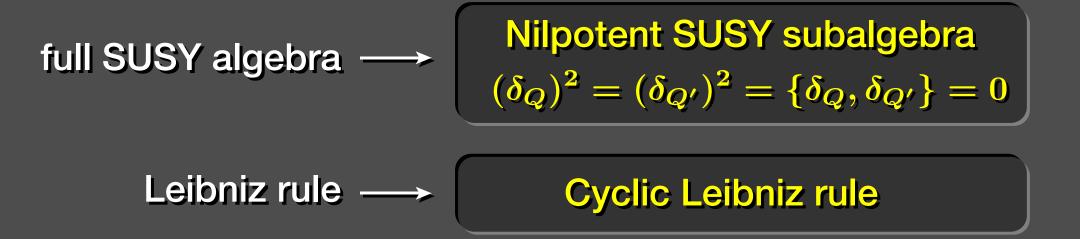
The No-Go theorem tells us that we *cannot* realize SUSY algebra with ∇ on lattice!

Our approach to construct lattice SUSY models 6

Our strategy to construct lattice SUSY models is



Our strategy to construct lattice SUSY models is



Complex SUSY quantum mechanics on lattice 7

Lattice action

$$egin{aligned} S &= (
abla \phi_{-},
abla \phi_{+}) - (F_{-}, F_{+}) - i(\chi_{-},
abla ar{\chi}_{+}) + i(
abla ar{\chi}_{-}, \chi_{+}) \ & -\lambda_{+}(F_{+}, \phi_{+} st \phi_{+}) + 2\lambda_{+}(\chi_{+}, ar{\chi}_{+} st \phi_{+}) \ & -\lambda_{-}(F_{-}, \phi_{-} st \phi_{-}) - 2\lambda_{-}(\chi_{-}, ar{\chi}_{-} st \phi_{-}) \end{aligned}$$

difference operator: $(\nabla \phi)_n \equiv \sum_m \nabla_{nm} \phi_m$ lattice points inner product: $(\phi, \psi) \equiv \sum_n \phi_n \psi_n$

Complex SUSY quantum mechanics on lattice 🧭

Lattice action

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difference operator: $(\nabla \phi)_n \equiv \sum_m \nabla_{nm} \phi_m$ lattice points inner product: $(\phi, \psi) \equiv \sum_n \phi_n \psi_n$

A crucial point is that the *field product* is generalized such that

field product:
$$\phi_n \psi_n \longrightarrow (\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$$

same lattice point Fields are allowed to interact between different lattice points!

Discrete translation invariance & Locality



difference operator: $(\nabla \phi)_n \equiv \sum_m \nabla_{nm} \phi_m$ field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$

We impose *translation invariance* & *locality* on difference operators and field products.

i) discrete translation invariance

$$abla_{nm} =
abla(n-m), \qquad M_{nlm} = M(l-n,m-n)$$

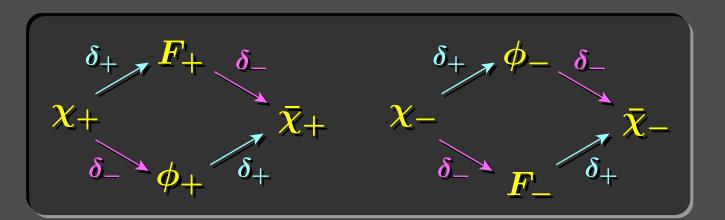
ii) locality

$$abla(n) \xrightarrow[ext{exponentially}]{|n| o \infty} 0, \qquad M(l,m) \xrightarrow[ext{exponentially}]{|l|, |m| o \infty} 0$$

The locality condition guarantees that the interactions become local in the continuum limit.



N=2 Nilpotent SUSYs: $(\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0$



$$\left\{egin{array}{l} \delta_+\phi_+=ar\chi_+\ \delta_+\chi_+=F_+\ \delta_+\chi_-=-i
abla\phi_-\ \delta_+F_-=-i
ablaar\chi_-\ \mathrm{others}\ =0 \end{array}
ight.$$

$$\left\{egin{array}{l} \delta_-\chi_+ = i
abla \phi_+\ \delta_-F_+ = -i
abla ar\chi_+\ \delta_-\phi_- = -ar\chi_-\ \delta_-\chi_- = F_-\ others \ = 0 \end{array}
ight.$$

N=2 nilpotent SUSYs and Cyclic Leibniz rule 10

We require that the lattice action is invariant under δ_{\pm} .

 $\delta_{\pm}S = 0$

N=2 nilpotent SUSYs and Cyclic Leibniz rule 10

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$$\delta_{\pm}S=0$$
 \Downarrow

 $(\nabla \bar{\chi}_{\pm}, \phi_{\pm} * \phi_{\pm}) + (\nabla \phi_{\pm}, \phi_{\pm} * \bar{\chi}_{\pm}) + (\nabla \phi_{\pm}, \bar{\chi}_{\pm} * \phi_{\pm}) = 0$

N=2 nilpotent SUSYs and Cyclic Leibniz rule 10

We require that the lattice action is invariant under δ_{\pm} .

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$$(\nabla \bar{\chi}_{\pm}, \phi_{\pm} * \phi_{\pm}) + (\nabla \phi_{\pm}, \phi_{\pm} * \bar{\chi}_{\pm}) + (\nabla \phi_{\pm}, \bar{\chi}_{\pm} * \phi_{\pm}) = 0$$

We call this Cyclic Leibniz rule.

Cyclic Leibniz rule vs. Leibniz rule

We have found that the *Cyclic Leibniz Rule* guarantees the N=2 nilpotent SUSYs.

 $\begin{array}{l} \hline Cyclic \ Leibniz \ Rule \ (CLR) \\ \hline (\nabla A, \ B * C) + (\nabla B, \ C * A) + (\nabla C, \ A * B) = 0 \\ \hline VS. \ Leibniz \ Rule \ (LR) \\ \hline (\nabla A, \ B * C) + (A, \ \nabla B * C) + (A, \ B * \nabla C) = 0 \end{array}$

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We have found that the *Cyclic Leibniz Rule* guarantees the N=2 nilpotent SUSYs.

Cyclic Leibniz Rule (CLR)

 $\overline{(
abla A, B * C)} + (
abla B, C * A) + (
abla C, A * B) = 0$

 $\begin{array}{ll} & \text{VS.} & \textit{Leibniz Rule (LR)} \\ (\nabla A, \ B * C) + (A, \ \nabla B * C) + (A, \ B * \nabla C) \swarrow 0 \\ & \text{No-Go theorem} \end{array}$

The Cyclic Leibniz Rule *can* be realized on lattice, but the Leibniz Rule *cannot!*

An example of Cyclic Leibniz rule



An explicit example of the Cyclic Leibniz Rule :

$$egin{aligned} (
abla \phi)_n &= rac{1}{2} ig(\phi_{n+1} - \phi_{n-1} ig) \ (\phi * \psi)_n &= rac{1}{6} ig(2 \phi_{n+1} \psi_{n+1} + 2 \phi_{n-1} \psi_{n-1} \ &+ \phi_{n+1} \psi_{n-1} + \phi_{n-1} \psi_{n+1} ig) \end{aligned}$$

M.Kato, M.S. & H.So, JHEP 05(2013)089 D.Kadoh & N.Ukita, PTEP 2015(2015)103B04 which satisfy i) discrete translation invariance, ii) locality and iii) Cyclic Leibniz Rule.

An example of Cyclic Leibniz rule



An explicit example of the Cyclic Leibniz Rule :

$$(\nabla \phi)_{n} = \frac{1}{2} (\phi_{n+1} - \phi_{n-1})$$

$$(\phi * \psi)_{n} = \frac{1}{6} (2\phi_{n+1}\psi_{n+1} + 2\phi_{n-1}\psi_{n-1} + \phi_{n+1}\psi_{n-1} + \phi_{n-1}\psi_{n+1})$$

M.Kato, M.S. & H.So, JHEP 05(2013)089 D.Kadoh & N.Ukita, PTEP 2015(2015)103B04 which satisfy i) discrete translation invariance, ii) locality and iii) Cyclic Leibniz Rule.

—The field product $(\phi * \psi)_n$ is non-trivial!



	CLR	no CLR
nilpotent SUSYs		
Nicolai maps		
non-renormalization theorem		
cohomology		

Advantages of Cyclic Leibniz rule (CLR)



	CLR	no CLR
nilpotent SUSYs	δ_+, δ	$\delta = \delta_+ + \delta$
Nicolai maps		
non-renormalization theorem		
cohomology		



	CLR	no CLR
nilpotent SUSYs	δ_+,δ	$\delta = \delta_+ + \delta$
Nicolai maps	2	1
non-renormalization theorem		
cohomology		



	CLR	no CLR
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non-renormalization theorem	0	×
cohomology		



	CLR	no CLR
nilpotent SUSYs	δ_+,δ	$\delta = \delta_+ + \delta$
Nicolai maps	2	1
non-renormalization theorem	0	×
cohomology	non-trivial	trivial



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Nicolai maps	2	1
non-renormalization theorem	0	×
cohomology	non-trivial	trivial

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One of the striking features of SUSY theories is the *non-renormalization theorem*.

4d N=1 Wess-Zumino model in continuum

 $S=\int\!\!d^4x\Big\{\int\!\!d^2 heta d^2ar{ heta}\;\Phi^\dagger(ar{ heta})\Phi(heta)+\int\!\!d^2 heta\,W(\Phi)+c.c.\Big\}$



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Non-renormalization Theorem

There is *no quantum correction to the F-terms* in any order of perturbation theory.

An important property is that the F term $W(\Phi)$ depends only on the *chiral superfield* $\Phi(x,\theta)$, which is defined by

$$ar{D}\Phi(x, heta)\equiv \Big(rac{\partial}{\partialar{ heta}}-i heta\sigma_\mu\partial_\mu\Big)\Phi(x, heta)=0$$
 in continuum

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$$ar{D}\Phi(x, heta) \equiv \left(rac{\partial}{\partialar{ heta}} - i heta\sigma_{\mu}\partial_{\mu}
ight)\Phi(x, heta) = 0$$
 in continuum $\sqrt{2}$
 $ar{D}\Phi(heta)_n \equiv \left(rac{\partial}{\partialar{ heta}} - i heta\sigma_{\mu}
abla_{\mu}
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 in continuum \bigvee $\mathbf{\hat{P}}$
 $ar{D}\Phi(heta)_n \equiv \left(rac{\partial}{\partialar{ heta}} - i heta\sigma_\mu
abla_\mu
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However, the above definition of the chiral superfield is *ill-defined* because any products of chiral superfields are not chiral due to the *breakdown of LR on lattice!*

$$ar{D}\Phi_1 = ar{D}\Phi_2 = 0 \implies ar{D}(\Phi_1\Phi_2) \stackrel{\mathsf{v}}{
eq} 0$$

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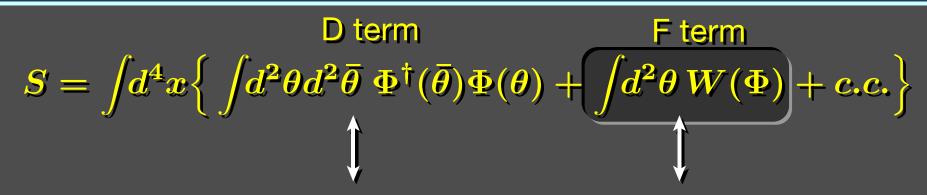
However, the above definition of the chiral superfield is *ill-defined* because any products of chiral superfields are not chiral due to the *breakdown of LR on lattice!*

$$ar{D} \Phi_1 = ar{D} \Phi_2 = 0 \implies ar{D} (\Phi_1 \Phi_2) \stackrel{ extsf{v}}{
eq} 0$$

We *cannot* introduce chiral superfields on lattice!

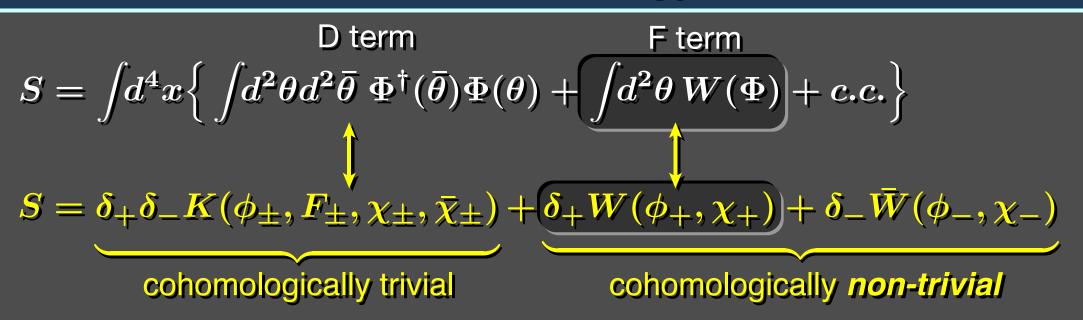
Q-exact form and cohomology





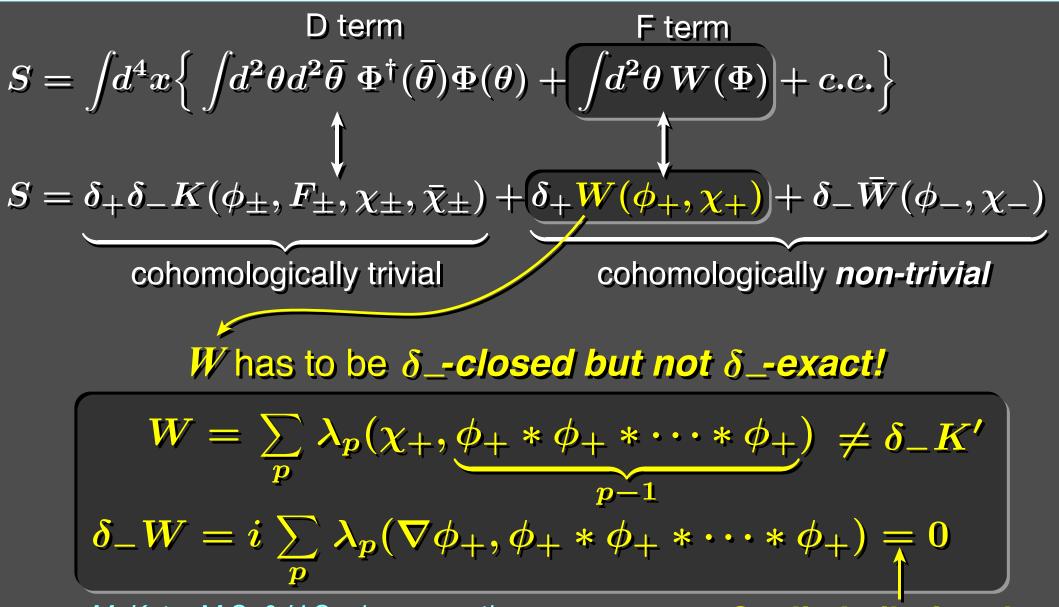
Q-exact form and cohomology





Q-exact form and cohomology



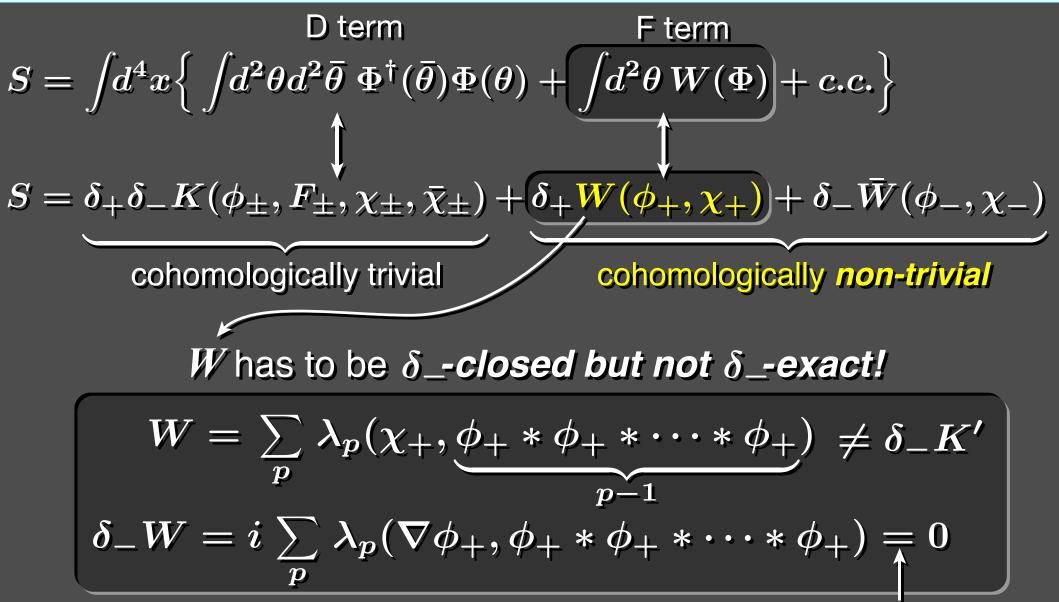


M. Kato, M.S. & H.So, in preparation

Cyclic Leibniz rule

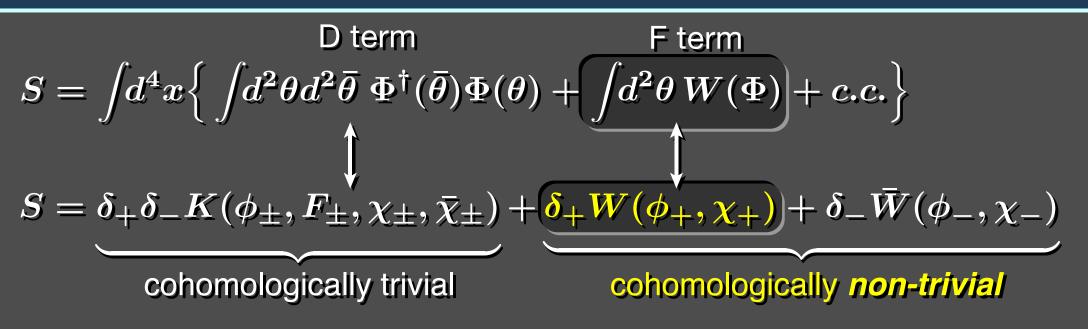
Q-exact form and cohomology





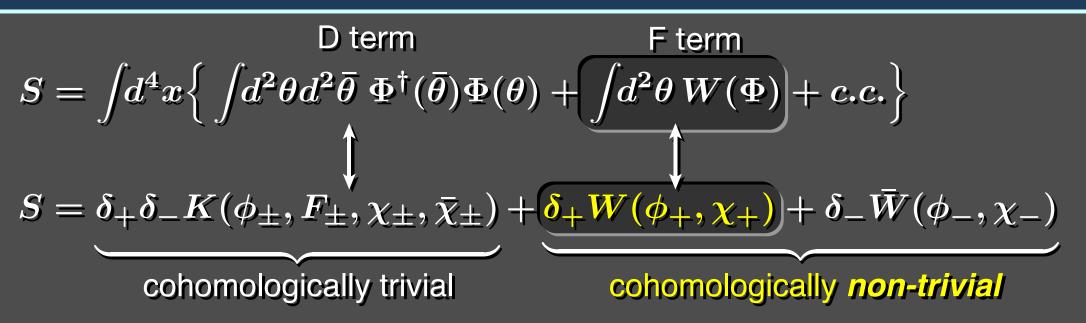
The Cyclic Leibniz rule is crucial to get — Cyclic Leibniz rule non-trivial cohomology! SUSY 2016 at the University of Melbourne, July 4 - 8, 2016

Non-renormalization theorem in our lattice model 17



We can prove the non-renormalization theorem for our lattice model!

Non-renormalization theorem in our lattice model 1



We can prove the non-renormalization theorem for our lattice model!

Non-renormalization Theorem on lattice

There is *no quantum correction* to *the cohomologically non-trivial terms* in any order of perturbation theory.

M. Kato, M.S. & H.So, in preparation



We have proved the No-Go theorem that the Leibniz rule cannot be realized on lattice under reasonable assumptions.

- ✓ We proposed a lattice SUSY model equipped with the *cyclic* Leibniz rule as a modified Leibniz rule.
- ✓ A striking feature of our lattice SUSY model is that the *non-renormalization theorem* holds for a finite lattice spacing.
- ✓ Our results suggest that the cyclic Leibniz rule grasps important properties of SUSY.



Extension to higher dimensions

We have to extend our analysis to higher dimensions. In particular, we need to find solutions to CLR in more than one dimension.

□ inclusion of gauge fields

□ Nilpotent SUSYs with CLR ↔ full SUSYs Are nilpotent SUSYs extended by CLR enough to guarantee full SUSYs ?



Appendix



 $\Psi_{\pm}(\theta_{\pm}, \theta_{-}) \equiv \chi_{\pm} + \theta_{\pm}F_{\pm} + \theta_{\mp}i\nabla\phi_{\pm} + \theta_{\pm}\theta_{\mp}i\nabla\bar{\chi}_{\pm}$ $\Lambda_{\pm}(\theta_{\pm}) \equiv \phi_{\pm} + \theta_{\pm}\chi_{\pm}$

transform under SUSY transformations δ_{\pm} as

$$\delta_{\pm} \mathcal{O}(heta_{\pm}) = rac{\partial}{\partial heta_{\pm}} \mathcal{O}(heta_{\pm})$$



Two Nicolai maps:

$$\xi_{\pm} \equiv
abla \phi_{-} \pm \phi_{+} * \phi_{+}$$

 $ar{\xi}_{\pm} \equiv
abla \phi_{+} \pm \phi_{-} * \phi_{-}$

Action:
$$S = S_{B} + S_{F}$$

 $S_{B} = (\overline{\xi}_{+}, \xi_{+}) = (\overline{\xi}_{-}, \xi_{-})$
 \uparrow
 $(\nabla \phi_{\pm}, \phi_{\pm} * \phi_{\pm}) = 0$
CLB



difference operator: $(\nabla \phi)_n \equiv \sum_m \nabla_{nm} \phi_m$ field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$



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abla \phi)_n \equiv \sum_m
abla_{nm} \phi_m$ field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$

i) translation invariance

 $abla_{nm} =
abla(n-m)$ $M_{nlm} = M(l-n,m-n)$



difference operator:
$$(
abla \phi)_n \equiv \sum_m
abla_{nm} \phi_m$$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$

$$\begin{split} \text{ii) locality} & \nabla(m) \xrightarrow{|m| \to \infty} 0 \text{ (exponentially)} \\ & M(l,m) \xrightarrow{|l|,|m| \to \infty} 0 \text{ (exponentially)} \\ \end{split} \\ \textbf{holomorphic representation} \\ & \widetilde{\nabla}(z) \equiv \sum_{m} \nabla(m) \, z^m \quad \text{on } 1 - \varepsilon < |z|, |w| < 1 + \varepsilon \\ & \widetilde{M}(z,w) \equiv \sum_{lm} M(l,m) \, z^l w^m \end{split}$$

 $\widetilde{
abla}(z), \widetilde{M}(z,w)$ have to be holomorphic on 1-arepsilon < |z|, |w| < 1+arepsilon



difference operator:
$$(
abla \phi)_n \equiv \sum_m
abla_{nm} \phi_m$$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m$

iii) Leibniz rule

$$abla(\phi * \psi) = (
abla \phi) * \psi + \phi * (
abla \psi)$$

$$\implies M(z,w)\left(
abla(zw)-
abla(z)-
abla(w)
ight)=0$$

$$\implies
abla(zw) -
abla(z) -
abla(w) = 0$$

 $\implies
abla(z) \propto \log z$

 $\implies \log z$ is non-holomorphic on $1 - \varepsilon < |z| < 1 + \varepsilon$.

 \implies The Leibniz rule cannot be realized on lattice!



Forward/Backward difference opreators:

$$(
abla^{(+)}\phi)_n\equiv\phi_{n+1}-\phi_n$$

 $(
abla^{(-)}\phi)_n\equiv\phi_n-\phi_{n-1}$

5

Forward/Backward difference opreators:

$$(
abla^{(+)}\phi)_n \equiv \phi_{n+1} - \phi_n$$

 $(
abla^{(-)}\phi)_n \equiv \phi_n - \phi_{n-1}$

 $\implies \nabla^{(+)}(\phi\psi)_n = \phi_{n+1}\psi_{n+1} - \phi_n\overline{\psi_n}$



Forward/Backward difference opreators:

$$(
abla^{(+)}\phi)_n \equiv \phi_{n+1} - \phi_n$$

 $(
abla^{(-)}\phi)_n \equiv \phi_n - \phi_{n-1}$

 $\Rightarrow \nabla^{(+)}(\phi\psi)_n = \phi_{n+1}\psi_{n+1} - \phi_n\psi_n$ $= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n)$

5

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5

Forward/Backward difference opreators:

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5

Forward/Backward difference opreators:

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 $\Rightarrow \nabla^{(+)}(\phi\psi)_n = \phi_{n+1}\psi_{n+1} - \phi_n\psi_n$ $= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n)$ $= (\nabla^{(+)}\phi)_n\psi_{n+1} + \phi_n(\nabla^{(+)}\psi)_n$ $\neq (\nabla^{(+)}\phi)_n\psi_n + \phi_n(\nabla^{(+)}\psi)_n$ $\nabla^{(\pm)} do \text{ not satisfy the Leibniz rule!}$