Cyclic Leibniz Rule, Cohomology and Non-renormalization Theorem in Lattice Supersymmetry

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Supersymmetry on lattice

It would be of great importance to reveal ***non-perturbative aspects*** of SUSY theories.

Great success of ***Lattice Gauge theory*** suggests that ***Lattice SUSY Theory***, if possible, could provide a powerful tool to analyze non-perturvative properties of the theories.

But • • •

Supersymmetry on lattice

For more than 30 years, no one has succeeded satisfactorily to realize ***supersymmetry algebra*** on lattice!

In my talk, I would like to discuss

- ❑ What are obstacles to construct Lattice SUSY?
- ❑ How can we circumvent them?

SUSY algebra and Leibniz rule

infinitesimal SUSY transf.

infinitesimal translation

Supersymmetry algebra: $$\boxed{\{\,\delta_Q\,,\,\delta_{Q'}\,\} = \delta_P}$$

SUSY algebra and Leibniz rule

infinitesimal SUSY transf. *infinitesimal translation*

Supersymmetry algebra: $\boxed{\{\,\delta_Q\,,\,\delta_{Q'}\,\} = \delta_P}$

$$\delta_P(\phi\psi) = (\delta_P\phi)\psi + \phi(\delta_P\psi) \longleftarrow \text{Leibniz rule}$$

SUSY algebra and Leibniz rule

infinitesimal SUSY transf.

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Supersymmetry algebra: $\boxed{\{\,\delta_Q\,,\,\delta_{Q'}\,\} = \delta_P}$

$$\delta_P(\phi\psi) = (\delta_P\phi)\psi + \phi(\delta_P\psi) \longleftarrow$$ Leibniz rule

linear realization of δ_P ***on lattice***

$$\delta_P\phi = \nabla\phi$$

equivalent to a certain difference operator on lattice

SUSY algebra and Leibniz rule

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Supersymmetry algebra: $\boxed{\{\,\delta_Q\,,\,\delta_{Q'}\,\} = \delta_P}$

$$\delta_P(\phi\psi) = (\delta_P\phi)\psi + \phi(\delta_P\psi)$$ ⟵ Leibniz rule

linear realization of δ_P ***on lattice***

$$\delta_P\phi = \nabla\phi$$

equivalent to a certain difference operator on lattice

$$\boxed{\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)}$$

SUSY algebra on lattice would require the Leibniz rule for ∇.

No-Go Theorem

No-Go Theorem

M.Kato, M.S. & H.So, JHEP 05(2008)057

There is ***no*** difference operator ∇ on lattice satisfying the following three properties:

i) discrete translation invariance

ii) locality

iii) Leibniz rule $\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$

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i) discrete translation invariance
ii) locality
iii) Leibniz rule $\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$

The No-Go theorem tells us that we ***cannot*** realize SUSY algebra with ∇ on lattice!

Our approach to construct lattice SUSY models

Our strategy to construct lattice SUSY models is

full SUSY algebra $\longrightarrow$

Leibniz rule $\longrightarrow$

Our approach to construct lattice SUSY models

Our strategy to construct lattice SUSY models is

full SUSY algebra $\longrightarrow$ **Nilpotent SUSY subalgebra**

$$(\delta_Q)^2 = (\delta_{Q'})^2 = \{\delta_Q, \delta_{Q'}\} = 0$$

Leibniz rule $\longrightarrow$ **Cyclic Leibniz rule**

Complex SUSY quantum mechanics on lattice

Lattice action

$$S = (\nabla\phi_-, \nabla\phi_+) - (F_-, F_+) - i(\chi_-, \nabla\bar{\chi}_+) + i(\nabla\bar{\chi}_-, \chi_+)$$
$$-\lambda_+(F_+, \phi_+ * \phi_+) + 2\lambda_+(\chi_+, \bar{\chi}_+ * \phi_+)$$
$$-\lambda_-(F_-, \phi_- * \phi_-) - 2\lambda_-(\chi_-, \bar{\chi}_- * \phi_-)$$

lattice points

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

inner product: $(\phi, \psi) \equiv \sum_n \phi_n\psi_n$

Complex SUSY quantum mechanics on lattice

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$$-\lambda_-(F_-, \phi_- * \phi_-) - 2\lambda_-(\chi_-, \bar{\chi}_- * \phi_-)$$

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$ *lattice points*

inner product: $(\phi, \psi) \equiv \sum_n \phi_n \psi_n$

A crucial point is that the ***field product*** is generalized such that

field product: $\phi_n\psi_n \longrightarrow \boxed{(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m}$

same lattice point

Fields are allowed to interact between different lattice points!

Discrete translation invariance & Locality

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$ — lattice points

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

We impose ***translation invariance*** & ***locality*** on difference operators and field products.

i) discrete translation invariance

$$\nabla_{nm} = \nabla(n-m)\,, \qquad M_{nlm} = M(l-n, m-n)$$

ii) locality

$$\nabla(n) \xrightarrow[\text{exponentially}]{|n|\to\infty} 0\,, \qquad M(l,m) \xrightarrow[\text{exponentially}]{|l|,|m|\to\infty} 0$$

The locality condition guarantees that the interactions become local in the continuum limit.

N=2 nilpotent SUSYs

N=2 Nilpotent SUSYs: $\boxed{(\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0}$

$$
\begin{tikzcd}[row sep=small]
 & F_+ \arrow[dr, "\delta_-"] & \\
\chi_+ \arrow[ur, "\delta_+"] \arrow[dr, "\delta_-"'] & & \bar{\chi}_+ \\
 & \phi_+ \arrow[ur, "\delta_+"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & \phi_- \arrow[dr, "\delta_-"] & \\
\chi_- \arrow[ur, "\delta_+"] \arrow[dr, "\delta_-"'] & & \bar{\chi}_- \\
 & F_- \arrow[ur, "\delta_+"'] &
\end{tikzcd}
$$

$$
\begin{cases}
\delta_+\phi_+ = \bar{\chi}_+ \\
\delta_+\chi_+ = F_+ \\
\delta_+\chi_- = -i\nabla\phi_- \\
\delta_+F_- = -i\nabla\bar{\chi}_- \\
\text{others } = 0
\end{cases}
\qquad
\begin{cases}
\delta_-\chi_+ = i\nabla\phi_+ \\
\delta_-F_+ = -i\nabla\bar{\chi}_+ \\
\delta_-\phi_- = -\bar{\chi}_- \\
\delta_-\chi_- = F_- \\
\text{others } = 0
\end{cases}
$$

N=2 nilpotent SUSYs and Cyclic Leibniz rule

We require that the lattice action is invariant under $\delta_{\pm}$.

$$\delta_{\pm} S = 0$$

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$$\Downarrow$$

$$\boxed{(\nabla\bar{\chi}_\pm\,,\phi_\pm * \phi_\pm) + (\nabla\phi_\pm\,,\phi_\pm * \bar{\chi}_\pm) + (\nabla\phi_\pm\,,\bar{\chi}_\pm * \phi_\pm) = 0}$$

N=2 nilpotent SUSYs and Cyclic Leibniz rule

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$$\delta_\pm S = 0$$

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We call this ***Cyclic Leibniz rule***.

Cyclic Leibniz rule vs. Leibniz rule

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We have found that the ***Cyclic Leibniz Rule*** guarantees the N=2 nilpotent SUSYs.

Cyclic Leibniz Rule (CLR)

$$(\nabla A,\, B * C) + (\nabla B,\, C * A) + (\nabla C,\, A * B) = 0$$

vs.

Leibniz Rule (LR)

$$(\nabla A,\, B * C) + (A,\, \nabla B * C) + (A,\, B * \nabla C) = 0$$

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$$(\nabla A,\ B * C) + (A,\ \nabla B * C) + (A,\ B * \nabla C) \cancel{=} 0$$

No-Go theorem

The Cyclic Leibniz Rule ***can*** be realized on lattice, but the Leibniz Rule ***cannot!***

An example of Cyclic Leibniz rule

An explicit example of the ***Cyclic Leibniz Rule*** :

$$(\nabla\phi)_n = \frac{1}{2}\big(\phi_{n+1} - \phi_{n-1}\big)$$

$$(\phi * \psi)_n = \frac{1}{6}\big(2\phi_{n+1}\psi_{n+1} + 2\phi_{n-1}\psi_{n-1} + \phi_{n+1}\psi_{n-1} + \phi_{n-1}\psi_{n+1}\big)$$

M.Kato, M.S. & H.So, JHEP 05(2013)089
D.Kadoh & N.Ukita, PTEP 2015(2015)103B04

which satisfy i) discrete *translation invariance*, ii) *locality* and iii) *Cyclic Leibniz Rule.*

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! *The field product* $(\phi * \psi)_n$ *is non-trivial!*

Advantages of Cyclic Leibniz rule (CLR)

Advantages of our lattice model with ***CLR*** are given by

	CLR	no CLR
nilpotent SUSYs		
Nicolai maps		
non-renormalization theorem		
cohomology		

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cohomology	**non-trivial**	**trivial**

Non-renormalization theorem in continuum

One of the striking features of SUSY theories is the ***non-renormalization theorem***.

❑ 4d N=1 Wess-Zumino model in continuum

$$S = \int d^4x \left\{ \int d^2\theta d^2\bar{\theta}\; \Phi^\dagger(\bar{\theta})\Phi(\theta) + \int d^2\theta\, W(\Phi) + c.c. \right\}$$

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D term F term

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Non-renormalization Theorem

There is ***no quantum correction to the F-terms*** in any order of perturbation theory.

Problem in defining chiral superfield on lattice

An important property is that the F term $W(\Phi)$ depends only on the ***chiral superfield*** $\Phi(x,\theta)$, which is defined by

$$\bar{D}\Phi(x,\theta) \equiv \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\sigma_\mu\partial_\mu\right)\Phi(x,\theta) = 0 \quad \text{in continuum}$$

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$$\bar{D}\Phi(\theta)_n \equiv \left(\frac{\partial}{\partial\bar{\theta}} - i\theta\sigma_\mu\nabla_\mu\right)\Phi(\theta)_n = 0 \quad \text{on lattice}$$

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However, the above definition of the chiral superfield is ***ill-defined*** because any products of chiral superfields are not chiral due to the ***breakdown of LR on lattice!***

$$\boxed{\bar{D}\Phi_1 = \bar{D}\Phi_2 = 0 \quad \Longrightarrow \quad \bar{D}(\Phi_1\Phi_2) \neq 0}$$

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We ***cannot*** introduce chiral superfields on lattice!

Q-exact form and cohomology

D term F term

$$S = \int d^4x \left\{ \int d^2\theta d^2\bar{\theta}\ \Phi^\dagger(\bar{\theta})\Phi(\theta) + \boxed{\int d^2\theta\, W(\Phi)} + c.c. \right\}$$

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$$\updownarrow \qquad\qquad \updownarrow$$

$$S = \underbrace{\delta_+\delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm)}_{\text{cohomologically trivial}} + \underbrace{\boxed{\delta_+ W(\phi_+, \chi_+)} + \delta_- \bar{W}(\phi_-, \chi_-)}_{\text{cohomologically } \textit{\textbf{non-trivial}}}$$

Q-exact form and cohomology

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W **has to be *δ_--closed but not δ_--exact!***

$$W = \sum_p \lambda_p(\chi_+, \underbrace{\phi_+ * \phi_+ * \cdots * \phi_+}_{p-1}) \neq \delta_- K'$$

$$\delta_- W = i \sum_p \lambda_p(\nabla\phi_+, \phi_+ * \phi_+ * \cdots * \phi_+) = 0$$

M. Kato, M.S. & H.So, in preparation

Cyclic Leibniz rule

Q-exact form and cohomology

D term   F term

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The Cyclic Leibniz rule is crucial to get non-trivial cohomology! $\longrightarrow$ ***Cyclic Leibniz rule***

Non-renormalization theorem in our lattice model

$$S = \int d^4x \Big\{ \overbrace{\int d^2\theta d^2\bar{\theta}\ \Phi^\dagger(\bar{\theta})\Phi(\theta)}^{\text{D term}} + \overbrace{\boxed{\int d^2\theta\, W(\Phi)}}^{\text{F term}} + c.c. \Big\}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$S = \underbrace{\delta_+\delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm)}_{\text{cohomologically trivial}} + \underbrace{\boxed{\delta_+ W(\phi_+, \chi_+)} + \delta_- \bar{W}(\phi_-, \chi_-)}_{\text{cohomologically } \textbf{\textit{non-trivial}}}$$

We can prove the non-renormalization theorem for our lattice model!

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We can prove the non-renormalization theorem for our lattice model!

Non-renormalization Theorem on lattice

There is ***no quantum correction*** to
the cohomologically non-trivial terms
in any order of perturbation theory.

M. Kato, M.S. & H.So, in preparation

Summary

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- ✓ We have proved the ***No-Go theorem*** that the Leibniz rule cannot be realized on lattice under reasonable assumptions.
- ✓ We proposed a lattice SUSY model equipped with the ***cyclic Leibniz rule*** as a modified Leibniz rule.
- ✓ A striking feature of our lattice SUSY model is that the ***non-renormalization theorem*** holds for a finite lattice spacing.
- ✓ Our results suggest that ***the cyclic Leibniz rule grasps important properties of SUSY.***

Remaining tasks

- Extension to higher dimensions

  We have to extend our analysis to higher dimensions. In particular, we need to find solutions to CLR in more than one dimension.

- inclusion of gauge fields

- Nilpotent SUSYs with CLR $\overset{?}{\longleftrightarrow}$ full SUSYs

  Are nilpotent SUSYs extended by CLR enough to guarantee full SUSYs ?

Appendix

SUSY transformations of superfields

$$\Psi_\pm(\theta_+, \theta_-) \equiv \chi_\pm + \theta_\pm F_\pm + \theta_\mp i\nabla\phi_\pm + \theta_\pm\theta_\mp i\nabla\bar{\chi}_\pm$$

$$\Lambda_\pm(\theta_\pm) \equiv \phi_\pm + \theta_\pm\chi_\pm$$

transform under SUSY transformations $\delta_\pm$ as

$$\delta_\pm \mathcal{O}(\theta_\pm) = \frac{\partial}{\partial\theta_\pm}\mathcal{O}(\theta_\pm)$$

Nicolai maps

Two Nicolai maps:

$$\xi_\pm \equiv \nabla\phi_- \pm \phi_+ * \phi_+$$
$$\bar{\xi}_\pm \equiv \nabla\phi_+ \pm \phi_- * \phi_-$$

Action: $S = S_{\mathrm{B}} + S_{\mathrm{F}}$

$$S_{\mathrm{B}} = (\bar{\xi}_+, \xi_+) = (\bar{\xi}_-, \xi_-)$$

$$(\nabla\phi_\pm,\ \phi_\pm * \phi_\pm) = 0$$

CLR

Proof of No-Go Theorem

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

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field product: $(\phi * \psi)_n \equiv \sum_{lm} M_{nlm}\phi_l\psi_m$

i) translation invariance

$$\nabla_{nm} = \nabla(n-m)$$

$$M_{nlm} = M(l-n, m-n)$$

Proof of No-Go Theorem

difference operator: $(\nabla\phi)_n \equiv \sum_m \nabla_{nm}\phi_m$

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ii) locality

$$\nabla(m) \xrightarrow{|m|\to\infty} 0 \quad \text{(exponentially)}$$

$$M(l,m) \xrightarrow{|l|,|m|\to\infty} 0 \quad \text{(exponentially)}$$

holomorphic representation

$$\widetilde{\nabla}(z) \equiv \sum_m \nabla(m)\, z^m$$

$$\widetilde{M}(z,w) \equiv \sum_{lm} M(l,m)\, z^l w^m$$

on $1-\varepsilon < |z|, |w| < 1+\varepsilon$

$\widetilde{\nabla}(z), \widetilde{M}(z,w)$ have to be holomorphic on $1-\varepsilon < |z|, |w| < 1+\varepsilon$

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iii) Leibniz rule

$$\nabla(\phi * \psi) = (\nabla\phi) * \psi + \phi * (\nabla\psi)$$

$$\Longrightarrow \quad M(z,w)\big(\nabla(zw) - \nabla(z) - \nabla(w)\big) = 0$$

$$\Longrightarrow \quad \nabla(zw) - \nabla(z) - \nabla(w) = 0$$

$$\Longrightarrow \quad \nabla(z) \propto \log z$$

$$\Longrightarrow \quad \log z \text{ is non-holomorphic on } 1-\varepsilon < |z| < 1+\varepsilon.$$

$$\Longrightarrow \quad \text{The Leibniz rule cannot be realized on lattice!}$$

Examples of difference opreators

Forward/Backward difference opreators:

$$(\nabla^{(+)}\phi)_n \equiv \phi_{n+1} - \phi_n$$

$$(\nabla^{(-)}\phi)_n \equiv \phi_n - \phi_{n-1}$$

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$$\Longrightarrow \nabla^{(+)}(\phi\psi)_n = \phi_{n+1}\psi_{n+1} - \phi_n\psi_n$$

Examples of difference opreators

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$$\Longrightarrow \nabla^{(+)}(\phi\psi)_n = \phi_{n+1}\psi_{n+1} - \phi_n\psi_n$$

$$= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n)$$

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$$\begin{aligned} \Longrightarrow \nabla^{(+)}(\phi\psi)_n &= \phi_{n+1}\psi_{n+1} - \phi_n\psi_n \\ &= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n) \\ &= (\nabla^{(+)}\phi)_n\psi_{n+1} + \phi_n(\nabla^{(+)}\psi)_n \end{aligned}$$

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\Longrightarrow \nabla^{(+)}(\phi\psi)_n &= \phi_{n+1}\psi_{n+1} - \phi_n\psi_n \\
&= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n) \\
&= (\nabla^{(+)}\phi)_n\psi_{n+1} + \phi_n(\nabla^{(+)}\psi)_n
\end{aligned}$$

Examples of difference opreators

Forward/Backward difference opreators:

$$(\nabla^{(+)}\phi)_n \equiv \phi_{n+1} - \phi_n$$

$$(\nabla^{(-)}\phi)_n \equiv \phi_n - \phi_{n-1}$$

$$\begin{aligned}
\Longrightarrow \nabla^{(+)}(\phi\psi)_n &= \phi_{n+1}\psi_{n+1} - \phi_n\psi_n \\
&= (\phi_{n+1} - \phi_n)\psi_{n+1} + \phi_n(\psi_{n+1} - \psi_n) \\
&= (\nabla^{(+)}\phi)_n\psi_{n+1} + \phi_n(\nabla^{(+)}\psi)_n \\
&\neq (\nabla^{(+)}\phi)_n\psi_n + \phi_n(\nabla^{(+)}\psi)_n
\end{aligned}$$

$\nabla^{(\pm)}$ do ***not*** satisfy the Leibniz rule!