Mass Insertions vs. Mass Eigenstates calculations in Flavour Physics

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Outline

Based on: Dedes, Paraskevas, JR, Suxho, Tamvakis, JHEP 1506 (2015) 151 and JR, Comput.Phys.Commun. 201 (2016) 144-158 .

Introduction

QFT models defined by specifying the Lagrangian - choice of "field basis" not unique!

Usual approach to construct QFT model:

- assume some symmetries, local or global
- choose particle content and their quantum numbers
- define interactions - add to Lagrangian all (or subset) of terms allowed by the symmetries of the theory and extra requirements - renormalizability etc.

Result: Lagrangian in the "interaction basis"

Fields in “interaction basis” Lagrangian may not represent the physical degrees of freedom!

Further steps may be required:

- spontaneous symmetry breaking
- rediagonalization of mass matrices
- ...

Result: Lagrangian in the “mass eigenstates” basis

Advantage: redefined fields represent physical degrees of freedom.

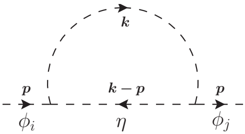
Disadvantage: couplings in “mass eigenstates” basis are usually complicated functions of the initial interaction basis parameters.

Toy example: self-energy in model with real scalar field η and complex scalar multiplet Φ_I:

$$L_{\rm int} = (\partial^\mu \Phi_I^\dagger)\,(\partial_\mu \Phi_I) - M^2_{IJ}\Phi_I^\dagger \Phi_J + \frac{1}{2}(\partial^\mu \eta)\,(\partial_\mu \eta) - \frac{1}{2} m_\eta^2 \eta^2 - Y_{IJ}\;\eta\,\Phi_I^\dagger\,\Phi_J$$

Transition to mass eigenstates basis: $\Phi_I = U_{Ii}\phi_i$

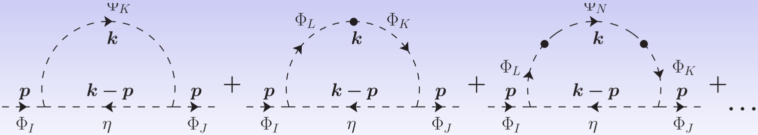
$$\begin{aligned} \mathbf{U}^\dagger\,\mathbf{M^2}\,\mathbf{U} &= \mathbf{m^2} = \mathbf{diag}(m_1^2, \ldots, m_N^2) \\ y_{ij} &= U^\dagger_{iI}(\mathbf{M^2})\,Y_{IJ}\,U_{Jj}(\mathbf{M^2}) \end{aligned}$$



Mass eigenstates basis: $\Sigma_{ji}(p) = -\frac{1}{(4\pi)^2}\, y_{j\ell}\, B_0(p; m_\ell^2, m_\eta^2)\, y_{\ell i}$

- simple diagram and compact expression
- physical states on external legs
- but: result in terms of $y = y(Y, M)$, $m = m(M)$

Interaction basis: thick dots represent "mass insertions" - off diagonal elements of M^2_{IJ}.



- infinite series of diagrams, complicated calculation and expression
- unphysical states on the external legs
- but: result in terms of the initial "interaction basis" parameters Y, M

$$\begin{aligned}\hat{\Sigma}_{JI}(p) &= -\frac{1}{(4\pi^2)} Y_{JK} Y_{LI} \Big(\delta_{KL} B_0(p; M_K^2, m_\eta^2) \\ &+ \hat{M}^2_{KL} C_0(0, p; M_K^2, M_L^2, m_\eta^2) \\ &+ \hat{M}^2_{KN} \hat{M}^2_{NL} D_0(0, 0, p; M_K^2, M_N^2, M_L^2, m_\eta^2) + \dots \Big),\end{aligned}$$

Transition to physical states $U^\dagger_{jJ}\hat{\Sigma}_{JI}(p)U_{Ii} = \Sigma_{ji}(p)$:

$$\begin{aligned} U_{K\ell}\, B_0(p, m_\ell^2, m_\eta^2)\, U^\dagger_{\ell L} &= \delta_{KL} B_0(p; M_K^2, m_\eta^2) \\ &+ \hat{M}^2_{KL} C_0(0, p; M_K^2, M_L^2, m_\eta^2) \\ &+ \hat{M}^2_{KN}\hat{M}^2_{NL} D_0(0, 0, p; M_K^2, M_N^2, M_L^2, m_\eta^2) \ + \ldots \end{aligned}$$

Very particular relation!

- Holds for 1-loop functions only?
- Can it be generalized ? How? $\rightarrow$ Flavor Expansion Theorem

Idea: Typical term in QFT mass-eigenstates amplitude: $U_{Ii} f(m_i^2) U_{Ji}^*$ can be expressed as an element of function of the matrix:

$$\begin{aligned} U_{Ii} f(m_i^2) U_{Ji}^* &= U_{Ii} \left(c_0 + c_1 m_i^2 + c_2 m_i^4 + \ldots \right) U_{Ji}^* \\ &= \left(c_0 + c_1 \mathbf{M^2} + c_2 \mathbf{M^4} + \ldots \right)_{IJ} = f(\mathbf{M^2})_{IJ} \end{aligned}$$

Disadvantage: each power of mass insertion appears in infinite number of terms of Taylor series!

$$(\mathbf{M^2})^n_{IJ} \supset \mathbf{M^2}_{II} \mathbf{M^2}_{II} \ldots \mathbf{M^2}_{IJ}$$

Can we derive another series (not Taylor) for $f(\mathbf{M^2})$ in powers of the off-diagonal elements of $\mathbf{M^2}$ only?

Answer - yes, on purely algebraic ground!

Flavor Expansion Theorem

Definition (Divided differences)

Divided differences are defined recursively as

$$
\begin{aligned}
f^{[0]}(x) &\equiv f(x) \\
f^{[1]}(x_0, x_1) &\equiv \tfrac{f(x_0)-f(x_1)}{x_0-x_1} \\
\dots \\
f^{[k+1]}(x_0, \dots, x_k, x_{k+1}) &\equiv \tfrac{f^{[k]}(x_0,\dots,,x_{k-1},x_k)-f^{[k]}(x_0,\dots,x_{k-1},x_{k+1})}{x_k-x_{k+1}}
\end{aligned}
$$

Properties:

- symmetric functions of all arguments
- smooth degeneracy limit

$$\lim_{\{x_0,\dots,x_m\}\to\{\xi,\dots,\xi\}} f^{[k]}(x_0, \dots, x_k) = \frac{1}{m!}\frac{\partial^m}{\partial\xi^m} f^{[k-m]}(\xi, x_{m+1} \dots, x_k) \ ,$$

Theorem (Flavor Expansion Theorem or "FET")

Let's decompose and Hermitian matrix $\mathbf{A}$ *as a sum of diagonal and non-diagonal part* $\mathbf{A} = \mathbf{A_0} + \hat{\mathbf{A}}$*:*

$$\begin{aligned} A_0^I &\equiv A_{II} \ , \\ \hat{A}_{IJ} &\equiv A_{IJ}, \qquad \hat{A}_{II} = 0 \ , \qquad (I, J = 1, \ldots, n) \ . \end{aligned}$$

Then, matrix element $f(\mathbf{A})_{IJ}$ *is given by (no sum over* I, J*):*

$$\begin{aligned} f(\mathbf{A})_{IJ} &= \delta_{IJ} f(A_0^I) + f^{[1]}(A_0^I, A_0^J)\, \hat{A}_{IJ} \\ &+ \sum_{K_1} f^{[2]}(A_0^I, A_0^J, A_0^{K_1})\, \hat{A}_{IK_1} \hat{A}_{K_1 J} \\ &+ \sum_{K_1, K_2} f^{[3]}(A_0^I, A_0^J, A_0^{K_1}, A_0^{K_2})\, \hat{A}_{IK_1} \hat{A}_{K_1 K_2} \hat{A}_{K_2 J} + \ldots \end{aligned}$$

Series coefficients: divided differences of $f(\mathbf{A_0})$

Expansion parameters: non-diagonal elements of $\hat{\mathbf{A}}$.

- **holds for any analytic function, not just loop functions - as long as the RHS is convergent!**
- formal proof: rather technical → *see arXiv:1504.00960*
- bonus features:
  - degenerate eigenvalues and/or diagonal matrix elements treated uniformly due to smooth degeneracy limit of the divided differences
  - natural relation to Passarino-Veltman loop functions

Common application of FET: for many processes leading order terms cancel and only higher ones are left - diagrammatic MI calculation tedious and error prone.

FET expansion of PV functions

Any 1-loop amplitude can be expressed in terms of Passarino-Veltman functions of the order n:

$$\frac{i}{(4\pi)^2} PV_n^{\mu_1 \dots \mu_l}(p_1, \dots, p_{n-1}; m_1^2, \dots, m_n^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_l}}{(k^2 - m_1^2) \prod_{j=2}^n ((k + p_1 + \dots + p_{j-1})^2 - m_j^2)}$$

Useful recursive relation: divided difference of 1-loop functions of the order n is a 1-loop function of the order $n+1$!

$$\frac{PV_n^X(\dots p_{j-1} \dots ; \dots m_j^2 \dots) - PV_n^X(\dots p_{j-1} \dots ; \dots {m'_j}^2 \dots)}{m_j^2 - {m'_j}^2} = PV_{n+1}^X(\dots p_{j-1}, 0 \dots ; \dots m_j^2, {m'_j}^2 \dots)$$

FET formula for PV functions:

$$
\begin{aligned}
\Big[PV^{(n)}(\ldots,\mathbf{M}^2,\ldots)\Big]_{IJ} &= \delta_{IJ}PV^{(n)}(\ldots,(M^2)_{II},\ldots) \\
&+ PV^{(n+1)}(\ldots,(M^2)_{II},(M^2)_{JJ},\ldots)\hat{M}^2_{IJ} \\
&+ \sum_K PV^{(n+2)}(\ldots,(M^2)_{II},(M^2)_{JJ},(M^2)_{KK},\ldots)\hat{M}^2_{IK}\hat{M}^2_{KJ} \\
&+ \ldots
\end{aligned}
$$

- immediately reproduces the relation discussed in the “toy model”
- applies to expansion with degenerate diagonal elements - PV functions not singular in this limit, no need to calculate derivatives
- explicit condition for FET series convergence

Convergence of FET expansion of PV functions

Most useful case for flavor physics - vanishing external momenta.

Define matrix of moduli of "dimensionless mass insertions"

$$\Delta_{IJ} = \frac{|\hat{M}^2_{IJ}|}{\sqrt{(M_0)^2_{II}(M_0)^2_{JJ}}} \qquad \Delta_{II} = 0$$

FET expansion of PV functions at vanishing external momenta converges if moduli of Δ matrix eigenvalues are smaller then 1 *(proof in arXiv:1504.00960)*

Simpler but weaker sufficient (but not necessary) condition: norm of any row or column of Δ must be smaller then 1.

FET expansion for fermionic amplitudes

FET applies to the function of Hermitian matrices - works for bosons (scalar or vector particles).

General mass term for chiral fermions:

$$-\bar{\mathbf{\Psi}}\Big(\mathbf{M}P_L + \mathbf{M}^\dagger P_R\Big)\mathbf{\Psi}$$

where $\mathbf{M}$ is **general complex** mass matrix diagonalized by **2 unitary rotations** U, V:

$$\Psi_{L\,A} = U_{Ai}\psi_{Li}\ , \qquad \Psi_{R\,A} = V_{Ai}\psi_{Ri}$$

$$\mathbf{V}^\dagger\,\mathbf{M}\,\mathbf{U} = \mathbf{m} = \mathbf{diag}(m_1, \ldots, m_N)$$

Can FET expansion be used for fermionic amplitudes?

Fermion propagator can be decomposed as

$$\frac{i}{\not{k} - \mathbf{M}P_L - \mathbf{M}^\dagger P_R} = (\mathbf{M}^\dagger P_L + \not{k}P_L)\frac{i}{k^2 - \mathbf{M}\mathbf{M}^\dagger} + (\mathbf{M}P_R + \not{k}P_R)\frac{i}{k^2 - \mathbf{M}^\dagger\mathbf{M}}$$

- Loop functions depend always on Hermitian matrices $\mathbf{M}\mathbf{M}^\dagger$ or $\mathbf{M}^\dagger\mathbf{M}$.
- Only some combinations of mixing matrices can appear in fermionic amplitudes:

$$\begin{aligned}
U_{Bi}\, f(m_i^2)\, U^\star_{Ai} &= f(\mathbf{M}^\dagger\mathbf{M})_{BA} \\
V_{Bi}\, f(m_i^2)\, V^\star_{Ai} &= f(\mathbf{M}\mathbf{M}^\dagger)_{BA} \\
U_{Bi}\, m_i f(m_i^2)\, V^\star_{Ai} &= M^\dagger_{BC}\, f(\mathbf{M}\mathbf{M}^\dagger)_{CA} = f(\mathbf{M}^\dagger\mathbf{M})_{BC}\, M^\dagger_{CA} \\
V_{Bi}\, m_i f(m_i^2)\, U^\star_{Ai} &= M_{BC}\, f(\mathbf{M}^\dagger\mathbf{M})_{CA} = f(\mathbf{M}\mathbf{M}^\dagger)_{BC}\, M_{CA}
\end{aligned}$$

- All can be expressed using FET formula, works for fermions as well!

Applies also to Majorana fermions – then $M = M^T$ and $U = V^*$

MassToMI Mathematica package

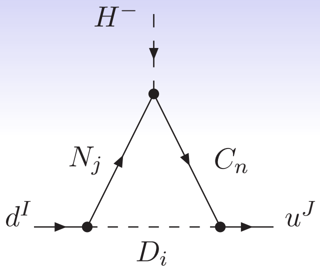
Great advantage of FET technique - can be easily automatized!

Prescription:

- calculate amplitude in the mass eigenstates basis - less diagrams, more compact expressions, better suited for numerical computations
- expand result using FET implemented in MassToMI package - recover direct analytic dependence on “interaction basis” parameters (better suited for understanding of various effects)

Avoids tedious and error-prone direct calculations of diagrams with mass insertions.

Example: $H^- \to d^I \bar{c}^J$ decay in the MSSM, the triangle diagram with chargino (Dirac) C_n, neutralino (Majorana) N_j and down squark D_i circulating in the loop.



Typical term (Z, O, U, V are scalar, Majorana and Dirac fermion mixing matrices):

$$A \supset Z_D^{Ii} Z_D^{Ji*} O_N^{Kj} O_N^{Lj*} V_C^{Mn*} U_C^{Nn} m_{C_n} c_0(p, q, m_{C_n}^2, m_{D_i}^2, m_{N_j}^2)$$

(summation convention assumed for the repeating indices)

MassToMI input syntax rules:

Z_D^{Ii}	$\rightarrow$	`SMIX[D,I,i]`	scalar
O_N^{Kj}	$\rightarrow$	`NMIX[N,K,j]`	Majorana fermion
V_C^{Mn*}	$\rightarrow$	`Conjugate[FMIXL[C,M,n]]`	left Dirac fermion
U_C^{Mn}	$\rightarrow$	`FMIXR[C,M,n]`	right Dirac fermion
m_{C_n}	$\rightarrow$	`MASS[C,n]`	physical (fermion) mass

Loop integral:

$c_0(p, q, m_{C_n}^2, m_{D_i}^2, m_{N_j}^2) \rightarrow$ LOOP[c0,{{C,n},{D,i},{N,j}},{p,q}]

MassToMI expression for amplitude:

$$A = Z_D^{Ii} Z_D^{Ji*} O_N^{Kj} O_N^{Lj*} V_C^{Mn*} U_C^{Nn} m_{C_n} c_0(p, q, m_{C_n}^2, m_{D_i}^2, m_{N_j}^2) \to$$

```
A =  SMIX[D,I,i] Conjugate[SMIX[D,J,i]]
     NMIX[N,K,j] Conjugate[NMIX[N,L,j]]
     Conjugate[FMIXL[C,M,n]] FMIXR[C,N,n] MASS[C,n]
     LOOP[c0,{{C,n},{D,i},{N,j}},{p,q}];
```

Control variables:

- **FetScalarList=**`{{D,2}}`. Mixing matrices for scalar D are expanded up to 2nd order in mass insertions.
- **FetFermionList=**`{{C,1,MHM},{O,1,MMH}}`. Mixing matrices for fermions C, O are expanded to 1st order in MI. Parameters `MHM,MMH` - final result is expressed in terms of $M^\dagger M$ or $MM^\dagger$.
- **FetMaxOrder=2**. Only mass insertion products of the total order `FetMaxOrder` or lower are kept in the final result.

Function FetExpand[A] automatically performs the MI expansion to required order!

MassToMI tested on realistic case set of LFV processes in the MSSM (to be published):

- Initial mass eigenstates expression for amplitude: few lines.
- Mathematica code/execution time: 300 lines/up to few hours on standard PC.
- Intermediate expressions: ~ 50000 MI terms! Equivalent to tens/hundreds of diagrams in the interaction basis.
- Final expanded interaction basis expressions after simplifications: few lines for leading terms.

Conclusions

1. Mass Insertion expansion can be done starting from mass eigenstates amplitude, without direct diagrammatic calculations.
2. “Flavor Expansion Theorem” applies to all types of amplitudes: scalar, vector and fermionic.
3. Natural relation of FET to recursive properties of the 1-loop functions (including explicit convergence criterion for the MI series).
4. FET technique automatized and implemented in MassToMI Mathematica package.

MassToMI can be downloaded from **www.fuw.edu.pl/masstomi**