Localization in Quiver Quantum Mechanics

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Introduction: motivation

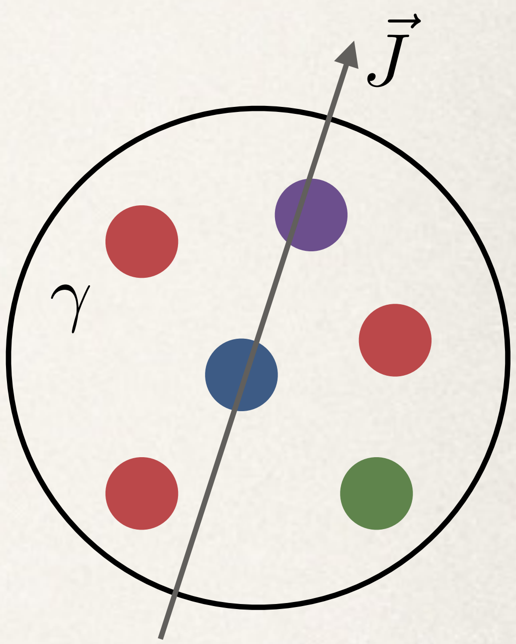
- We would like to count the number (degree) of the BPS states (multi-centered black holes) in 4d $\mathcal{N}$=2 SUGRA (IIA/B on CY$_3$) via the "index" (second helicity supertrace)

moduli　　　　　　　　　　spin

$$\Omega(\gamma; z_a) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_\gamma(z_a)}(-1)^{2J_3}(2J_3)^2$$

charge

$$= \mathrm{Tr}_{\mathcal{H}'_\gamma(z_a)}(-1)^{2J_3}$$



or the refined one

$$\Omega_{\mathrm{ref}}(\gamma; z_a, y) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_\gamma(z_a)}(-1)^{2J_3}(2J_3)^2 y^{2J_3}$$

$$= \mathrm{Tr}_{\mathcal{H}'_\gamma(z_a)}(-1)^{2J_3} y^{2J_3}$$

where $\mathcal{H}_\gamma(z_a)$: Hilbert space one-particle states with charge γ

and $\mathcal{H}_\gamma(z_a) = \left(\left[\frac{1}{2}\right] + 2[0]\right) \otimes \mathcal{H}'_\gamma(z_a)$

center of mass degrees of freedom

Introduction: Hall halo

- There also exist the Hall halo bound state

(# of bound states) = (lowest Landau level degeneracies of the quantum Hall halo)

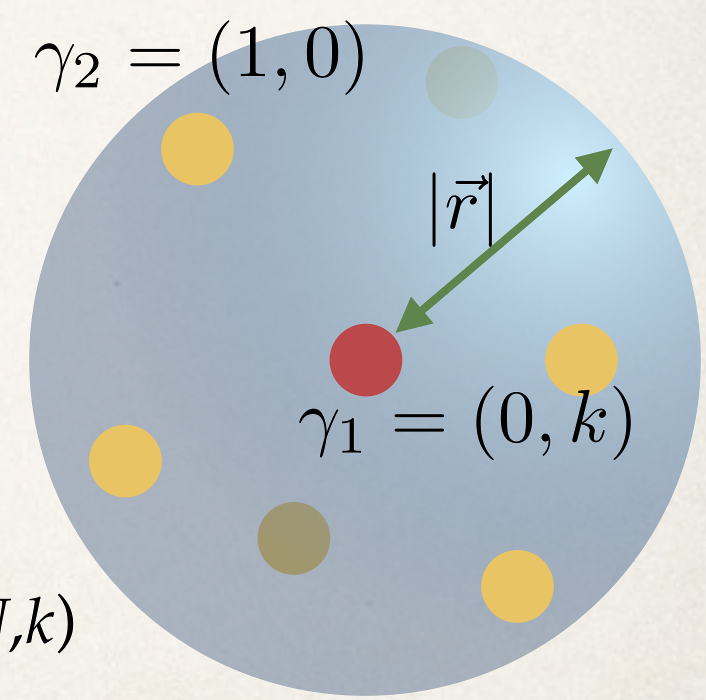
$$\Omega_{\mathrm{ref}}(y) = \sum_{n=-N(k-N)/2}^{N(k-N)/2} d_n y^{2n}$$

$$= y^{-N(k-N)} \boxed{\frac{\prod_{j=1}^{k}(1-y^{2j})}{\prod_{j=1}^{N}(1-y^{2j})\prod_{j=1}^{k-N}(1-y^{2j})}}$$

Poincaré polynomial of the Grassmannian *Gr*(*N*,*k*)

Higgs branch moduli of the *U*(1)×*U*(*N*) quiver theory with *k* arrows

N electrons + monopole



Introduction: Denef's conjecture

- Denef conjectured that there is a correspondence between the BPS bound states and quiver quantum mechanics (QQM)

BPS bound state (SUGRA)	QQM (LσM)
particles	nodes
# of particles	rank of gauge groups
DSZ product	# of arrows
phase of the central change	FI parameters (ζ)
spin	charges (Ω-background)
index	partition function

Examples: two Abelian nodes

- I concentrate on a simplest example today

- Two Abelian nodes (G=$U(1)\times U(1)$) with k arrows:

$$\underset{(Z,\phi)}{\textcircled{1}} \xrightarrow[q_a]{k} \underset{(\tilde{Z},\tilde{\phi})}{\textcircled{1}} \qquad (a=1,\cdots,k)$$

- Vector multiplets (nodes)

$$Z,\bar{Z},\phi=\sigma+iA_\tau, \quad \tilde{Z},\bar{\tilde{Z}},\tilde{\phi}=\tilde{\sigma}+i\tilde{A}_\tau$$ Coulomb branch moduli

- Chiral multiplets (arrows)

$$q_a \quad (a=1,\ldots,k)$$ Higgs branch moduli

$$\mathcal{M}_{\mathrm{Higgs}}=\mathbb{C}P^{k-1}$$

Q-exact action

- The supersymmetric action is written in the Q-exact form, where Q is a specific supercharge

$$\begin{aligned} S_{\mathrm{QQM}} &= \frac{1}{g^2} S_{\mathrm{vector}} + S_{\mathrm{chiral}} \\ &= Q \int d\tau \left(\frac{1}{g^2} V_{\mathrm{vector}} + V_{\mathrm{chiral}} \right) \end{aligned}$$

$\Downarrow$

Independent of the couplings (WKB, 1-loop exact)

$\Downarrow$

The path integral is localized at Q-transformation and D&F-term constraints

Fixed point equations

- The partition function is localized at the solutions to:

$$\partial_\tau Z = \partial_\tau \tilde{Z} = \partial_\tau \sigma = \partial_\tau \tilde{\sigma} = 0$$

$$(\partial_\tau + \phi - \tilde{\phi}) q_a = 0$$

$$\mu_D = g^2 \sum_{a=1}^{k} (|q_a|^2 - \zeta) = 0$$

$$\tilde{\mu}_D = g^2 \sum_{a=1}^{k} (-|q_a|^2 + \zeta) = 0$$

$$\mu_{F,a} = (Z - \tilde{Z}) q_a = 0$$

All fields are constant

and

$$Z - \tilde{Z} \neq 0$$
$$\phi - \tilde{\phi} \neq 0$$
$$q_a = 0 \quad (\zeta \ll 1)$$

Coulomb

or

$$Z - \tilde{Z} = 0$$
$$\phi - \tilde{\phi} = 0$$
$$q_a \neq 0 \quad (\zeta \gg 1)$$

Higgs

Overview: Coulomb vs Higgs

- We can evaluate exactly the partition function in two different regions, but the localization says that these should give the same quantity because of the coupling (parameter) independence

$$S_{\mathrm{QQM}} = \frac{1}{g^2} S_{\mathrm{vector}} + S_{\mathrm{chiral}}$$

Coulomb

$$Z - \tilde{Z} \neq 0$$
$$\phi - \tilde{\phi} \neq 0$$
$$q_a = 0 \quad (\zeta \ll 1)$$

$$g \to \infty$$

Higgs

$$Z - \tilde{Z} = 0$$
$$\phi - \tilde{\phi} = 0$$
$$q_a \neq 0 \quad (\zeta \gg 1)$$

$$S^{\mathrm{eff}}_{\mathrm{Coulomb}} = \frac{1}{2}\int dt\, g_C^{ij} \dot{x}_i \dot{x}_j + \cdots + A_i(x)\dot{x}^i$$

monopole flux!

$$S^{\mathrm{eff}}_{\mathrm{Higgs}} = \frac{1}{2}\int dt\, g_H^{i\bar{\jmath}} \dot{z}_i \dot{\bar{z}}_{\bar{\jmath}} + \cdots$$

Localization & integral over all zero modes

$$Z_{\mathrm{Coulomb}} \sim \int_{\mathcal{M}_C} \hat{A}(T\mathcal{M}_C)\mathrm{ch}(\mathcal{F}) \;\; = \;\; Z_{\mathrm{Higgs}} \sim \chi(\mathcal{M}_H) = \sum_l (-1)^l \dim H^l(\mathcal{M}_H)$$

Ω-backgrounds

- The Coulomb and Higgs branch fixed points of the original model are highly degenerated (many flat directions)
- We can turn on the Ω-backgrounds by gauging the global symmetries in $SU(2)_R \times U(k)$ (isometries of each branch) in order to resolve the degeneracy
- The parameters of the Ω-backgrounds gives the refined parameters (fugacities)

Coulomb: $$SU(2)_R \supset U(1) \quad \Rightarrow \quad \epsilon \quad \Rightarrow \quad y \equiv e^{i\beta\epsilon}$$

Higgs: $$U(k) \supset U(1)^k \quad \Rightarrow \quad \epsilon_a \quad \Rightarrow \quad x_a \equiv e^{i\beta\epsilon_a}$$

where β is the periodicity of the time-direction: $\tau \sim \tau + \beta$

Fixed points: Higgs branch

- Let us consider more concrete examples

- Two Abelian nodes with *k* arrows:

$$
\begin{tikzcd}
\underset{(Z,\phi)}{\textcircled{1}} \arrow[r, Rightarrow, "k", "q_a"'] & \underset{(\tilde{Z},\tilde{\phi})}{\textcircled{1}}
\end{tikzcd}
\qquad (a = 1, \cdots, k)
$$

$$\mathcal{M}_{\text{Higgs}} = \mathbb{C}P^{k-1}$$

$$\boxed{\begin{aligned} & Z = \tilde{Z} = 0 \\ & (\phi - \tilde{\phi} + i\epsilon_a) q_a = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} & q_a = (0, \cdots, 0, q_l, 0, \cdots, 0) \\ & \phi - \tilde{\phi} + i\epsilon_l = 0 \end{aligned}}$$

Fixed point equation

choice of $l \Rightarrow k$ fixed points

Evaluation of the index: Higgs branch

- For the BRST fixed points, D-term (moment map) equation says:

$$|q_l|^2 = \zeta$$

$\zeta > 0$ ⟶ $q_l \neq 0$ ⟶ we can choose poles at $\phi - \tilde{\phi} + i\epsilon_l = 0$

$\zeta < 0$ ⟶ no solution of q_l ⟶ The contour does not contain any pole

Thus we find

$$\mathcal{Z} = \left(\frac{\beta}{2i\sin\frac{\beta}{2}\epsilon}\right)^2 \int \frac{d\phi_c}{2\pi i}\int_{\mathcal{C}} \frac{d\phi_r}{2\pi i}\prod_{a=1}^{k}\frac{\sinh\frac{\beta}{2}(\phi_r + i(\epsilon+\epsilon_a))}{\sinh\frac{\beta}{2}(\phi_r + i\epsilon_a)}$$

where

$$\boxed{\begin{aligned}\phi_c &\equiv (\phi+\tilde{\phi})/2\\ \phi_r &\equiv (\phi-\tilde{\phi})/2\end{aligned}}$$

$$= \begin{cases} \mathcal{Z}_c \displaystyle\sum_{l=1}^{k}\prod_{a\neq l}\frac{x_l - y x_a}{x_l - x_a} = \mathcal{Z}_c \frac{1-y^k}{1-y} & \text{for } \zeta > 0 \\ 0 & \text{for } \zeta < 0 \end{cases}$$

Poincaré polynomial of $\mathbf{C}P^{k-1}$

Wall crossing phenomena

Effective potential in Coulomb branch

- Quantum effects generate the 1-loop effective potential

$$\mu_D \to U(\vec{r}, \vec{\tilde{r}}) \sim \sum_{a=1}^{k} \langle |q_a|^2 \rangle - \zeta$$

$$\sim \frac{k}{|\vec{r} - \vec{\tilde{r}}|} - \zeta$$

where

$$\boxed{\begin{aligned} \vec{r} &= (Z, \bar{Z}, \sigma) \\ \vec{\tilde{r}} &= (\tilde{Z}, \bar{\tilde{Z}}, \sigma) \end{aligned}}$$

$$\Big\downarrow \quad \epsilon \neq 0 \quad (Z = \tilde{Z} = 0)$$

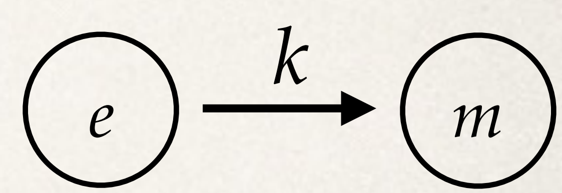
$$\frac{k}{|\sigma|} = \zeta \quad \left(\sigma = \pm \frac{k}{\zeta} \right)$$

Evaluation of the index: Coulomb branch

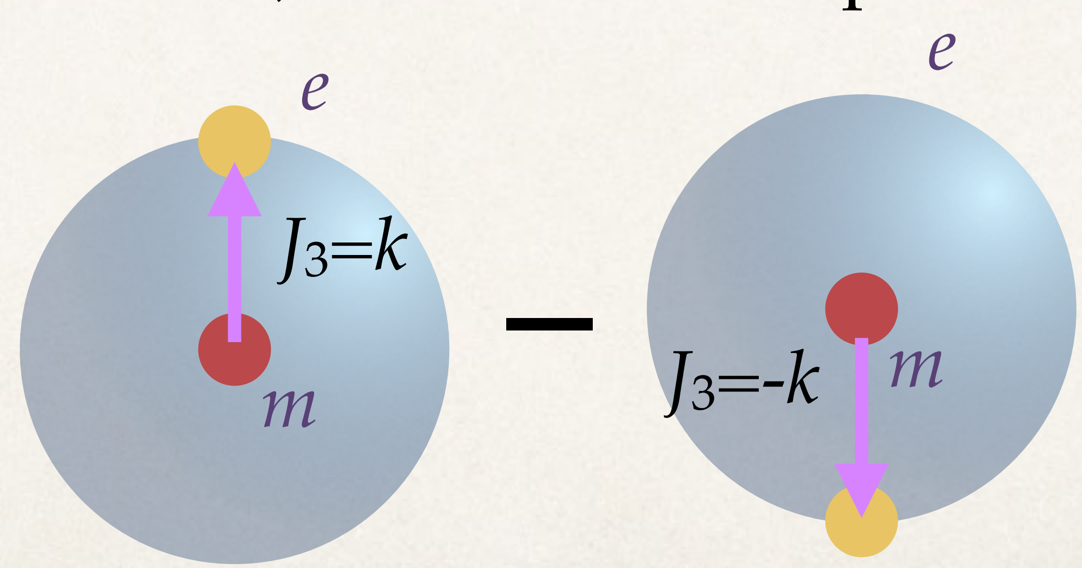
- Turning on the Ω-background, the Coulomb branch is localized at the isometry fixed points, and the localization formula reduces to

$$\mathcal{Z} = \frac{1}{(\sin\beta\epsilon)^{n-1}} \sum_{r^*} \mathrm{sign}(U'(r^*)) e^{i\beta\epsilon J_3}$$

- For example, the bound states of a single monopole and a single electron ($U(1)\times U(1)$ with k arrows) have two fixed points in the Coulomb picture



$\mathcal{Z} =$



$$= \frac{1}{y - y^{-1}}(y^k - y^{-k}) \sim \frac{1 - y^k}{1 - y}$$

agrees with the index in the Higgs branch

Conclusion and Discussion

Conclusion:

- We exactly evaluate the partition function of the supersymmetric quiver quantum mechanics by using the localization
- The partition function gives the (refined) index of the BPS bound states (Poincaré polynomial (Dirac genus) of the Higgs branch moduli)
- The partition function also gives configurations of the BPS particles as the Coulomb branch fixed points

Conclusion and Discussion

Future problems:

- Extension to the quiver theory with loops (including the superpotentials)
- Relation and application to the integrable systems (brane tilings, dimer models, etc)
- Deeper understandings of the Coulomb branch picture from the holographic point of view
- Asymptotic behaviour of the partition function (generating function) ⇔ Bubbling geometry, BH entropy, etc.