

Scattering Amplitudes

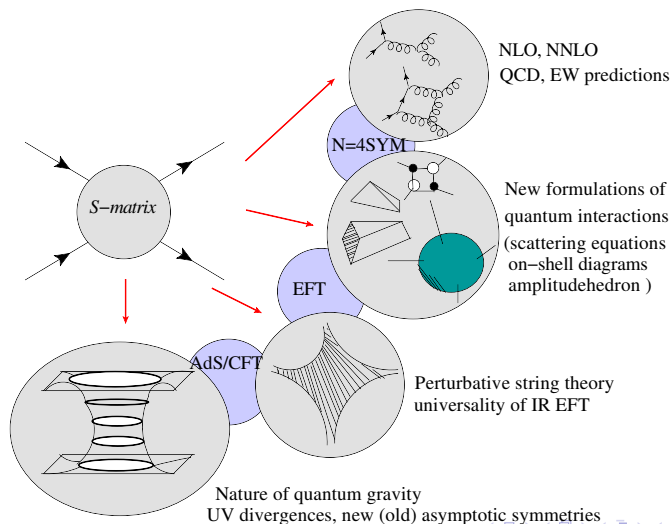
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SUSY 2016 Melbourne

Prelude



- Classical Squaring Relations Between Gauge Theory and Gravity
- The Three, Four-Loop Cusp Anomalous Dimension in QCD and in N=4 SYM
- Complete Two-Loop All-Plus Yang-Mills Amplitudes
- Holography, Soft Theorems, and Infinite-Dimensional Symmetries of Flat Space

Amplitudes 2016 International Conference

4-8 July 2016

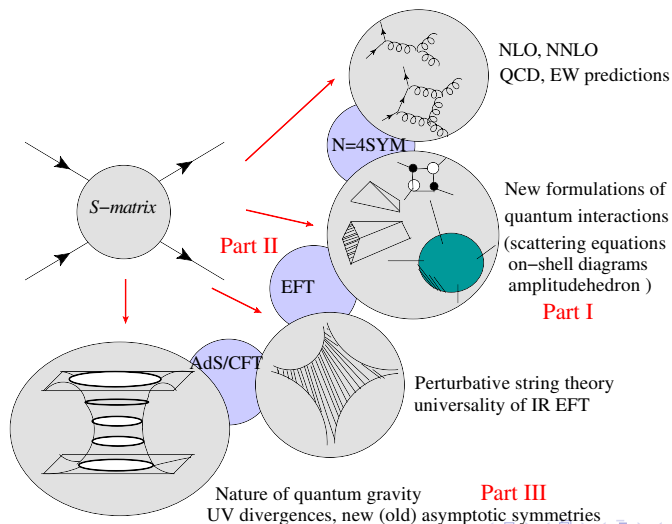
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Prelude



(I) Twistor String Revolution

Massless states: $p^\mu \rightarrow p^{\alpha\dot{\alpha}}$

$$p^2 = 0 \rightarrow p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \quad M_n(i^-, j^-) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad \langle ij \rangle \equiv \lambda_i^\alpha \lambda_{j\dot{\alpha}}$$

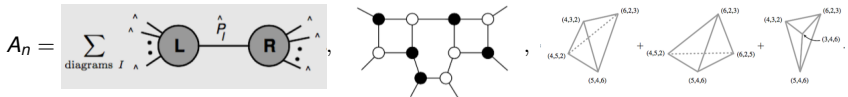
- **Witten's twistor string theory**: a worldsheet model for gluon tree amplitudes
amps = string correlators fixed by a map from \mathbb{CP}^1 to $\mathbb{CP}^{3|4}$ (twistor space) [Witten, 2003]
- Key observation: [Nair, 88] Parke-Taylor MHV amps = correlator on \mathbb{CP}^1

$$\lambda_i^\alpha \sim (z_i, 1), \quad PT_n := \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)}, \quad j_A(z)j_B(z') = \frac{f_{AB}^C j_C}{z - z'} + \text{double poles} + \dots$$

(I) Twistor String Revolution

- **Witten's twistor string theory**: a worldsheet model for gluon tree amplitudes
 $\text{amps} = \text{string correlators}$ fixed by a map from \mathbb{CP}^1 to $\mathbb{CP}^{3|4}$ (twistor space) [Witten, 2003]

Inspired an explosion of structural exploration for massless amplitudes: **BCFW tree-recursion formulae**, **Grassmannians**, **Loop-level recursions**, **Amplituhedron**, e.t.c

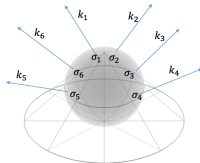


but highly dependent on 4D SUSY!

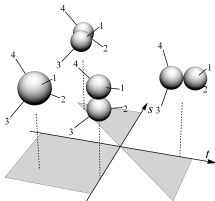
(I) Twistor String \rightarrow Scattering equations

Lesson: $\lambda_i^\alpha \rightarrow (z_i, 1)$ kinematics determine coordinates of the sphere!

Scattering Equations :
$$E_a = \sum_{i \neq a} \frac{k_i \cdot k_a}{\sigma_i - \sigma_a} = 0$$



Massless kinematics parameterizes the moduli space of n -punctured Riemann spheres



Factorization kinematics maps to degenerate limits of the moduli space.

(I) CHY formulas

$$M_n = \int \underbrace{\frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k, \epsilon, \sigma\}) = \sum_{\{\sigma\} \in \text{sols.}} \frac{\mathcal{I}(\{k, \epsilon, \sigma\})}{J(\{\sigma\})}$$

- Amplitude as an integral over moduli space localized by scattering equations, or equivalently a sum over the solutions, with certain “CHY integrand” [CHY 2013]
- n-3 integrals with n-3 delta functions; $J = \det'(\partial E / \partial \sigma)$ is the Jacobian

What are the integrands?

(I) CHY formulas

- Yang-Mills:

$$M_n^{\text{YM}}[\pi] = \int d\mu_n \text{PT}[\pi] \text{Pf}' \Psi, \quad \text{PT}[\pi] = \frac{1}{\prod_i (\sigma_{\pi_i} - \sigma_{\pi_{i+1}})}$$

- Gravity:

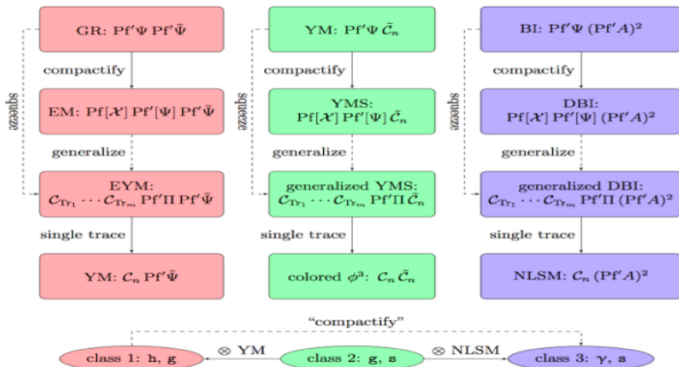
$$M_n^{h+B+\phi} = \int d\mu_n \text{Pf}' \Psi(\epsilon) \text{Pf}' \Psi(\epsilon')$$

where $\text{Pf}'[\psi]$ is manifest gauge invariant

$$\text{Pf}' \Psi := \frac{\text{Pf}[\Psi]_{i,j}^{i,j}}{\sigma_{i,j}}, \quad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$
$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

(I) CHY formulas

Massless unification



(I) CHY formulas

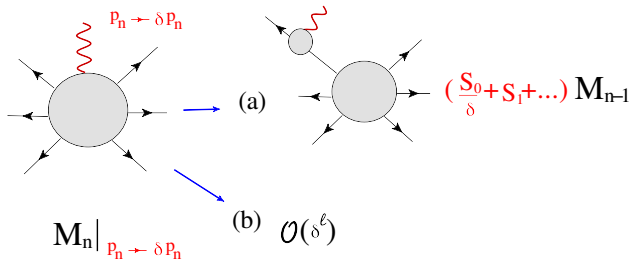
- Yields scattering amplitudes as a single object (no diagrammatic expansion, triangulation e.t.c)
- Extensions to multi-loop, higher genus vs forward limits yields new representations for loop integrand [see Adamo et al](#); [Geyer et al](#); [Feng et al](#); [CHY...](#)
- Manifest universal **soft behaviours** $M_n|_{p_n \rightarrow \epsilon} \rightarrow (\sum_i S_i(\epsilon))M_{n-1}$
- **What is special about the chosen theories?**
- **How does a world-sheet with massive excitation turned into just massless particles ?**

(II) Soft-theorems and EFT

Soft-behaviours and symmetries

(II) Soft-theorems and EFT

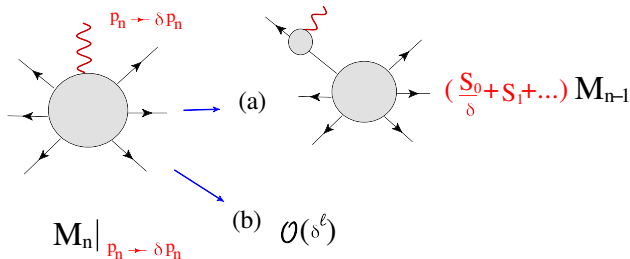
For massless particles, as $p_n \rightarrow 0$, the amplitude tends to have universal behaviour: ind of helicities, multiplicity, e.t.c



Fundamental interactions of massless particles are highly constrained by symmetries
(local gauge symmetries, spontaneous broken global symmetries)

(II) Soft-theorems and EFT

For massless particles, as $p_n \rightarrow 0$, the amplitude tends to have universal behaviour: ind of helicities, multiplicity, e.t.c

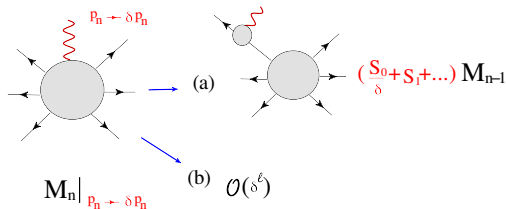


(a) Ward identities, QED: [Low](#), Gravity: [Weinberg,Gross,Jackiw](#)

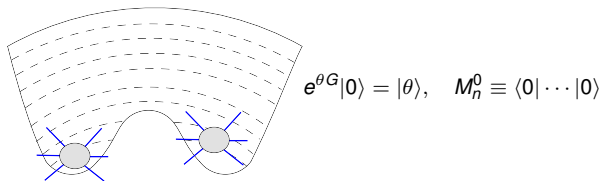
$$A_n \Big|_{p_n \rightarrow \delta p_n} = \frac{1}{\delta} S_0 A_{n-1} + S_1 A_{n-1} + \mathcal{O}(\delta), \quad S_0 = \sum_i \frac{\epsilon_n \cdot p_i}{p_n \cdot p_i}$$

$$M_n \Big|_{p_n \rightarrow \delta p_n} = \frac{1}{\delta} S_0 M_{n-1} + S_1 M_{n-1} + \delta S_2 M_{n-1} + \mathcal{O}(\delta^2), \quad S_0 = \sum_i \frac{\epsilon_n \cdot p_i p_i}{p_n \cdot p_i}$$

(II) Soft-theorems and EFT



(b) Adler's zero:



$$M_n^\theta \equiv \langle \theta | \dots | \theta \rangle = M_n^0 + M_{n+\pi}^0 + M_{n+\pi+\pi}^0 + \dots$$

The U(1) goldstone bosons are derivatively coupled: $\mathcal{L}(\partial\phi)$

(II) Soft-theorems and EFT

Ward identity

$$\partial_\mu \langle J^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = - \sum_i \delta(x - x_i) \langle \phi(x_1) \cdots \delta\phi(x_i) \cdots \phi(x_n) \rangle$$

Spontaneous symmetry breaking implies $J^\mu|0\rangle = p^\mu|phys\rangle$

- LHS: performing LSZ reduction on $i = 1, \dots, n \rightarrow M_n(\pi_1 \cdots) |_{p_1 \rightarrow 0}$
- RHS: $\begin{cases} = 0 & \text{if } \delta\phi \neq |phys\rangle \\ \neq 0 & \text{if } \delta\phi = |phys\rangle \end{cases}$

Conventional spontaneous symmetry breaking: $\delta\phi = \text{constant}$ hence Adler's zero

For systems with multiple GB modes, things become tricky

(II) Soft-theorems and EFT

Spontaneous broken dilation and conformal boost generator leads to single dilaton,

$$[K, D] \sim K$$

The dilaton transforms linearly under the broken generator \rightarrow non-vanishing soft-limits:

Boels, Wormsbecher, Y-t Wen, Di Vecchia, Marotta, Mojaza, Nohle

$$M_n|_{p_n \rightarrow 0} = \left(\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)} \right) M_{n-1} + \mathcal{O}(p_n^2),$$

$(\mathcal{S}_n^{(0)}, \mathcal{S}_n^{(1)})$ are universal soft functions

$$\mathcal{S}_n^{(0)} = \sum_{i=1}^{n-1} \left(p_i \cdot \frac{\partial}{\partial p_i} + \frac{d-2}{2} \right) - d,$$

$$\mathcal{S}_n^{(1)} = p_n^\mu \sum_{i=1}^{n-1} \left[p_i^\nu \frac{\partial^2}{\partial p_i^\nu \partial p_i^\mu} - \frac{p_{i\mu}}{2} \frac{\partial^2}{\partial p_{i\nu} \partial p_i^\nu} + \frac{d-2}{2} \frac{\partial}{\partial p_i^\mu} \right].$$

(II) Soft-theorems and EFT

There's more! In general CFTs with scalar moduli space has “flavor” symmetry, which will be spontaneously broken along with conformal symmetry \rightarrow pions

Exp: $\mathcal{N} = 4$ SYM on Coulomb branch, 6 massless scalars (1 dilaton φ , 5 $\text{SO}(6) \rightarrow \text{SO}(5)$ GBs ϕ^I)

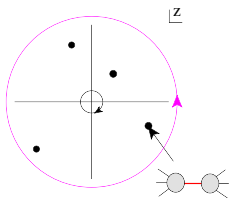
$$A_n(\phi_1, \dots, \phi_n^I) \Big|_{p_n \rightarrow 0} = \sum_i A_{n-1}(\dots, \delta^I O, \dots) + \mathcal{O}(p_n^1).$$

where $\delta^I \varphi = \phi^I$ and $\delta^I \phi^J = -\delta^{IJ} \varphi$.

(II) Soft-theorems and EFT

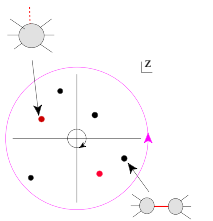
Using the fact that S-matrix are analytic functions, we start with: Britto, Cachazo, Feng, Witten

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{z} = - \oint_{|z|=z^*} dz \frac{A_n(z)}{z},$$



The constraint from soft-theorems can be utilized via augmented recursion: Cheung, Kampf, Novotny, Shen, Trnka

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{zF(z)} = - \oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)} - \oint_{|z|=z^*} dz \frac{A_n(z)}{zF(z)},$$



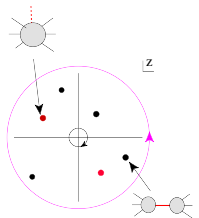
(II) Soft-theorems and EFT

Take

$$A(z) = A|_{p_i \rightarrow (1-za_i)p_i}, \quad F_n(z) = \prod_{i=1}^n [(1-za_i)]^{d_i}$$

with $\sum_i a_i p_i = 0$

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{z F_n(z)} = - \oint_{|z|=z^*} dz \frac{A_n(z)}{z F_n(z)} - \oint_{|z|=z^*} dz \frac{A_n(z)}{z F_n(z)},$$



The residue of $F(z)$ is determined

$$A(z) \rightarrow A_0 + A_1 q + A_2 q^2 + \dots A_d q^{d-1}$$

where $q = (1 - za_i)p_i$

(II) Soft-theorems and EFT

The residue of $F(z)$ is determined

$$A(z) \rightarrow A_0 + A_1 q + A_2 q^2 + \dots + A_d q^{d-1}$$

Since for the pure dilaton sector

$$M_n|_{p_n \rightarrow 0} = \left(S_n^{(0)} + S_n^{(1)} \right) M_{n-1} + \mathcal{O}(p_n^2),$$

we have $d = 2$.

The pure dilaton amplitude can be constructed using recursion

$$A_n(0) = \oint_{|z|=0} dz \frac{A_n(z)}{z \prod_i (1 - za_i)^2}$$

The denominator $\sim z^{2n}$, while $A_n(z) \sim z^{2m}$ for order $\partial^{2m} \rightarrow$ we need $n > m$

(II) Soft-theorems and EFT

The pure dilaton sector is highly constrained:

$s^n \setminus \# \text{ of points}$	4	5	6	7	8	...
2	×	✓	✓	✓	✓	✓
3	×	✓	✓	✓	✓	✓
4	×	✓	✓	✓	✓	✓
5	✓	×	✓	✓	✓	✓
6	✓	✓	×	✓	✓	✓
7	✓	✓	✓	×	✓	✓
8	✓	✓	✓	✓	×	✓
⋮

At s^n , the dilaton EFT is determined up to coefficients for operators $\partial^{2n}\varphi^n$

(II) Soft-theorems and EFT

When is a theory soft constructable ? $\rightarrow A_4$ determines the complete effective theory

Cheung, Kampf, Novotny, Shen, Trnka

With $p_i \rightarrow (1 - za_i)p_i$, dimension m $A_n(z) \sim z^m$ and $A_n \sim p^\sigma$ as $p \rightarrow 0$,

$$\frac{A_n(z)}{E_n(z)} \sim z^{m-n\sigma}, \quad m/n < \sigma$$

Examples: NLSM, Dirac-Born-Infeld, special Galileon

(III) Constraints on perturbative completions

If we assume EFT are perturbatively completed, how constraining is it ?

For gravity

$$\mathcal{M}_4|_{s \rightarrow \infty} \sim \frac{s^2}{t} \sim E^2$$

- Assuming that the high energy behaviour is tamed while weakly coupled (true for many known examples **in nature**)
- **Implies** new degrees of freedom come in at tree-level
- **Implies** new degrees of freedom $M \ll M_{Plank}$

How constraining is the space of possible solutions?

(III) Constraints on perturbative completions

We expect the existence of massless particles with three-point interactions to be more constraining



There are finite number of consistent three-point operators

$$\mathcal{L} = \int d^D x \sqrt{g}(R + R^2 + R^3), \quad D > 2$$

$$\mathcal{L} = \int d^D x F^2 + F^3, \quad D > 4$$

$$\mathcal{L} = \int d^D x (\partial\phi)^2 + \phi^3, \quad D > 6$$

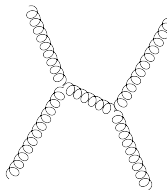
Residues of massless poles are severely constrained.

(III) Constraints on perturbative completions

Constraints from causality: Camanho, Edelstein, Maldacena, Zhiboedov

$$S = \frac{1}{l_p^{D-2}} \int d^D x \sqrt{g} (R + \alpha_1 R^2 + \alpha_2 R^3 + \dots)$$

Causality: if $\alpha_{1,2} \gg l_p$ a particle traveling pass a shock wave will experience time advancement


$$\sim \frac{t^2 + \alpha_1^2 t^4 + \alpha_2^2 t^6}{s}$$

Can only be cured by an infinite $J > 2$ massive particles

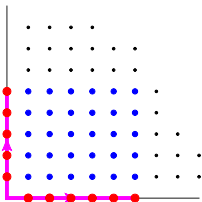
(III) Constraints on perturbative completions

$$M \sim \frac{(s^2 + t^2 + u^2)^2}{stu} \frac{f(s, t)}{(s-m_1)(t-m_2)\cdots} \Big|_{s=m_1} \rightarrow \frac{(s^2 + t^2 + u^2)^2}{stu} \frac{f(m_1, t)}{(t-m_2)\cdots}$$

Unitarity requires the function $f(m_1, t)$ to have a zero when $t = m_2$, and all other t -channel poles.

$f(s, t)$ is a bounded polynomial function that has zero for each pair of $(s, t) = (m_i, m_j)!$

$$f(m_1, m_1) = f(m_1, m_2) = \cdots = f(m_i, m_j) = 0$$



There are more zeros than poles!

(III) Constraints on perturbative completions

$$M \sim \frac{(s^2 + t^2 + u^2)^2 \prod_{i,j} (s + b_{i,j})(t + b_{i,j})(u + b_{i,j})}{stu \prod_i (s - a_i)(t - a_i)(u - a_i)}$$

with $b_{i,j} \equiv a_i + a_j$. When $s = a_i, t = a_j$, we have a requisite zero in the numerator, since $s + t - a_i - a_j \sim u + b_{i,j}$

But, the massless residue must be polynomial:

$$\text{Res}[M]_{s \rightarrow 0} \sim -4t^2 \frac{\prod_{i,j} (b_{i,j})(t + b_{i,j})(t - b_{i,j})}{\prod_i (a_i)(t - a_i)(t + a_i)}$$

There must be a 1 – 1 map between a_i and b_{ij} : $a_j = i\alpha$

Constraints on perturbative completions

The solution is given as:

$$M \sim \frac{(s^2 + t^2 + u^2)^2}{stu} \frac{\prod_{i=1}^{\infty} (s+i)(t+i)(u+i)}{\prod_{i=0}^{\infty} (s-i)(t-i)(u-i)} = (s^2 + t^2 + u^2)^2 \frac{\Gamma[-u]\Gamma[-t]\Gamma[-s]}{\Gamma[1+t]\Gamma[1+u]\Gamma[1+s]}$$

TypeII superstring



Fig.3: Particle scattering processes (left), string scattering processes (right).

Another solution: unitarize each massless pole separately:

$$\frac{(s^2 + t^2 + u^2)^2}{stu} = (s^2 + t^2 + u^2) \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right)$$

The absence of massless poles in other channels leads to the absence of zeros

$$(s^2 + t^2 + u^2) \frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]} \left(\frac{tu}{1+s} + \frac{su}{1+t} + \frac{st}{1+u} \right)$$

Heterotic superstring

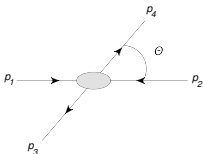


(III) Constraints on perturbative completions

Is there more solutions?

(III) Constraints on perturbative completions

Unitarity requires the exchange coefficient of each spin is positive:



The residue at $s = m^2$ is a function of $t = -\frac{s}{2}(1 - \cos \theta)$, i.e. $r(\cos \theta)$ Spin decomposition corresponds to expanding $r(\cos \theta)$ on Gegenbauer polynomial basis:

$$r(\cos \theta) = \sum_{\ell} c_{\ell} C_{\ell}^{\alpha}(\cos \theta)$$

where $\alpha = (D - 2)/2$

$$\frac{1}{(1 - 2x \cos \theta + x^2)^{\alpha}} = \sum_{\ell=0}^{\infty} x^{\ell} C_{\ell}^{\alpha}(\cos \theta)$$

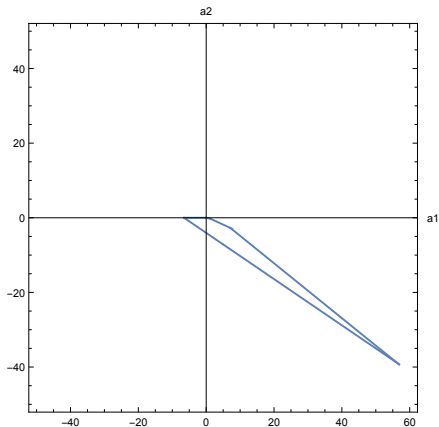
$c_{\ell} > 0$ are known as **positive functions**

Possible deformations

$$\text{Ansatz} = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$

$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

s=1

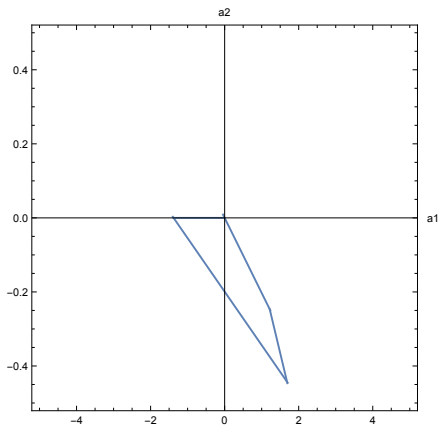


Possible deformations

$$\text{Ansatz} = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$

$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

s=2

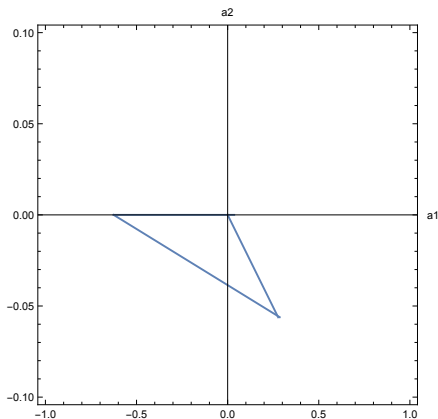


Possible deformations

$$\text{Ansatz} = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$

$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

s=3



(III) Constraints on perturbative completions

What do we expect of $r(x)$ ($x = \cos \theta$)?

- Permutation invariance: $\rightarrow f(x) = f(-x)$, since

$$t = -\frac{s}{2}(1-x), \quad u = -\frac{s}{2}(1+x)$$

- If $r(x)$ is positive in D , it is also positive in $D' < D$
- If $r(x)$ is positive (corollary of Schoenberg Theorem)

$$|r(x)| < r(1)$$

If $r(x)$ has root at $x = 1$, it must be a negative function

(III) Constraints on perturbative completions

$$(t+1)(t+2)\cdots(t+n-1) \rightarrow \left(x^2 - \frac{1}{n^2}\right) \left(x^2 - \frac{9}{n^2}\right) \cdots \left(x^2 - \frac{(n-2)^2}{n^2}\right)$$

- As $n \rightarrow \infty$ it's a boundary positive function
- But

$$stu = \frac{s^3}{4}(1 - x^2)$$

So anything with an stu factor cannot be positive!!!!

- A general solution:

$$A_4 = A_4^{\text{Ttree}} \frac{f(\sigma_3, \sigma_2)}{\prod_i (s - a_i)(t - a_i)(u - a_i)}$$

$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

Massless residue fixes the purely σ_2 part:

$$f(\sigma_3, \sigma_2) = \{\text{String}\} + \sigma_3 \Delta$$

As $n \rightarrow \infty$ $\{\text{String}\} \rightarrow$ near negative!

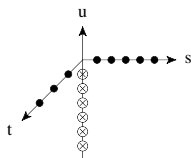
New view of string theory

Let's consider the four-point closed superstring answer

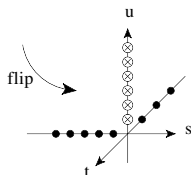
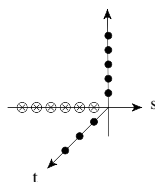
$$M^{\text{closed}}(s, t) = (\sin \pi t) A^{\text{open}}(s, t) A^{\text{open}}(t, u)$$

The poles and zero are determined from unitarity

(I) $A(s, t)$



(II) $A(u, t)$



All massive state cancels !

New view of string theory

It's the four-point supergravity amplitude! [Y-t, Siegel, Yuan](#)

$$M^{\text{sugra}}(s, t) = (\sin \pi t) A^{\text{open}}(-s, -t) A^{\text{open}}(t, u)$$

What does this mean on the world sheet ?

$$M^{\text{closed}}(s, t) = \int d^2 z |1 - z|^{2s} |z|^{2t} = \int d^2 z (1 - z)^s z^t (1 - \bar{z})^s \bar{z}^t$$

$$\rightarrow M^{\text{sugra}}(s, t) = \int d^2 z (1 - z)^{-s} z^{-t} (1 - \bar{z})^s \bar{z}^t$$

A change of boundary condition for Greens function

$$\ln z \bar{z} \rightarrow \ln z \bar{z} - 2 \ln z = \ln \bar{z} - \ln z$$

New view of string theory

Supergravity is simply perturbative string theory with chiral boundaries (no limit on α' is needed)

Free to take $\alpha' \rightarrow \infty$ He

$$\int dz^2(\dots) e^{\alpha'[-s \log z - t \log(1-z) + s \log \bar{z} + t \log(1-\bar{z})]}$$

Relation to CHY

For superstring the final result is α' independent, $\alpha' \rightarrow \infty$ He

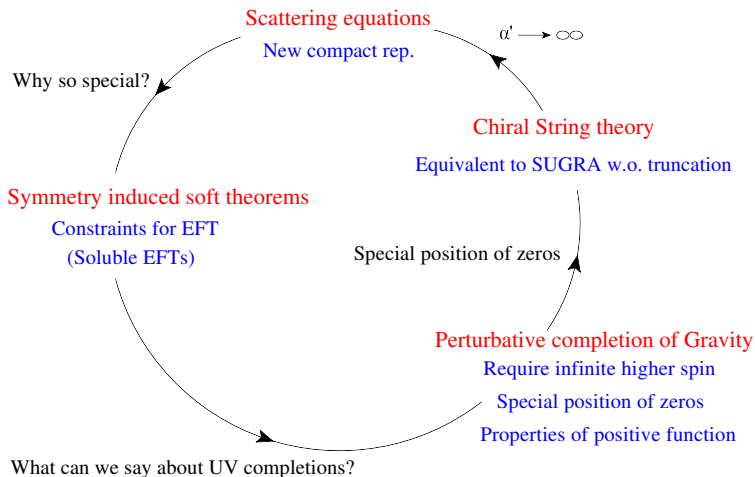
$$\int dz^2 (\dots) e^{\alpha' [-s \log z - t \log(1-z) + s \log \bar{z} + t \log(1-\bar{z})]}$$

Depending on kinematics, saddle point localizes on

$$\frac{s}{z} + \frac{t}{1-z} = 0$$

Precisely the scattering equations!

Conclusion



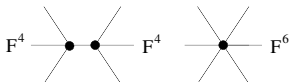
(II) Soft-theorems and EFT

Maximal SUSY is known to give exact results:

- s^2 : F^4 operator one-loop exact $\lambda = \left(\frac{g^4 N}{8\pi^2 m^4} \right)$
- For the pure field-strengths [Chen, Y-t, Wen](#)

$$\mathcal{L}_{\text{eff}} = \sum_{p,q=1} c_0^{p,q} \frac{(F_+^2)^p (F_-^2)^q}{(M^2)^{2(p+q-1)}} + \sum_{m=1} \sum_{p,q=1} c_m^{p,q} \frac{D^{2m} (F_+^2)^p (F_-^2)^q}{(M^2)^{2(p+q-1)+m}} + \dots$$

There are no local susy matrix elements that encode $F_-^2 F_+^{n-2} \rightarrow$ **must have zero coefficient**



One obtains an exact recursion formula for the operators $F_-^2 F_+^{2q}$

$$c_0^{1,q} = 4^{q-1} (c_0^{1,1})^q.$$