Principal Component Analysis of Flow

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Outline

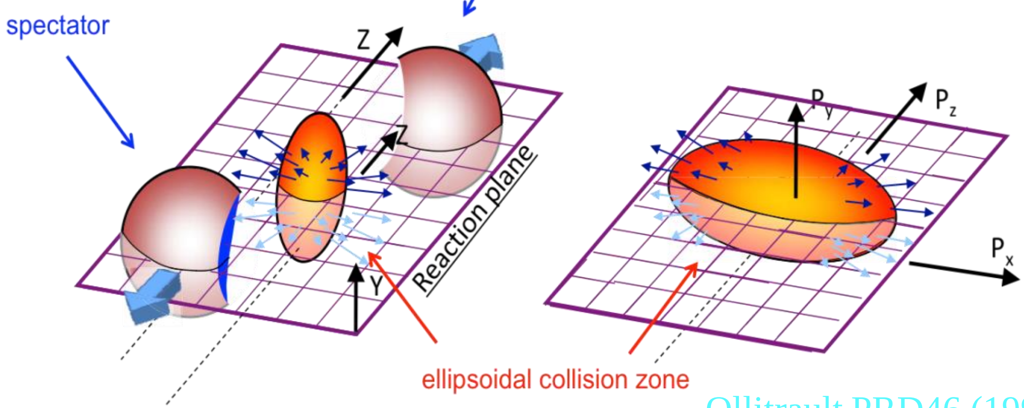
- Introduction and standard methods to determine anisotropic flow
- Principal Component Analysis (PCA)
- Application to 2-particle correlation matrix
- Results in AMPT (η, p_T) and ALICE (p_T)
- Conclusions

Flow analysis in heavy ion collision

Analysis of anisotropic flow v_n

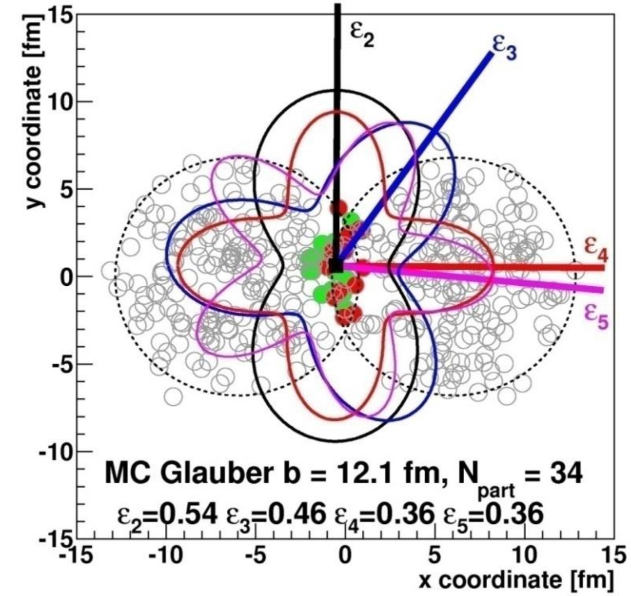
- Methods currently in use (event-plane, cumulant): devised before the importance of flow fluctuations was recognized
- New method: extraction of flow fluct. directly from expt. data on 2-particle correlations
- Based on Principal Component Analysis (PCA) — applied to the 2-particle correlation matrix, $\langle \cos n\Delta\phi \rangle$
- Leading eigenmode $\longleftrightarrow$ usual v_2, v_3
  Subleading modes of v_2, v_3 revealed for the 1st time

Anisotropic flow v_n



Ollitrault PRD46 (1992) 229

Origin of elliptic flow v_2: spatial anisotropy and re-interaction



Harmonic flow vector relates to various moments of initial spatial anisotropy (eccentricities) of nucleons/partons

$$S_n \equiv \varepsilon_n e^{in\Phi_n} = -\left\langle r^n e^{in\phi} \right\rangle / \left\langle r^n \right\rangle$$

ε_n = magnitude (eccenticities)
Φ_n = phase (participant-plane angle)

Origin of triangular flow v_3: fluctuations in the position of participant nucleons

Alver & Roland, PRC81(2010) 054905

EbyE initial distribution: $p(\varepsilon_n, \varepsilon_m, \ldots, \Phi_n, \Phi_m, \ldots)$

$\Downarrow$ Collective/Hydro expansion (σ, EoS, η/s, ζ/s, …)

Flow observable

$$p(v_n, v_m, \ldots, \Psi_n, \Psi_m, \ldots) = \frac{1}{N_{ev}} \frac{N_{ev}}{dv_n dv_m \ldots d\Psi_n d\Psi_m \ldots}$$

Standard methods to estimate v_n

Event Plane method

Ollitrault PRD46 (1992) 229

v_n are Fourier coeffecients in φ distrn of particles wrt reaction plane

$$E\frac{dN}{d^3p} \approx \frac{1}{2\pi}\frac{dN}{p_T dp_T d\eta}\left[1+2\sum_{n=1}^{\infty} v_n \cos[n(\phi-\Psi_n)]\right]$$

Event Plane angle:

$$\psi_n = \frac{1}{n}\arctan\frac{\langle p_T \sin(n\phi)\rangle}{\langle p_T \cos(n\phi)\rangle}$$

$$v_n = \langle e^{in(\phi-\Psi_n)}\rangle = \langle \cos n(\phi-\Psi_n)\rangle, \quad n = 1,2,3,4\ldots$$

Entire Flow vector can be expressed in a complex plane:

$$V_n \equiv v_n e^{in\psi_n}(p_T,\eta) = \int_0^{2\pi} d\varphi \frac{dNe^{in\varphi}}{d\varphi dp_T d\eta} \Bigg/ \int_0^{2\pi} d\varphi \frac{dN}{d\varphi dp_T d\eta} \equiv \langle e^{in\phi}\rangle$$

$v_n(p_T,\eta)$ = magnitude (anisotropic flow)

$\Psi_n(p_T,\eta)$ = phase (event-plane angle along minor axis)

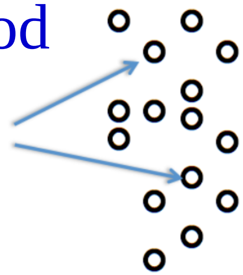
NB: flow fluctuations can modify v_n and Ψ_n

Standard methods to estimate v_n

Multiparticle Correlation method

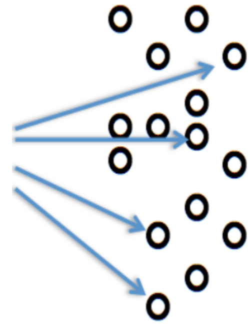
2-particle correlation method

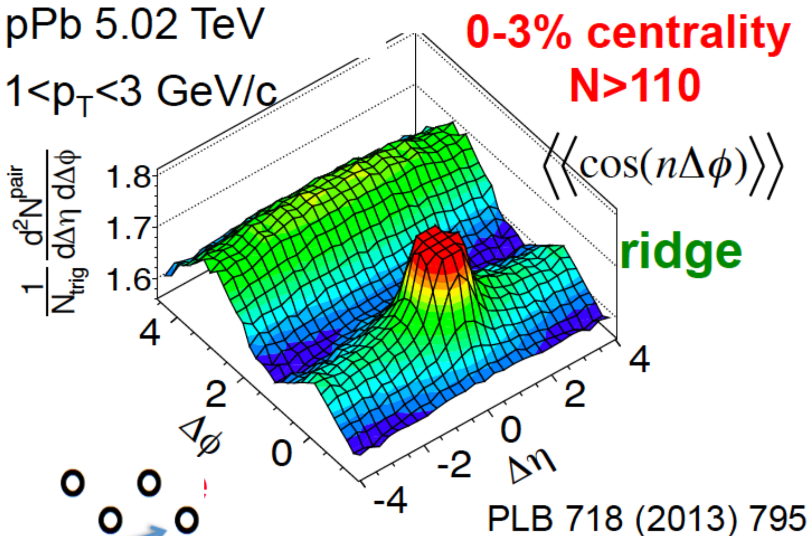
$$v_n\{2\} = \left\langle e^{in(\varphi_1-\varphi_2)} \right\rangle$$



4-particle correlation method

$$v_n\{4\} = \left\langle e^{in(\varphi_1+\varphi_2-\varphi_3-\varphi_4)} \right\rangle$$





2, 4 particle azimuthal correlations:

$$\langle \cos n(\varphi_1 - \varphi_2) \rangle = \langle e^{in(\varphi_1-\varphi_2)} \rangle = v_n^2\{2\} + \delta_n$$

nonflow part
(**short range**: resonance decay, BE correlation, etc)

$$\langle \cos n(\phi_1 + \phi_2 - \phi_3 - \phi_4) \rangle = 2v_n\{2\}^4 - v_n\{4\}^4$$

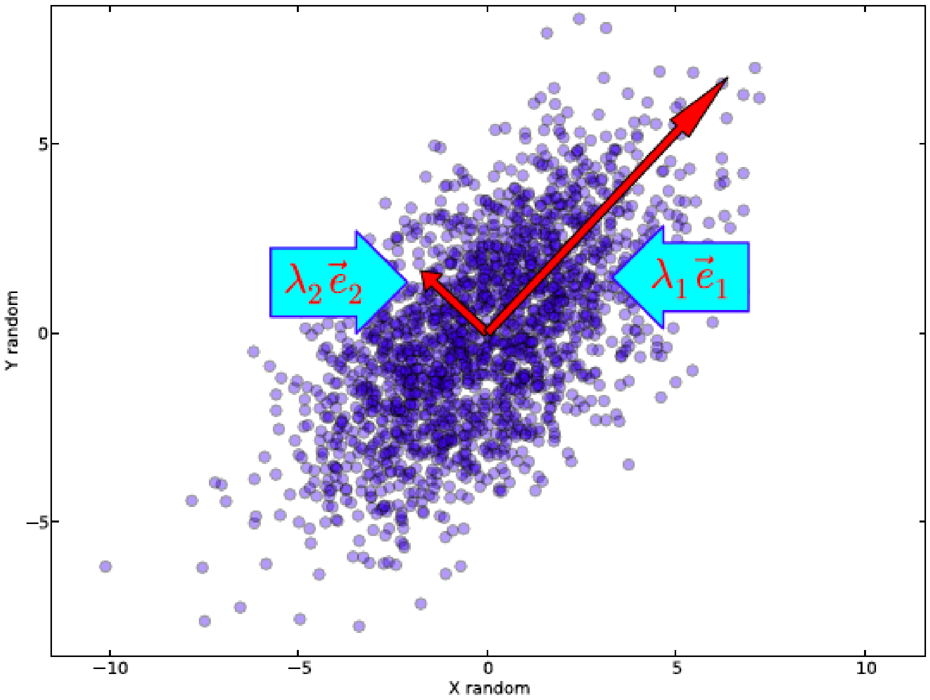
Multiparticle (n >2) correlations remove non-flow correlations

Principal Component Analysis (PCA)

- Statistical procedure to elucidate the underlying covariance structure in the multi-dimensional data
- To identify the directions (PC) where there is the most variance, and possibly reduce the dimension of data
- Diagonalize the covariance matrix: Eigenvector with the largest eigenvalue is the direction of greatest variance; that with the 2nd largest eigenvalue …
- PCA can be thought of as fitting a hyper-ellipsoid to the cloud of data points — each of its axes representing a PC

Principal Component Analysis (PCA)

A simple 2D example Milosevic, Quark Matter 2015



- Random data generated by 2D multivariate Gauss distribution

$$\vec{X}_n = (x_1, x_2, \ldots, x_n)$$
$$\vec{Y}_n = (y_1, y_2, \ldots, y_n)$$

- a matrix

$$\Sigma = \begin{bmatrix} \mathrm{var}(X) & \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,Y) & \mathrm{var}(Y) \end{bmatrix}$$

- eigenvectors $\boldsymbol{e}_i$ and eigenvalues λ_i by diagonalizing Σ

$$[e]^T \Sigma [e] = diag(\lambda_1, \lambda_2)$$

- First Principal Component: eigenvector $\boldsymbol{e}_1$ points to maximum variance of data cloud. Its magnitude is $\sqrt{\lambda_1} e_1$

- Second Principal Component: eigenvector $\boldsymbol{e}_2$ points to the next maximum variance of data cloud. Its magnitude is $\sqrt{\lambda_2} e_2$

Flow picture

- Single-particle distribution in an event:
  $\frac{dN}{d\mathbf{p}} = \sum_{n=-\infty}^{\infty} V_n(p) \exp(in\phi)$ Fourier coeff $V_n(p) \equiv V_n(p_T, \eta)$
- Pair distribution averaged over events:
  $\left\langle \frac{dN_{pairs}}{d\mathbf{p_1} d\mathbf{p_2}} \right\rangle = \left\langle \frac{dN}{d\mathbf{p_1}} \frac{dN}{d\mathbf{p_2}} \right\rangle + \mathcal{O}(N) \leftarrow$ nonflow correl.
  $= \sum_{n=-\infty}^{\infty} V_{n\Delta}(p_1, p_2) \exp(in(\phi_1 - \phi_2))$
- Two-particle correlation matrix:
  $V_{n\Delta}(p_1, p_2) = \langle V_n(p_1) V_n^*(p_2) \rangle$,
  neglecting non-flow correlations

Note: $n = 0$ multiplicity; $n \neq 0$: anisotropic flow

Bhalerao, Ollitrault, SP, Teaney: PRL 114 (2015) 152301

Covariance matrix $V_n(p)$

- $V_{n\Delta}(p_1, p_2) = \langle V_n(p_1) V_n^*(p_2) \rangle$ is a covariance matrix.

- Covariance matrix is symmetric, positive semidefinite, and its eigenvalues are non-negative.

- PCA yields $V_{n\Delta}(p_1, p_2) \simeq \sum_{\alpha=1}^{k} V_n^{(\alpha)}(p_1) V_n^{(\alpha)*}(p_2)$
  $=$ sum over modes of flow fluct.

- If no flow fluctuations, then factorization occurs:
  $V_{n\Delta}(p_1, p_2) \simeq V_n^{(1)}(p_1) V_n^{(1)*}(p_2)$

I.e. one component (k=1) exits corresp. to usual anisotropic flow.

Flow fluctuations break factorization and $k \leq N_b$ (= # of bins in p)

Method to calculate flow in PCA

- Divide the detector acceptance into several bins in p_T and/or η. Let the bin index $p = (p_T, \eta)$

- Flow vector in an event: $Q_n(p) \equiv \sum_{j=1}^{M(p)} \exp(in\phi_j)$

- Pair distribution $V_{n\Delta}(p_1, p_2)$

subtracts self-correl

singles out fluc.

$$\equiv \langle Q_n(p_1) Q_n^*(p_2) \rangle - \langle M(p_1) \rangle \delta_{p_1 p_2} - \langle Q_n(p_1) \rangle \langle Q_n^*(p_2) \rangle$$

- In practice $V_{n\Delta}(p_1, p_2) \equiv \sum_\alpha \lambda^{(\alpha)} \psi^{(\alpha)}(p_1) \psi^{(\alpha)*}(p_2)$
$$= \sum_\alpha V_n^{(\alpha)}(p_1) V_n^{(\alpha)*}(p_2)$$

- PC are obtained by diagonalizing $V_{n\Delta}(p_1, p_2)$:
Eigenvalues: $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)} \ldots$
Eigenvectors: $\psi^{(\alpha)}(p) = V_n^{(\alpha)}(p) / \sqrt{\lambda^{(\alpha)}}$

Method to calculate flow in PCA

- In a given event, flow can be decomposed as

  $$V_n(p) = \sum_{\alpha=1}^{k} \zeta^{(\alpha)} V_n^{(\alpha)}(p)$$

  Real vectors $V_n^{(\alpha)}(p)$ record rms values of (sub)leading flows.
  Complex coefficients $\zeta^{(\alpha)}$ indicate orientation wrt flow.

- To compare with the usual flow definition (or with experiments), we define

  $$v_n^{(\alpha)}(p) \equiv \frac{V_n^{(\alpha)}(p)}{\langle M(p) \rangle}$$

  Thus $v_0^{(\alpha)}(p)$: relative multiplicity fluctuations,

  $v_n^{(\alpha)}(p)$ for $n \neq 0$: fluctuations of anisotropic flow

A MultiPhase Transport model (AMPT)

Inclusive hadron distribution – calculable in pQCD

$$dN_h = f_{a/A}(x_a,Q^2)\otimes f_{b/A}(x_b,Q^2)\otimes d\sigma_{ab\to cX}\otimes \Delta\mathbf{E} \otimes D_{c\to h}(z_c,Q^2)$$

Energy loss

Lin, Ko, Li, Zhang, SP, PRC72 (2005) 064901

SP, Bleicher, PLB 709 (2012) 82

- ❑ Initial particle distribution obtained from updated HIJING 2.0 model

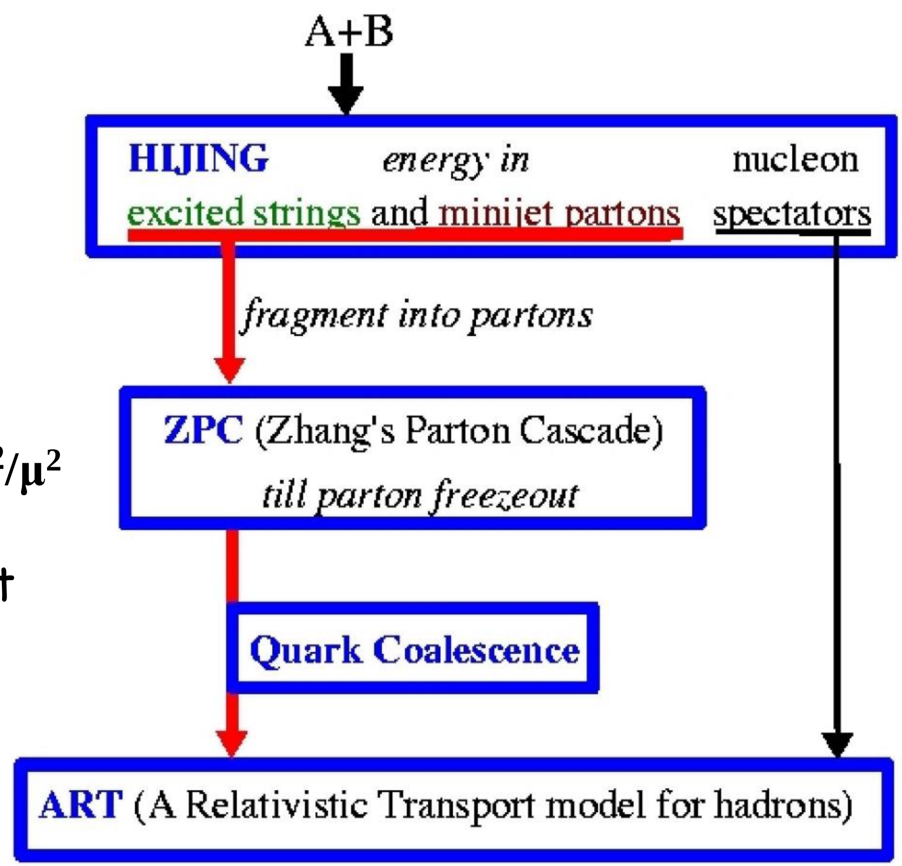
- ❑ Strings from HIJING converted to valence (anti)quarks – String Melting

- ❑ Partons scatter in ZPC model with elastic scattering cross section: $\sigma \approx 9\pi\alpha_s^2/\mu^2$

- ❑ Phase-space coalescence of freeze-out partons produce the hadrons

- ❑ Hadrons evolve with (in)-elastic scatterings via ART transport model

Structure of AMPT model with string melting



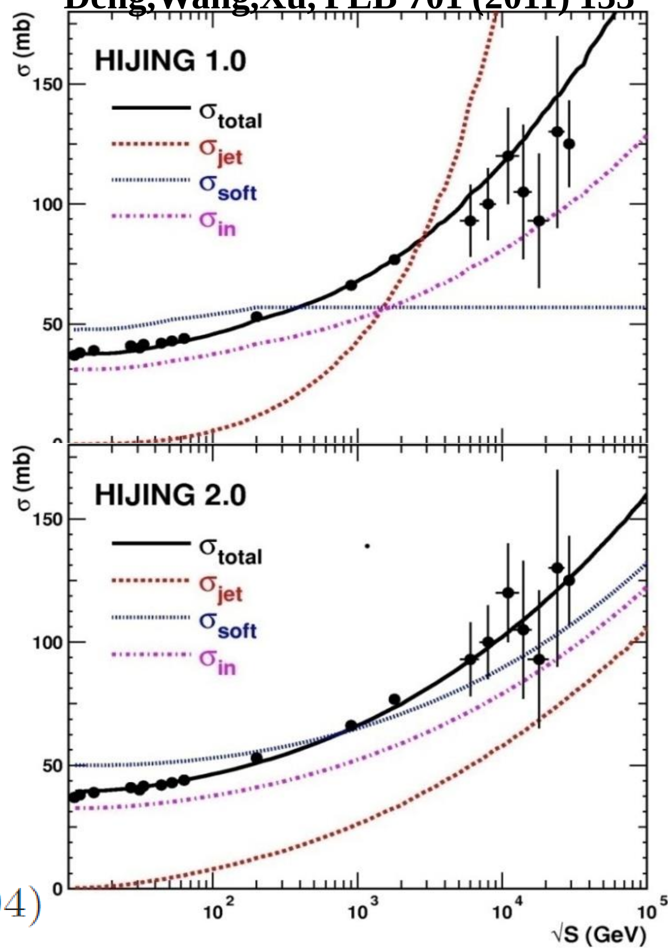
AMPT with updated HIJING 2.0

- GRV parametrization of parton distribution function
- c.m. energy dependence in 2-component HIJING 2.0

$$\sigma_{soft} = 55.316 - 4.1126\log(\sqrt{s}) + 0.854\log^2(\sqrt{s}) - 0.0307\log^3(\sqrt{s}) + 0.00328\log^4(\sqrt{s}),$$

$$p_0 = 2.62 - 1.084\log(\sqrt{s}) + 0.299\log^2(\sqrt{s}) - 0.0292\log^3(\sqrt{s}) + 0.00151\log^4(\sqrt{s}),$$

Deng,Wang,Xu, PLB 701 (2011) 133



PDF in nucleus: $f_a^A(x,Q^2) = AR_a^A(x,Q^2)f_a^N(x,Q^2),$

$$R_q^A(x,b) = 1.0 + 1.19\log^{1/6}A\,(x^3 - 1.2x^2 + 0.21x) - s_q(b)\,(A^{1/3}-1)^{0.6}(1-3.5\sqrt{x})\exp(-x^2/0.01),$$

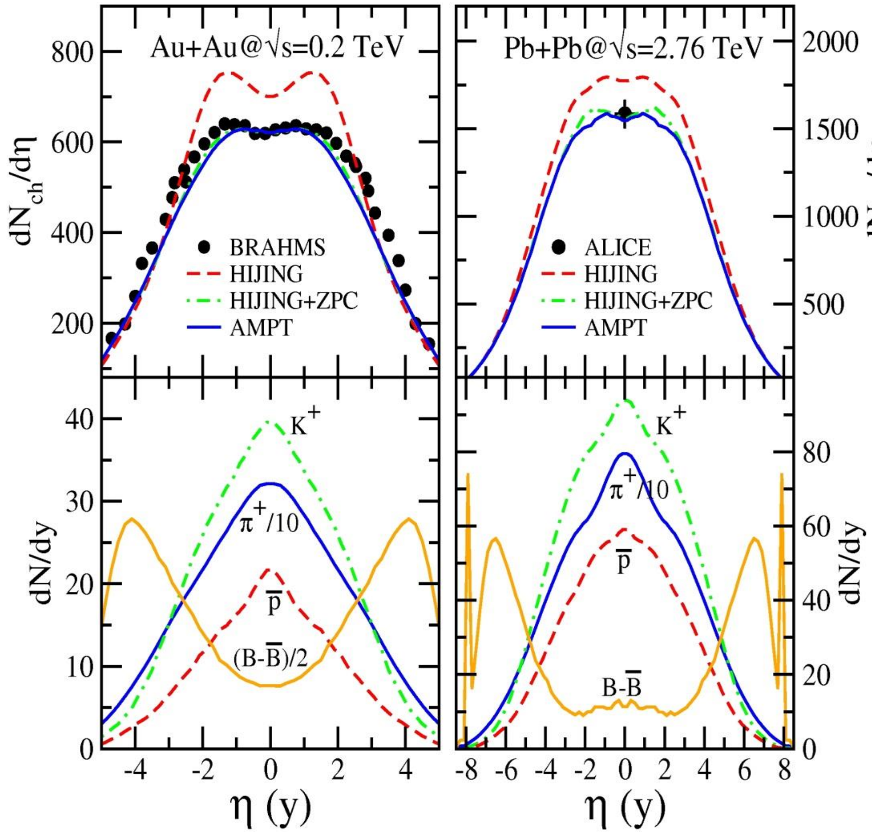
$$R_g^A(x,b) = 1.0 + 1.19\log^{1/6}A\,(x^3 - 1.2x^2 + 0.21x) - s_g(b)\,(A^{1/3}-1)^{0.6}(1-1.5x^{0.35})\exp(-x^2/0.004)$$

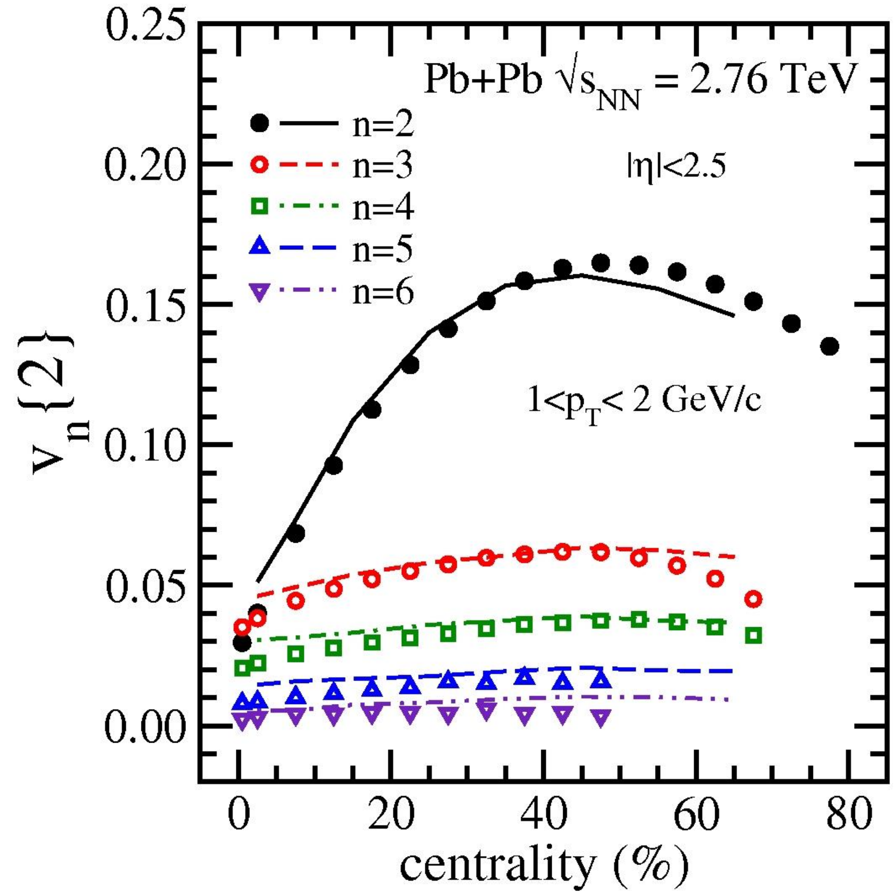
Impact parameter dependent shadowing

$$s_a(b) = s_a\frac{5}{3}(1 - b^2/R_A^2)$$

s_q = 0.1 (fixed) from deep-inelastic-scattering data off nuclear targets.
s_g fitted to centrality dependence of measured dN_{ch}/dy in A+A collision.

dN_{ch}/dy and $v_n\{2\}$ from AMPT





HIJING: $dN_{ch}/d\eta|_{|\eta|<0.5}$ = 705 (RHIC)
= 1775 (LHC)

❖ Parton scatterings lead to ~15% decrease in dN_{ch}/dy at RHIC & LHC!!

❖ Hadron scattering insensitive to $dN/d\eta$.

- $v_2 > v_3 > v_4 > v_5 > v_6$
- v_n from AMPT agrees with LHC data

Flow fluctuations with moments

Flow vector $Q_n \equiv \frac{1}{N}\sum_j e^{in\varphi_j}$ for two subevents (A,B) about midrapidity

Stat. properties of flow V_n contained in moments

$$\mathcal{M} \equiv \left\langle \prod_n (V_n)^{k_n} (V_n^*)^{l_n} \right\rangle = \left\langle \prod_n (Q_{nA})^{k_n} (Q_{nB}^*)^{l_n} \right\rangle$$

Testing the hypothesis: $V_4 = V_{4l} + \beta V_2^2$

Corr: $(\mathbf{v}_2)^2\,\mathbf{v}_4$

with $(\mathbf{v}_2)^2$ $$\frac{\langle V_4 (V_2^*)^2 v_2^2\rangle}{\langle V_4 (V_2^*)^2\rangle \langle v_2^2\rangle} = \frac{\langle v_2^6\rangle}{\langle v_2^4\rangle \langle v_2^2\rangle}$$

with $(\mathbf{v}_2)^4$ $$\frac{\langle V_4 (V_2^*)^2 v_2^4\rangle}{\langle V_4 (V_2^*)^2\rangle \langle v_2^4\rangle} = \frac{\langle v_2^8\rangle}{\langle v_2^4\rangle^2}$$

fluctuations

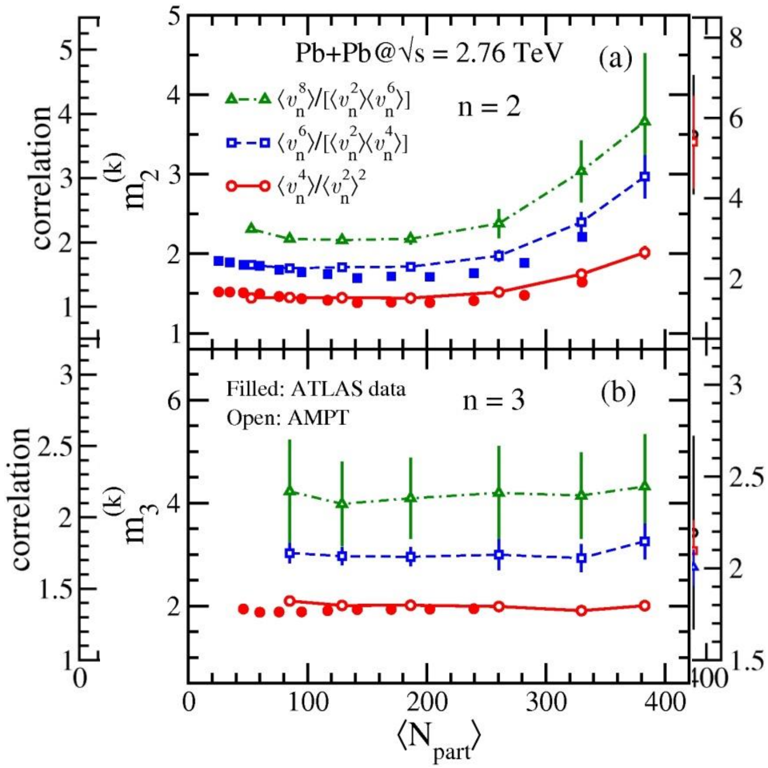
Testing the hypothesis: $V_5 = V_{5l} + \beta' V_2 V_3$

Corr: $\mathbf{v}_2\mathbf{v}_3\mathbf{v}_5$

with $(\mathbf{v}_2)^2$ $$\frac{\langle V_5 V_2^* V_3^* v_2^2\rangle}{\langle V_5 V_2^* V_3^*\rangle \langle v_2^2\rangle} = \frac{\langle v_2^4 v_3^2\rangle}{\langle v_2^2 v_3^2\rangle \langle v_2^2\rangle}$$

with $(\mathbf{v}_3)^2$ $$\frac{\langle V_5 V_2^* V_3^* v_3^2\rangle}{\langle V_5 V_2^* V_3^*\rangle \langle v_3^2\rangle} = \frac{\langle v_2^2 v_3^4\rangle}{\langle v_2^2 v_3^2\rangle \langle v_3^2\rangle}$$

Bhalerao, Ollitrault, SP, PLB 742 (2015) 94



AMPT supports conjectured nonlinear correlation at all centralities

→ Can be tested experimentally

Event plane correlations p(Ψ_n,Ψ_m,…)

EP correlators involve ≥ 3-particles → higher order correlations

v_n measured with single EP: $v_n\{\mathrm{EP}\} \equiv \dfrac{\langle \cos n(\phi_i - \Psi_n^A)\rangle_{\mathrm{particles,events}}}{\sqrt{\langle \cos n(\Psi_n^A - \Psi_n^B)\rangle} \equiv \mathrm{Res}\ \Psi_n}$

$\xrightarrow{\text{low res.}} v_n\{2\}$

$\xrightarrow{\text{high res.}} \langle v_n\rangle$

Correlations from multi-particles:

$$\cos(k_1\Psi_1 + 2k_2\Psi_2 \ldots + nk_n\Psi_n)\{\mathrm{EP}\} \equiv \frac{\langle v_1^{k_1}\ldots v_n^{k_n}\cos(k_1\Psi_1 + \ldots + nk_n\Psi_n)\rangle}{\sqrt{\langle v_1^{2k_1}\rangle\langle v_2^{2k_2}\rangle\ldots\langle v_n^{2k_n}\rangle}}$$

$k_1+2k_2+\ldots+nk_n=0$

Experimental analysis: (i) each EP in different η-window; (ii) windows pairwise separated by gaps $\Rightarrow$ decrease statistics

2-subevent method for EP correlators

Bhalerao, Ollitrault, SP, PRC88 (2013) 024909

Consider 2 subevents (A,B) separated by a η-gap.

Construct flow vector for each subevent: $Q_n = |Q_n| e^{in\Psi_n} \equiv \dfrac{1}{N}\sum_j e^{in\phi_j}$

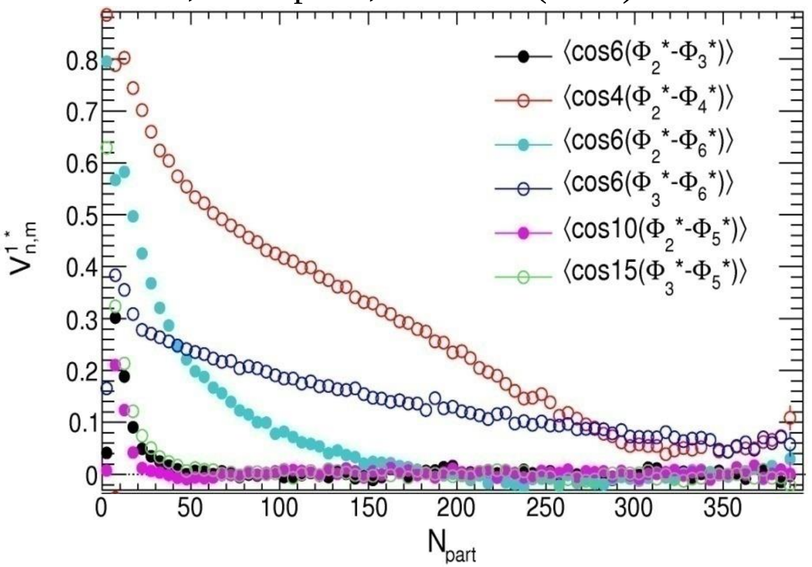
EP method: $\langle\cos 4(\Phi_2 - \Phi_4)\rangle \equiv \dfrac{\left\langle \frac{Q_{2A}^2}{|Q_{2A}^2|}\frac{Q_{4B}^*}{|Q_{4B}|}\right\rangle}{\sqrt{\left\langle \frac{Q_{4A}}{|Q_{4A}|}\frac{Q_{4B}^*}{|Q_{4B}|}\right\rangle}\sqrt{\left\langle \frac{Q_{2A}^2}{|Q_{2A}^2|}\frac{Q_{2B}^{*2}}{|Q_{2B}^2|}\right\rangle}}$ Resolution dependent

Scalar product method: $c\{2,2,-4\} \equiv \dfrac{\langle Q_{2A}^2 Q_{4B}^*\rangle}{\sqrt{\langle Q_{4A}Q_{4B}^*\rangle}\sqrt{\langle Q_{2A}^2 Q_{2B}^{*2}\rangle}}$ Well-defined flow observable

3-plane correlators

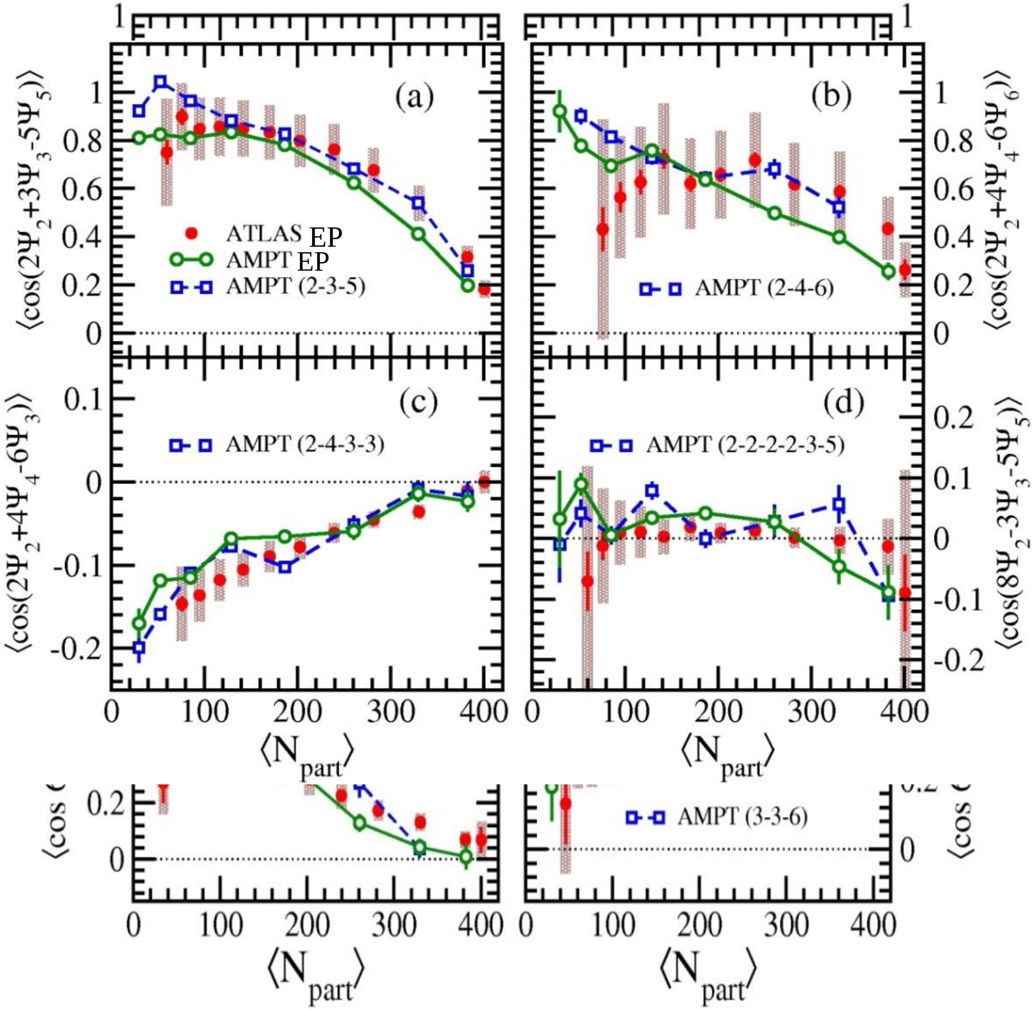
Initial state correlators – MC Glauber

Jia, Mohapatra, EPJC 73 (2013) 2510



Final state correlators – AMPT

Bhalerao, Ollitrault, SP, PRC 88 (2013) 024909



- Strong corr. between Φ_2 and Φ_4 from fluctuation & almond shape ε_2
- Weak corr. between Φ_2 and Φ_3
- EP corr. in AMPT agree with data
- Final-state corr. retain the initial info
- Strong corr. between:
  Ψ_2 & Ψ_4 as $v_4 \propto (v_2)^2$
  Ψ_2 & Ψ_6 as $v_6 \propto (v_2)^3$

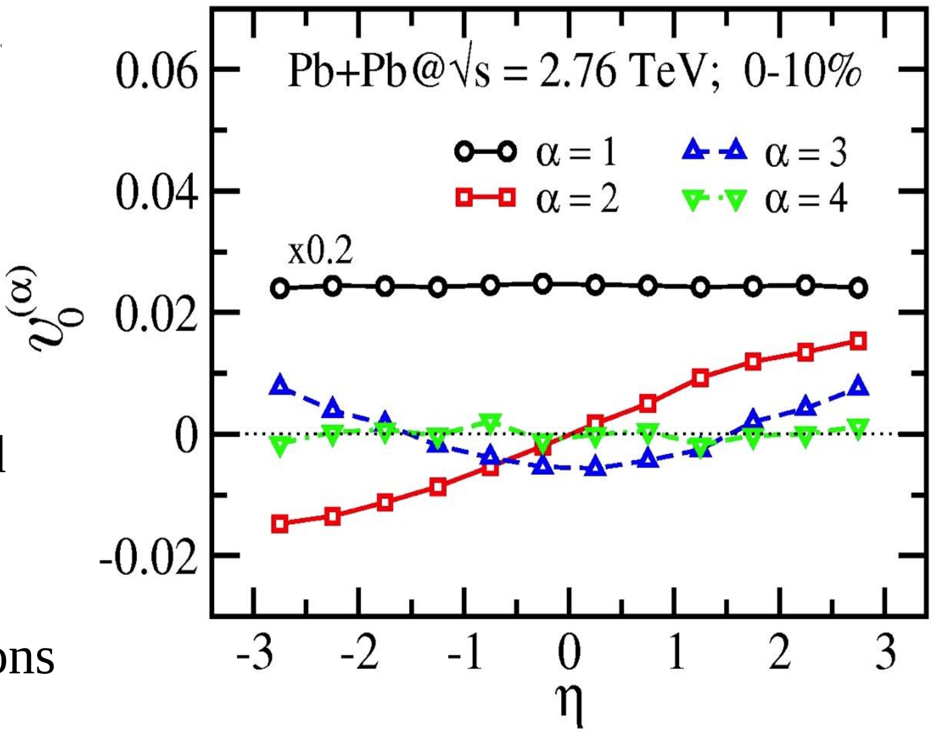
PC vs pseudorapidity (η) in AMPT

Within AMPT:

Construct a pair distrb in $-3 \leq \eta \leq 3$ with $\Delta\eta = 0.5$

Diagonalize the 12×12 matrix: $V_{n\Delta}(\eta_1, \eta_2)$

Bhalerao, Ollitrault, SP, Teaney: PRL 114 (2015)



n = 0: PC of multiplicity fluctuations

* Leading mode $v_0^{(1)}(\eta)$: global relative multiplicity fluctuation.

* Next-to-leading mode $v_0^{(2)}(\eta)$: is odd and arises due to small difference in N_{part} in projectile and target

➔ direct new info on longitudinal fluctuations

- Subleading modes << Leading modes: Eigenvalues $\lambda^{(3)} << \lambda^{(2)} << \lambda^{(1)}$
- PC have alternating parities as they are mutually orthogona $\sum_\eta V_n^{(\alpha)}(\eta) V_n^{(\beta)*}(\eta) = 0 \quad \text{if } \alpha \neq \beta.$
- Higher modes fall within statistical fluctuations

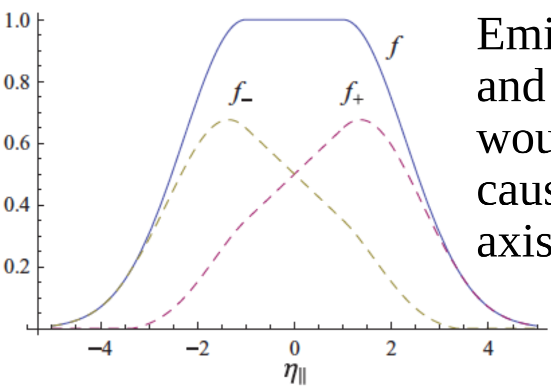
PC vs η in AMPT

PC: Elliptic (n = 2), Triangular (n=3) flow fluctuations

* Leading modes $v_2^{(1)}$, $v_3^{(1)}$: usual elliptic and triangular flow that weakly depend on η.

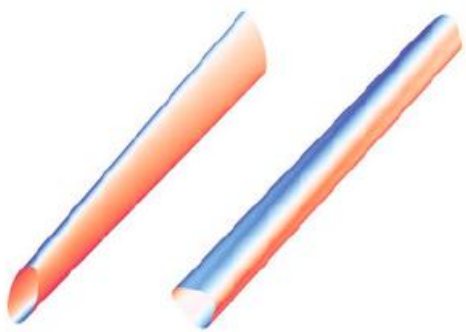
❖ Subleading modes << Leading modes.

❖ Next-to-leading mode $v_{2,3}^{(2)}(\eta)$: **torqued flow**



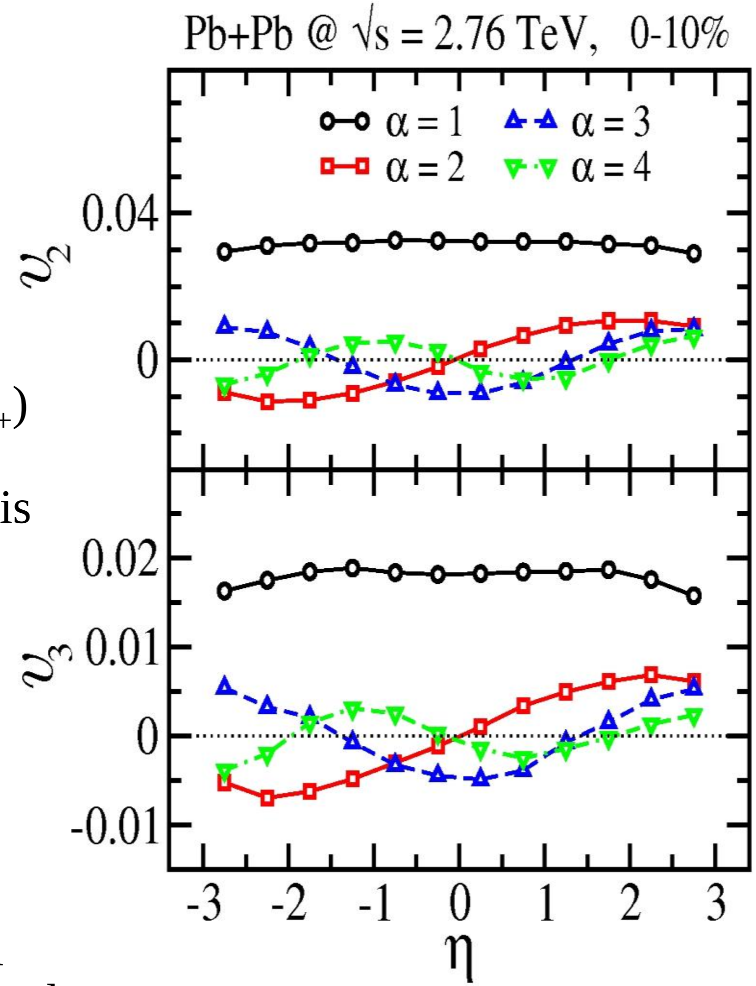
Emission profiles of forward (f_+) and backward (f_-) moving wounded nucleons in EbyE basis causes a torque of the principal axis (or participant plane).

Bozek, Broniowski, Moreira, PRC 83 (2011) 034911



Elliptic Triangular

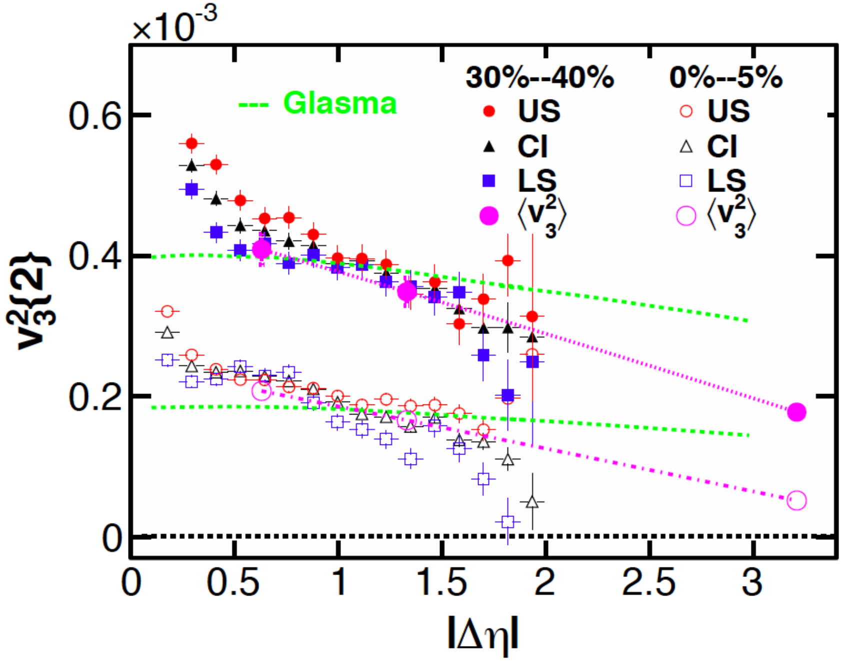
Direction of principal axis in transverse plane rotates as rapidity increases. The initial torque is converted into torqued flow $v_2^{(2)}(\eta)$ and $v_3^{(2)}(\eta)$.

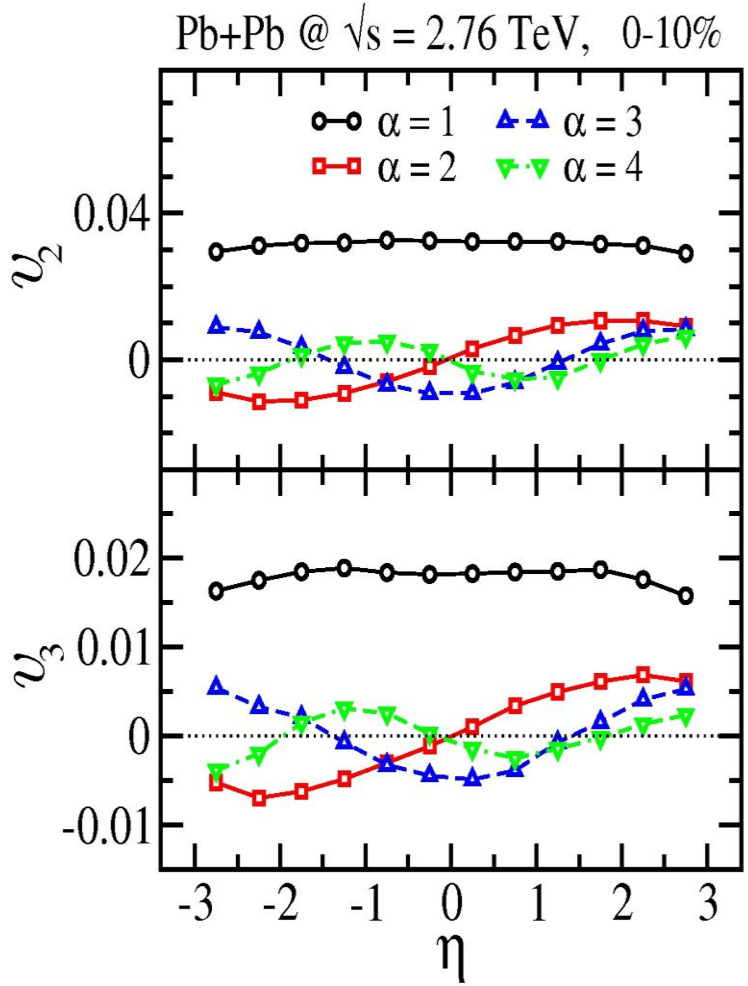


$v_{2,3}^{(2)}(\eta)$: Longitudinal flow fluctuations

- In a given event, the plane of v_n depends slightly on rapidity as the participant planes are not identical for target and projectile participants.

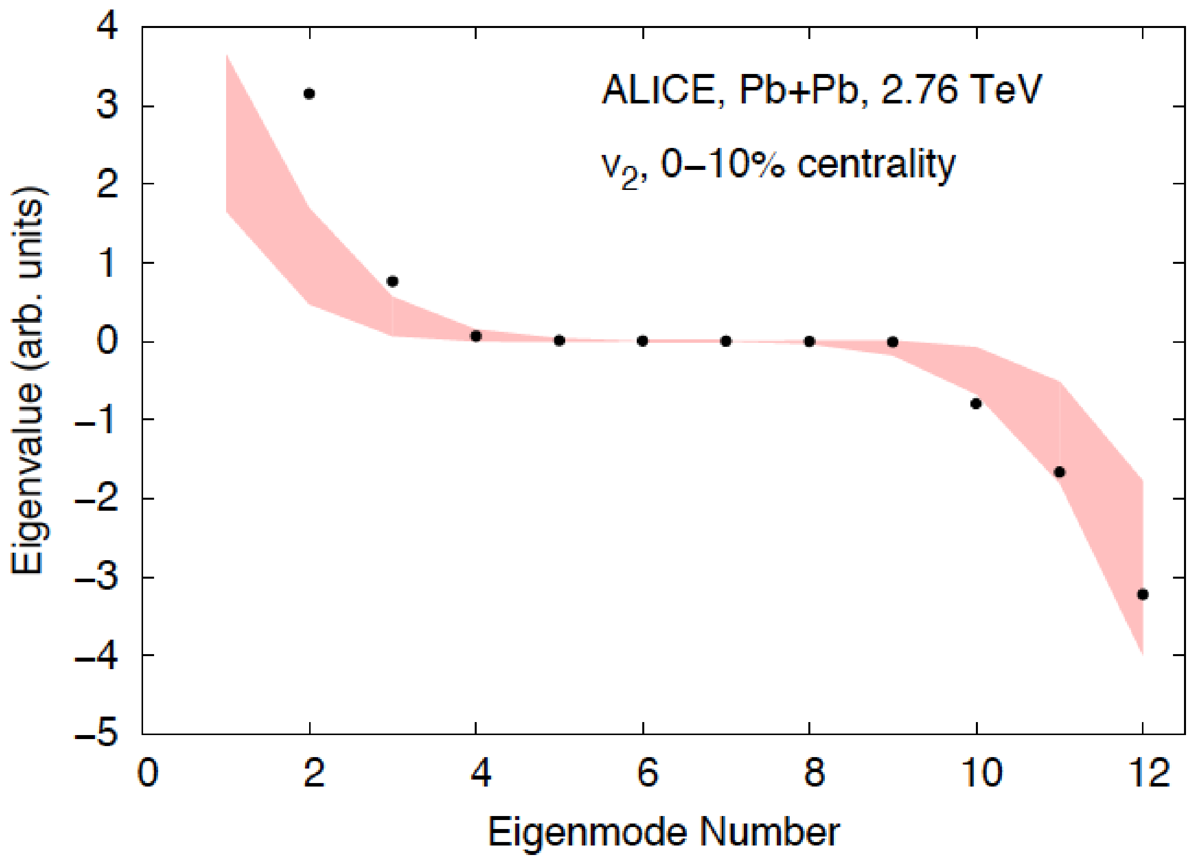
- Probably explains why STAR v_3 depends on $\Delta\eta$





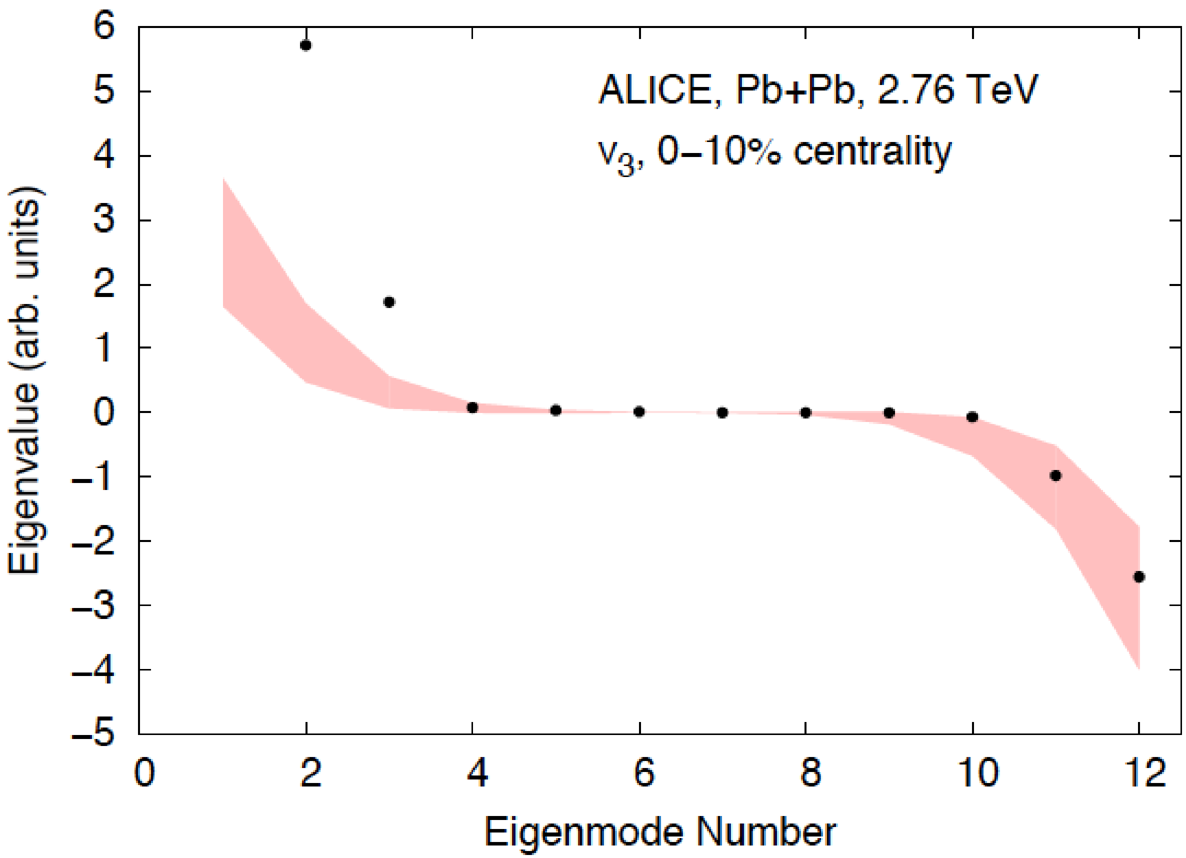
This rapidity dependence has never been measured directly and can falsify models of initial state.

PCA: Eigenvalues of $v_2(p_T)$ in ALICE



- Band: PCA applied to purely statistical fluctuations
- Negative eigenvalues of $V_{n\Delta}(p_1, p_2)$ are compatible with those of large random matrices. Can be attributed to stat. fluct.
- Note the few leading eigenmodes which clearly stand out

PCA: Eigenvalues of $v_3(p_T)$ in ALICE

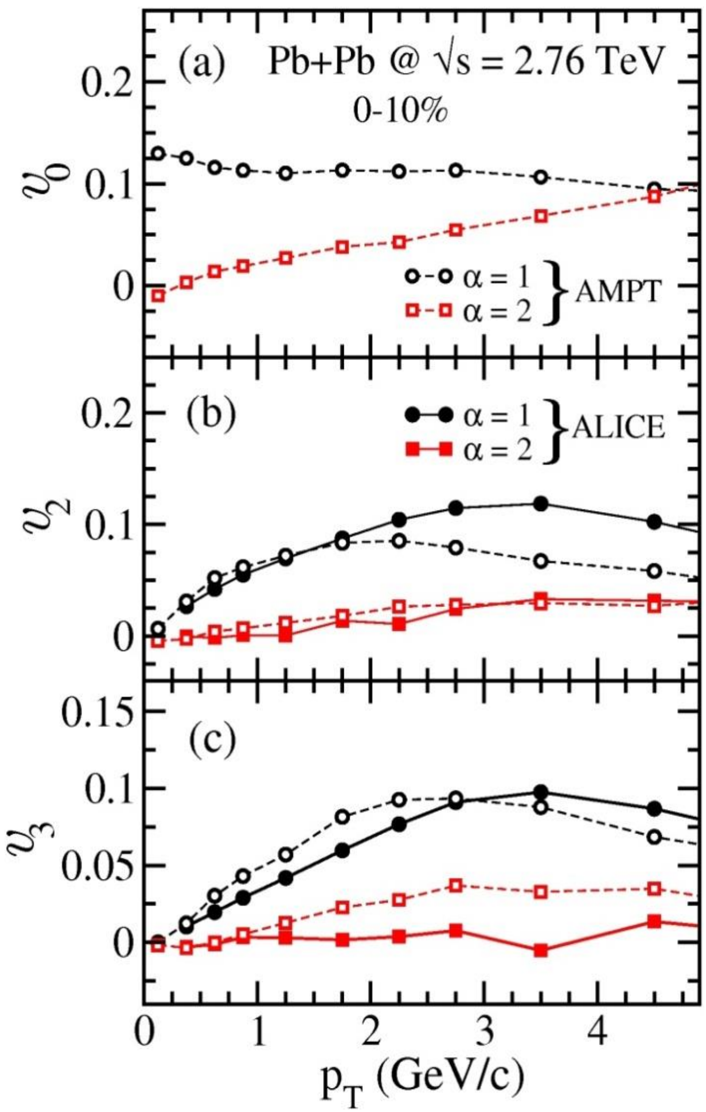


- Band: PCA applied to purely statistical fluctuations
- Negative eigenvalues of $V_{n\Delta}(p_1, p_2)$ are compatible with those of large random matrices. Can be attributed to stat. fluct.
- Note the few leading eigenmodes which clearly stand out

PC vs p_T in AMPT/ALICE

Within AMPT:
Construct a pair distrb in p_T bins
Diagonalize the matrix: $V_{n\Delta}(p_{T1}, p_{T2})$.

ALICE data: used $V_{n\Delta}(p_{T1}, p_{T2})$ in PCA
PLB:708 (2012) 249.



n = 0: PC of multiplicity fluctuations

$v_0^{(1)}(p_T)$ gives fluctuation in total multip.

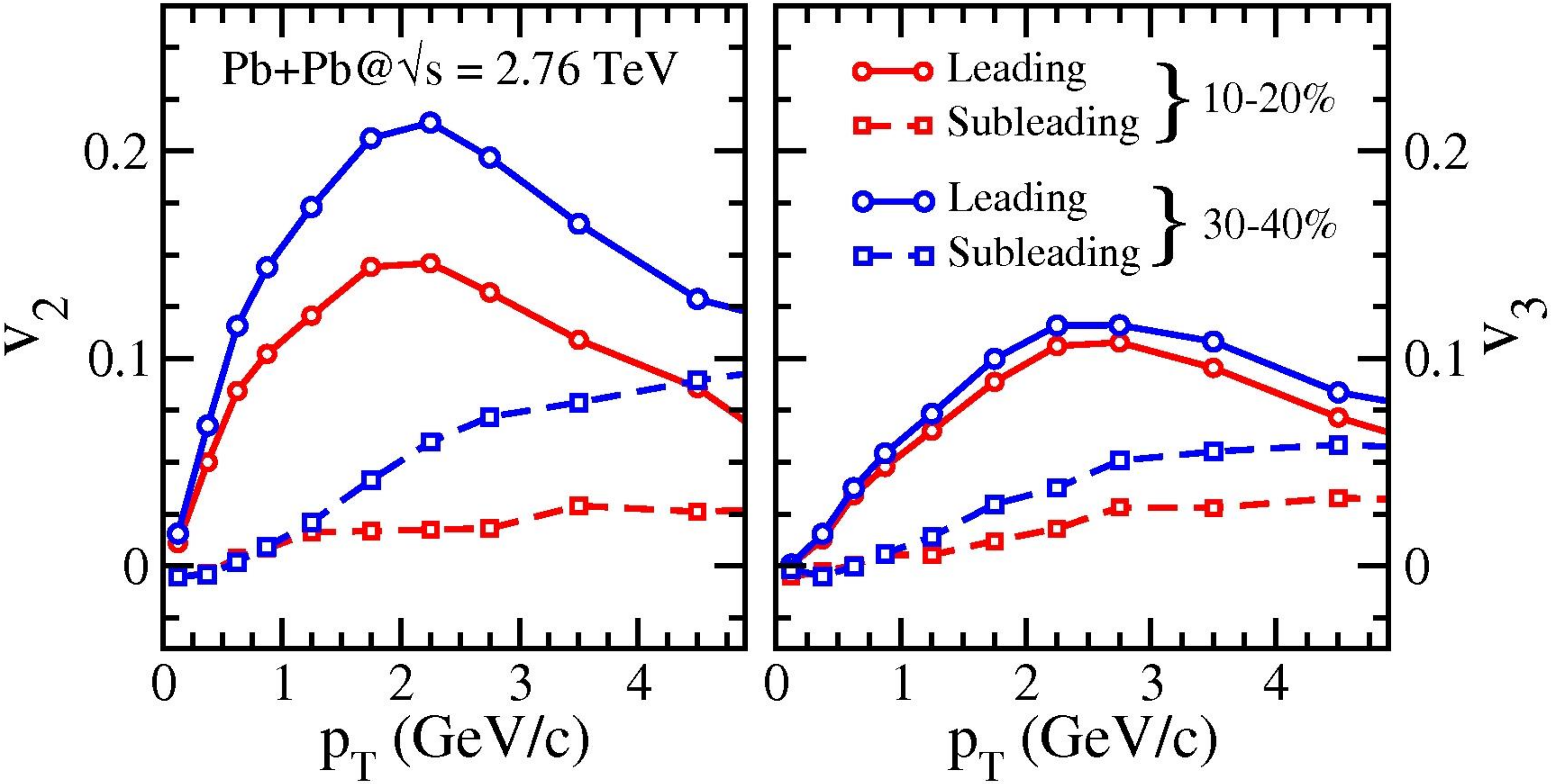
$v_0^{(2)}(p_T)$ increases linearly with p_T
$\rightarrow$ radial flow fuctuation

n = 2, 3 modes

Leading $v_2^{(1)}$, $v_3^{(1)}$ identical to usual $v_2\{2\}$, $v_3\{2\}$.

Subleading $v_2^{(2)}$, $v_3^{(2)}$ have smaller magnitudes and significant at large p_T as phase $\Psi_n(p_T)$

AMPT: PCA versus p_T with centrality



Leading and subleading modes increases with centrality of collision

Ideal and Dissipative Hydrodynamics

Fluid dynamics: An effective theory describing the long-wavelength, low-frequency limit of the microscopic dynamics of a system.

Ideal	Dissipative
$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$ $N^\mu = nu^\mu$ Unknowns: $\underbrace{\epsilon,\ P,\ n,\ u^\mu}_{1+1+1+3} = 6$	$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$ $N^\mu = nu^\mu + n^\mu$ $\underbrace{\epsilon,\ P,\ n,\ u^\mu,\ \Pi,\ \pi^{\mu\nu},\ n^\mu}_{1+1+1+3+1+5+3} = 15$
Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0,\quad \partial_\mu N^\mu = 0,\quad EOS}_{4\ +\ 1\ +\ 1} = 6$	
Closed set of equations	9 more equations required

Israel, Stewart, Ann. Phys. 118 (1979) 341.
Muronga, PRC69 (2004) 034903.
Romatschke, Int. J. Mod. Phys. E19 (2010) 1.

Huovinen, Petreczky, NPA 837 (2010) 26
s95p-PCE EoS matching lattice data to hadron reson gas at $T_{PCE} \approx 165$ MeV

- In terms of the distribution function: $f = f_0 + \delta f$,

$$\Pi = -\frac{\Delta_{\alpha\beta}}{3}\int dp\, p^\alpha p^\beta\, \delta f\,, \quad n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f\,, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f\,.$$

Hadron spectra at freeze-out temp. $T_{dec} \approx 120$ MeV obtained by Cooper-Frye formula

$$E\frac{dN}{dp^3} = \frac{g}{(2\pi)^3}\int_\sigma d\sigma_\mu p^\mu f(x,p)$$

Non-equilibrium distribution function

The equilibrium distribution function is: $f_0 = [\exp(\beta u\cdot p + \alpha) + r]^{-1}$

$$= [\exp\{y_0(x,p)\} + r]^{-1},\ r = 0, \pm 1$$

- Away from equilibrium, $f = [\exp\{y(x,p)\} + r]^{-1}$, where

$$\phi(x,p) \equiv y(x,p) - y_0(x,p) = \varepsilon(x) - \varepsilon_\mu(x)p^\mu + \varepsilon_{\mu\nu}(x)p^\mu p^\nu + \cdots$$

Grad, Comm. Pure App. Math 2 (1949) 2

For a system close to local equilibrium, Grad's approximation: ε, ε_μ and $\varepsilon_{\mu\nu}$ expressed in terms of Π, n_μ and $\pi_{\mu\nu}$

$$f = f_0 + f_0\tilde{f}_0\phi, \quad \phi = c_0\Pi + c_1 n_\mu p^\mu + c_2\pi_{\mu\nu}p^\mu p^\nu, \quad \phi \ll 1$$

$$\delta f_G^{(1)} = \frac{f_0\tilde{f}_0}{2(\epsilon + P)T^2}\, p^\alpha p^\beta \pi_{\alpha\beta}$$

Chapman-Enskog like derivation

- Boltzmann equation with relaxation-time approx in collision term

$$p^\mu\partial_\mu f = -\frac{u\cdot p}{\tau_R}(f - f_0) \ \Rightarrow\ f = f_0 - (\tau_R/u\cdot p)\, p^\mu\partial_\mu f$$

- Chapman-Enskog like expansion of $f(x,p)$ about its equilibrium value in powers of space-time gradients: $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$,

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\, p^\mu\partial_\mu f_0\ , \quad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu\Big(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\Big)\ .$$

$$\delta f_{CE}^{(1)} = \frac{5 f_0\tilde{f}_0}{8PT(u\cdot p)}\, p^\alpha p^\beta \pi_{\alpha\beta}$$

Bhalerao, Jaiswal, SP, Sreekanth, PRC89 (2014) 055903

Geometry origin of $V_3^{(\alpha)}$: 2+1 viscous hydro

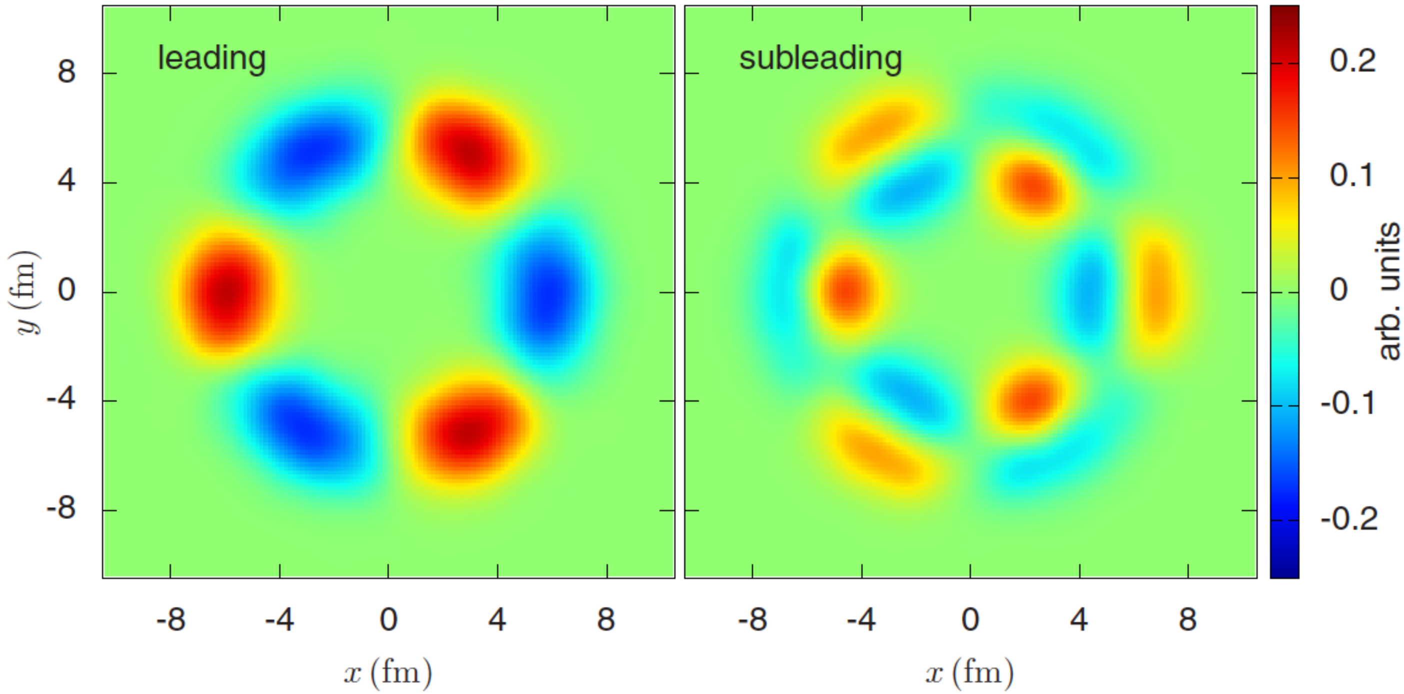
In a given event, flow can be decomposed as $V_n(p) = \sum_{\alpha=1}^{k} \zeta^{(\alpha)} V_n^{(\alpha)}(p)$

Real vectors $V_n^{(\alpha)}(p)$ record rms values of (sub)leading flows.

Complex coefficients $\zeta^{(\alpha)}$ indicate orientation wrt flow.

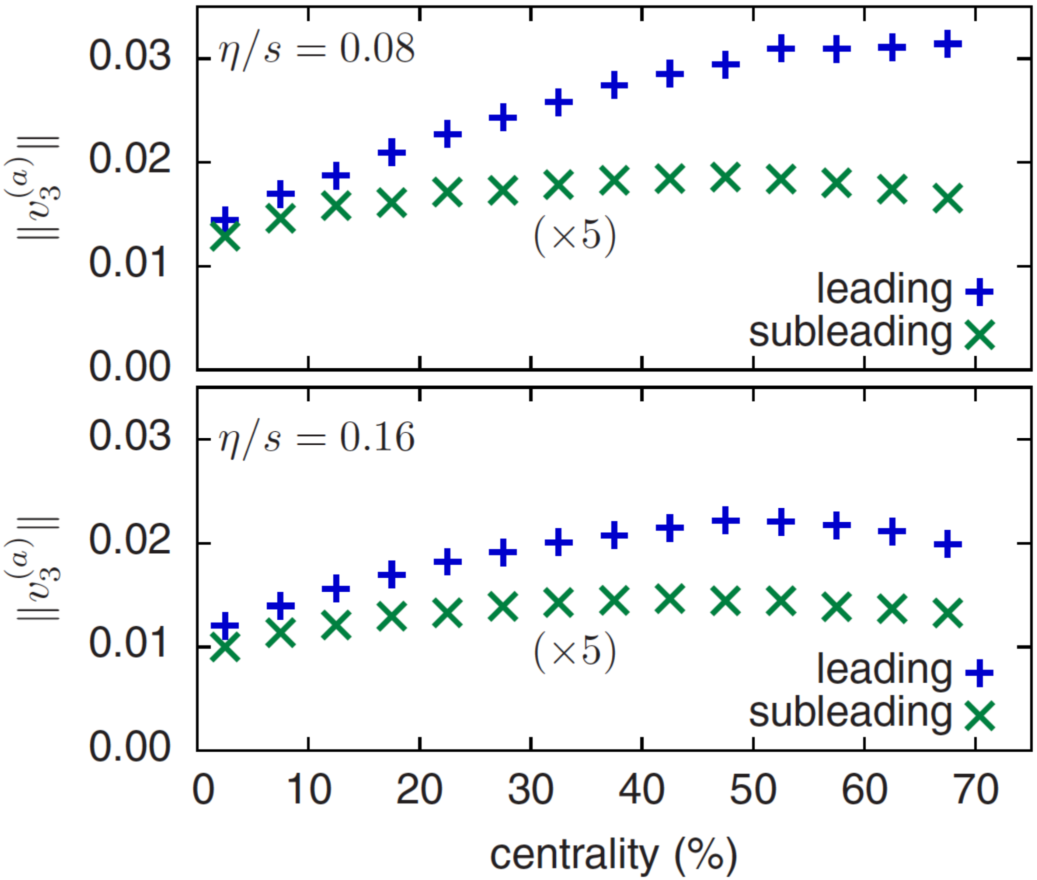
Average initial geometry for leading & subleading PC where each event is oriented along $\zeta^{(\alpha)}$ plane.

Mazeliauskas, Teaney, PRC91 (2015) 044902



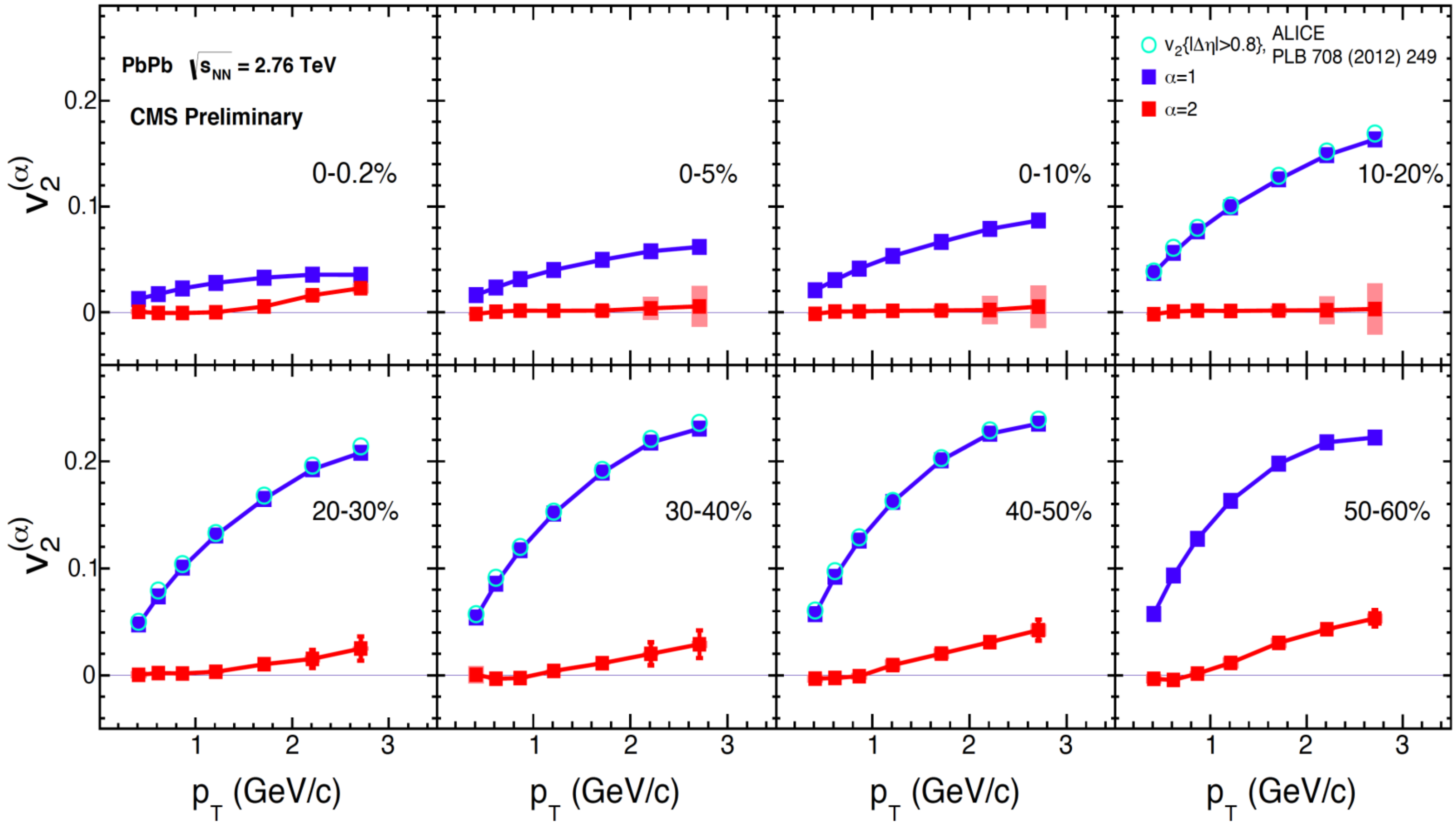
- Leading mode $V_3^{(1)}$ strongly correlated with triangular comp. of initial geom.
- Subleading mode $V_3^{(2)}$ correlated with radial excitations of this geometry.

Viscosity dependence of PC for v_3



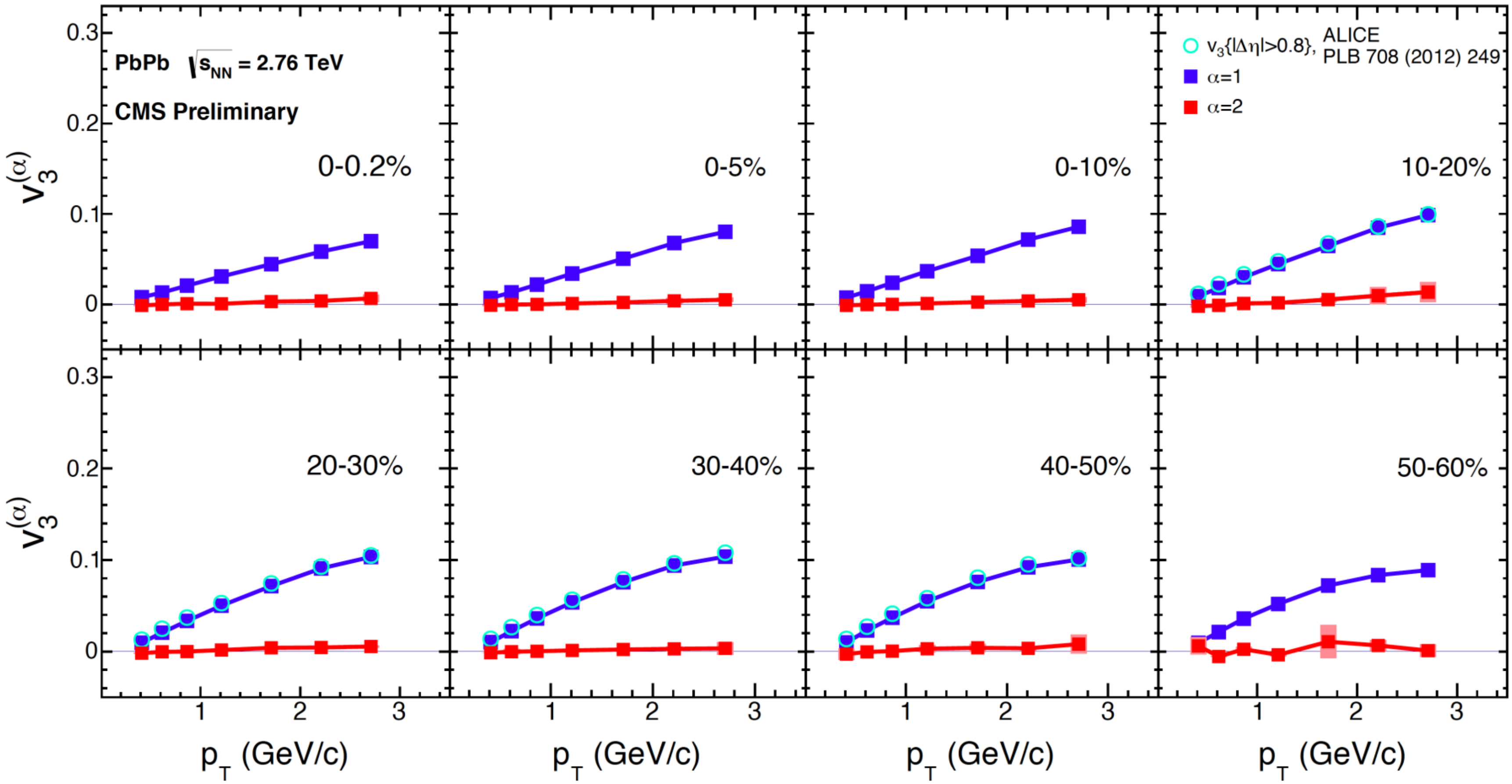
- Leading mode $v_3^{(1)}$ strongly suppressed with shear viscosity.
- Subleading flow $v_3^{(2)}$ depends weakly on viscosity and centrality.

PCA experimental v_2 in Pb+Pb



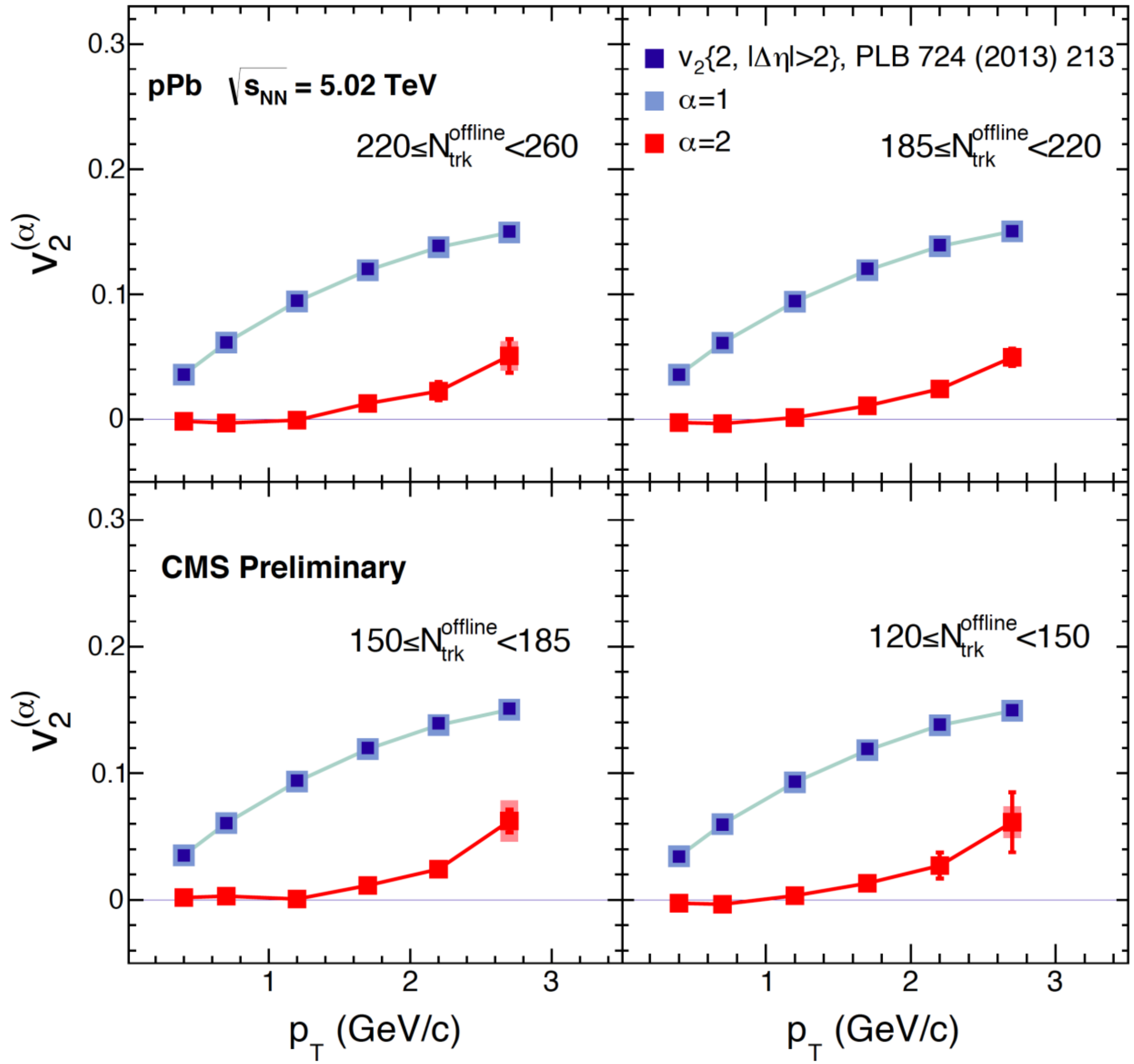
- Leading mode v_2 equal to $v_2\{2\}$
- Sub-leading mode (α=2) v_2 is positive for ultra-central collision, and at above 20% centrality.

PCA experimental v_3 in Pb+Pb



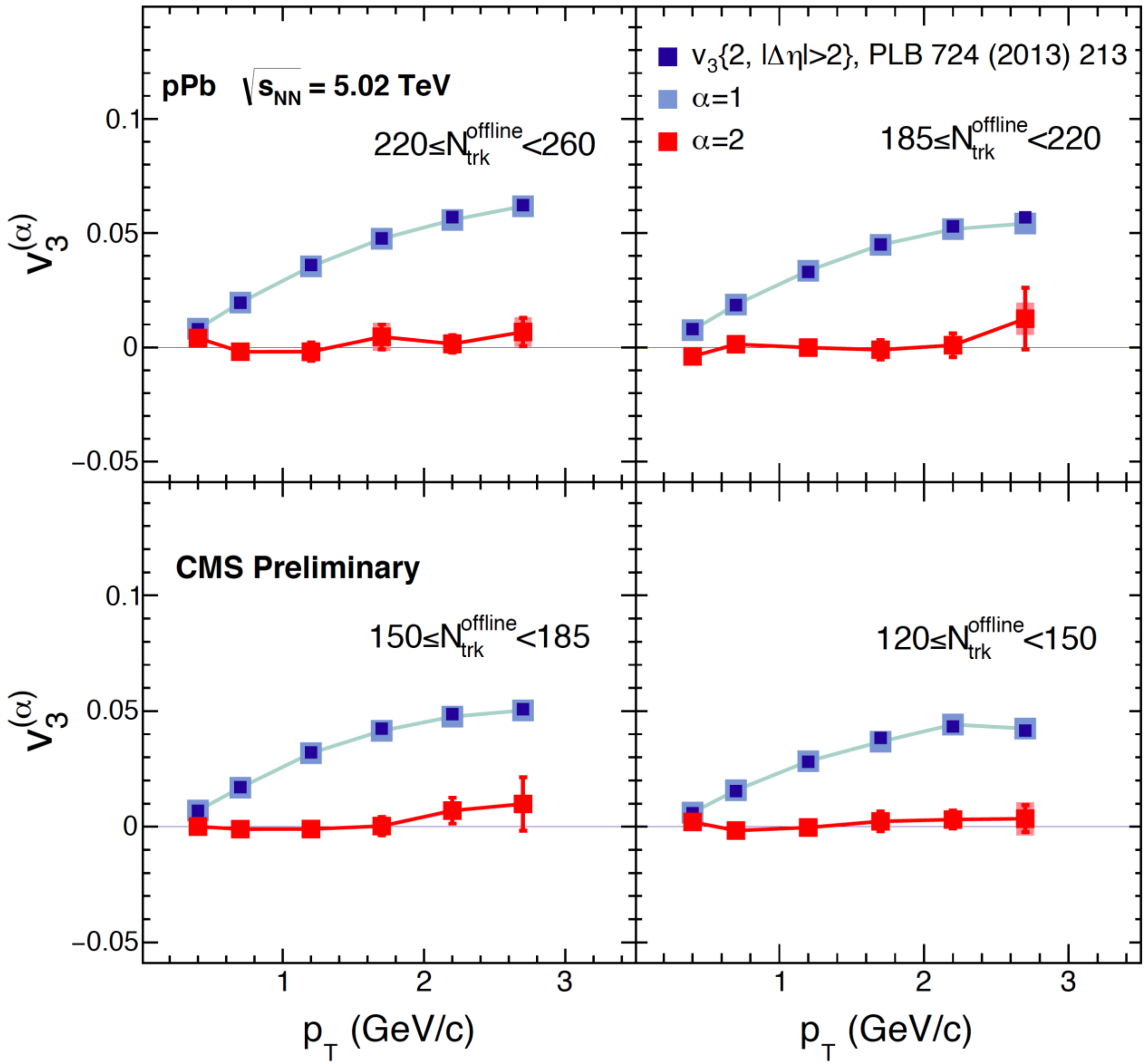
- ❑ Leading mode v_3 equal to $v_3\{2\}$
- ❑ Sub-leading mode (α=2) v_3 is zero within uncertainties.

PCA experimental v_2 in p+Pb



- Leading mode v_2 equal to $v_2\{2\}$.
- Sub-leading mode (α=2) v_2 is about zero at small p_T and increases by ~5% for p_T > 1 GeV.

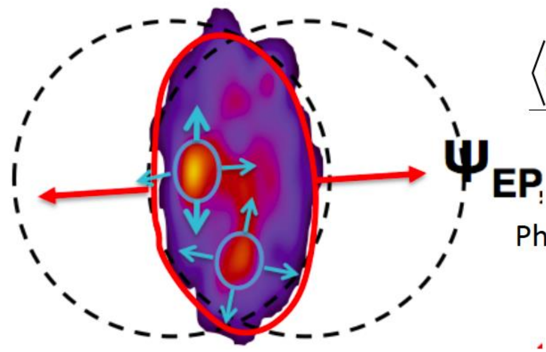
PCA experimental v_3 in p+Pb



- Leading mode v_3 equal to $v_3\{2\}$.
- Sub-leading mode (α=2) v_3 is nearly zero at all p_T.

Factorization breaking thru PCA

- Initial state fluctuations causes the EP $\Psi_n(p_T,\eta)$ to depend on p_T and η
- Breaks the factorization of azimuthal correlations



Factorization breaking characterized via Pearson coefficient

$$r = \frac{V_{n\Delta}(p_1,p_2)}{\sqrt{V_{n\Delta}(p_1,p_1)V_{n\Delta}(p_2,p_2)}} = \left\{ \begin{array}{c} 1 \\ <1 \\ >1 \end{array} \right\} \begin{array}{l} \text{holds} \\ \text{brakes} \\ \text{non-flow} \end{array}$$

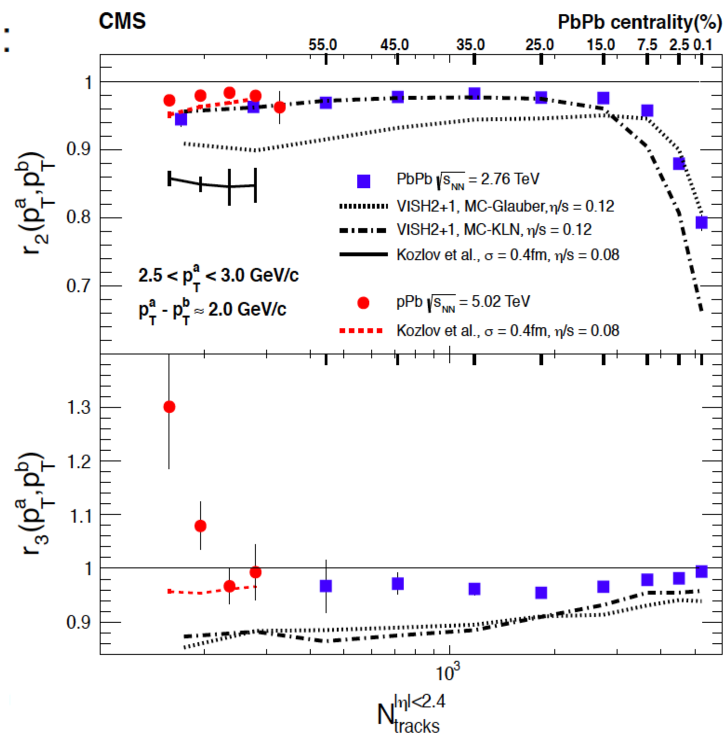
If no flow fluctuations, then factorization occurs:
$V_{n\Delta}(p_1,p_2) \simeq V_n^{(1)}(p_1)V_n^{(1)*}(p_2)$

Flow flucts. are dominated by α = 2 mode with $|V_n^{(2)}(p)| \ll |V_n^{(1)}(p)|$. In this limit:

$$1 - r \simeq \frac{1}{2}\left|\frac{V_n^{(2)}(p_1)}{V_n^{(1)}(p_1)} - \frac{V_n^{(2)}(p_2)}{V_n^{(1)}(p_2)}\right|^2$$

→ r ≤ 1 (Cauchy-Schwarz inequality)

Factorization breaking is induced by relative diff between subleading and leading modes



10.1103/PhysRevC.92.034911
(arXiv:1503.01692)

Summary & Conclusions

- Two-particle azimuthal correlations depend on momenta of both particles. Traditional methods: one of the momenta integrated over. New method: Makes use of both the momenta.
- PCA has revealed subleading modes in multiplicity, elliptic flow, and triangular flow fluctuations.

PCA can provide further constraints on the properties of QGP !