Resummations and Non-Perturbative Corrections

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Y.Hatsuda and KO [arXiv:1505.07460]

Non-perturbative effects in M-theory

- We have a detailed understanding of non-perturbative effects in M-theory on AdS_4 via holography

$$U(N)_k \times U(N)_{-k} \text{ ABJM theory on } S^3$$
$$\Updownarrow$$
$$\text{M-theory on } AdS_4 \times S^7/\mathbb{Z}_k$$

- In particular, membrane instantons in ABJM theory are determined by the refined topological string in the Nekrasov-Shatashvili limit [Hatsuda-Marino-Moriyama-KO]

Non-perturbative corrections in string coupling

- Membrane instanton corrections are non-perturbative in the string coupling g_s

$$F_{\text{non-pert}} \sim e^{-\frac{1}{g_s}}$$

- Such non-perturbative corrections are expected from the perturbative genus expansion [Shenker]

$$F_{\text{pert}} = \sum_{g=0}^{\infty} F_g \, g_s^{2g-2}$$

- Perturbative genus expansion is an asymptotic series

$$F_g \sim (2g)!$$

Membrane instantons from resummation?

- A useful way to analyze perturbative series is Borel resummation

$$F_{\mathrm{Borel}} = \int_0^\infty d\zeta \, e^{-\zeta} \sum_g \frac{F_g \zeta^2}{(2g)!} (g_s \zeta)^{2g-2}$$

- Free energy of ABJM theory on S^3 is Borel summable!

- However, Borel sum of ABJM free energy does not agree with the exact free energy [Grassi-Marino-Zakany]

$$F_{\mathrm{Borel}} \neq F_{\mathrm{exact}}$$

Question

- Is there any example such that
  - perturbative expansion is Borel summable
  - reproduces exact result
- Answer is yes

pure Chern-Simons theory on S^3

$$\Updownarrow$$

topological string on resolved conifold

- Interestingly, resummation of pure CS free energy contains membrane instanton corrections

Pure CS/resolved conifold

- Partition function of $U(N)_k$ pure CS theory [Witten]

$$Z_{\text{CS}} = \left(\frac{g_s}{2\pi}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} \left(2\sin\frac{g_s j}{2}\right)^{N-j}, \quad g_s = \frac{2\pi}{k+N}$$

- Pure CS free energy is equivalent to the genus expansion of resolved conifold free energy $F(g_s, t)$ in the 't Hooft limit [Gopakumar-Vafa]

$$N \to \infty, \quad g_s \to 0, \quad t = iNg_s \;\; \text{fixed}$$

Genus expansion of pure CS/resolved conifold

- Genus g free energy of pure CS/resolved conifold is

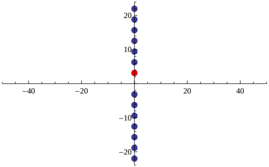
$$F(g_s, t) = \sum_{g=0}^{\infty} F_g(t)\, g_s^{2g-2}$$

$$F_{g\geq 2}(t) = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} \mathrm{Li}_{3-2g}(e^{-t}) + \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}$$

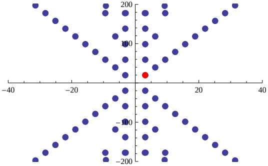
- This expansion is Borel summable as long as $\mathrm{Im}(t) \neq 0$

Singularity on the Borel plane

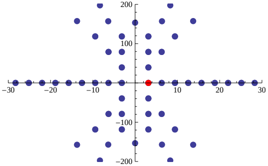
- Singularity at $\zeta = \frac{2\pi t}{g_s}$ (red dot) on the Borel ζ-plane rotates as we change the phase of t



$t =$ pure imaginary



$\mathrm{Im}(t) = \pi$



$t =$ real

- Free energy is Borel summable for $t \neq$ real

Generalization of Borel resummation

- It is more convenient to use a generalization of Borel resummation, known as the moment method

$$\text{Borel resummation:} \quad n! = \int_0^\infty d\zeta\, e^{-\zeta}\zeta^n$$

$$\text{moment method:} \quad \mu_n = \int_0^\infty d\zeta\, \mu(\zeta)\zeta^n$$

- Bernoulli number is ubiquitous in topological string

$$B_{2g} = 4g(-1)^{g-1}\int_0^\infty dx \frac{x^{2g-1}}{e^{2\pi x}-1}$$

- We will use a moment method for B_{2g}

Resummation of pure CS

- Free energy of physical pure CS theory is Borel summable

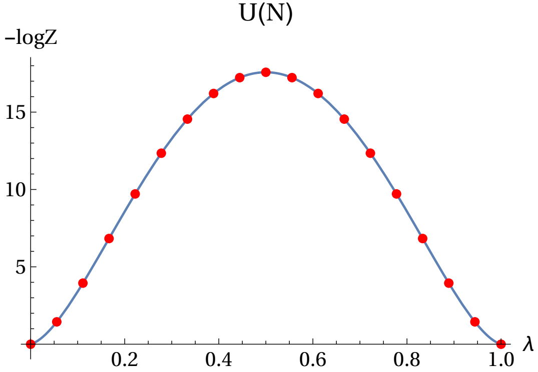
$$t = 2\pi i\lambda = \text{pure imaginary}$$
$$(\lambda = \frac{g_s N}{2\pi} = \frac{N}{k+N} = \text{real})$$

- Resummation using the *Bernoulli moment method* gives

$$F_{\text{CS}}^{\text{resum}} = \frac{\text{Re}[\text{Li}_3(e^{-2\pi i\lambda})] - \zeta(3)}{g_s^2} + \int_0^\infty dx \frac{x}{e^{2\pi x}-1} \log\left(\frac{\sinh^2 \frac{g_s x}{2}}{\sinh^2 \frac{g_s x}{2} + \sin^2 \pi\lambda}\right)$$

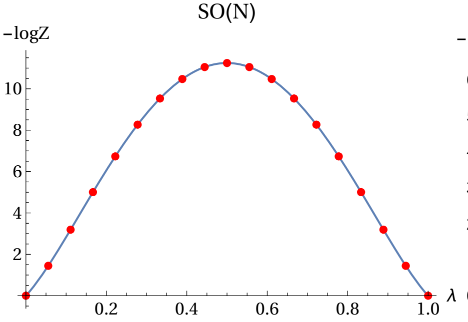
Plot of free energy as a function of $\lambda = \frac{g_s N}{2\pi}$

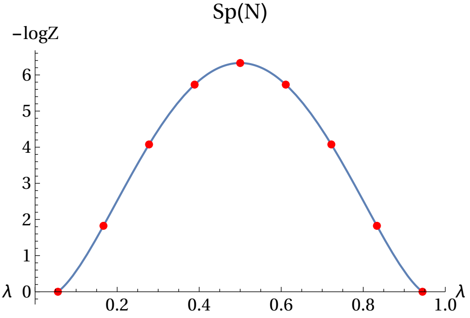
- Resummation of pure CS free energy reproduces the exact free energy at finite N (we set $g_s = \frac{\pi}{9}$ in this figure)



Free energy of $SO(N)$ and $Sp(N)$ pure CS

- Resummation of $SO(N)$ and $Sp(N)$ pure CS free energy also reproduces exact values at finite N





Large T expansion of free energy

- Free energy is Borel summable also in the large radius limit of resolved conifold, if we turn on a B-field

$$t = T + \pi i \quad (T \gg 1)$$

- Resummation of T-dependent part of free energy is

$$F_{\text{resum}} = \frac{\mathrm{Li}_3(-e^{-T})}{g_s^2} - \int_0^\infty dx \frac{x}{e^{2\pi x} - 1} \log(1 + e^{-2T} + 2e^{-T} \cosh g_s x)$$

- We consider M-theoretic expansion $=$ Large T expansion at finite g_s

Non-perturbative corrections in $F_{\rm resum}$

- We find $F_{\rm resum}$ contains membrane instanton corrections

$$F_{\rm resum} = F_{\rm WS}(\text{worldsheet inst.}) + F_{\rm M2}(\text{membrane inst.})$$

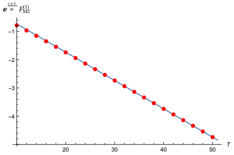
$$F_{\rm resum} = \frac{\mathrm{Li}_3(-e^{-T})}{g_s^2} - \int_0^\infty dx \frac{x}{e^{2\pi x}-1} \log(1 + e^{-2T} + 2e^{-T}\cosh g_s x)$$

$$F_{\rm WS} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n \sin^2 \frac{ng_s}{2}} e^{-nT}$$

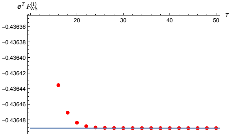
$$F_{\rm M2} = -\sum_{m=1}^{\infty} \frac{1}{4\pi m^2 \sin \frac{2\pi^2 m}{g_s}} \left[\frac{2\pi m}{g_s} T + \frac{2\pi^2 m}{g_s} \cot \frac{2\pi^2 m}{g_s} + 1\right] e^{-\frac{2\pi m T}{g_s}}$$

Numerical check at $g_s = 8$

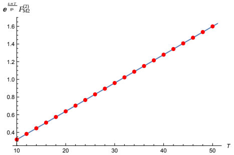
- For $g_s = 8$, we have $e^{-\frac{2\pi T}{g_s}} > e^{-T} > e^{-\frac{4\pi T}{g_s}} > e^{-2T} > \cdots$



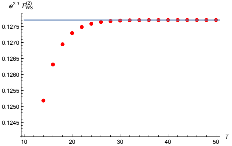
membrane 1-instanton $e^{-\frac{2\pi T}{g_s}}$



worldsheet 1-instanton e^{-T}



membrane 2-instanton $e^{-\frac{4\pi T}{g_s}}$



worldsheet 2-instanton e^{-2T}

Pole cancellation mechanism

- Free energy of resolved conifold can be partially resummed *a lá* Gopakumar-Vafa

$$F_{\mathrm{WS}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{4n\sin^2\frac{ng_s}{2}}e^{-nT}$$

- But, this is not the end of the story since F_{WS} has poles at some g_s

- Membrane instantons F_{M2} precisely cancel these poles and the total free energy is always finite [Hatsuda-Moriyama-KO, Hatsuda-Marino-Moriyama-KO]

$$F_{\mathrm{M2}}=-\frac{\partial}{\partial g_s}\sum_{m=1}^{\infty}\frac{g_s}{4\pi m^2\sin\frac{2\pi^2 m}{g_s}}e^{-\frac{2\pi m T}{g_s}}$$

Resummation of ABJM revisited

- Genus expansion of ABJM theory can be systematically computed by solving the holomorphic anomaly equation
- Borel sum of ABJM free energy does not agree with the exact one

- It is convenient to introduce approximated grand potential J_{Borel} (grand potential is a natural object in Fermi gas formalism)

$$e^{F_{\text{Borel}}} = \int d\mu e^{J_{\text{Borel}}(\mu) - N\mu}$$

- J_{Borel} contains membrane-like corrections, but they are different from the genuine membrane instantons in ABJM theory

Membrane-like corrections

- We find membrane-like correction in J_{Borel}

$$J_{\text{Borel}}(\mu) = \frac{1}{\pi \sin \frac{\pi k}{2}} \left(2\mu + \frac{\pi k}{2} \cot \frac{\pi k}{2} + 1 \right) e^{-2\mu} + \cdots$$

- This is different from the genuine membrane instantons in ABJM

$$J_{\text{M2}}(\mu) = \Big[a_1(k)\mu^2 + b_1(k)\mu + c_1(k) \Big] e^{-2\mu} + \cdots$$

- Membrane-like correction in J_{Borel} is rather similar to the membrane instantons in resolved conifold, and cancels the poles of worldsheet instantons at $k \in 2\mathbb{Z}$

Conclusion

- String perturbation theory is sometimes Borel summable
- However, this does not necessarily mean that Borel resummation reproduces the exact answer

$$\begin{aligned} \text{pure CS:} &\quad \text{Borel sum} = \text{exact} \\ \text{ABJM:} &\quad \text{Borel sum} \neq \text{exact} \end{aligned}$$

- Resummation of pure CS free energy naturally contains membrane instanton corrections
- Such membrane instantons are necessary for the cancellation of apparent poles at some g_s