QCD and Monte Carlo methods
Lecture II: Proton structure and Parton Showers

The 2009 Hadron Collider Physics Summer School
CERN, June 2009

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Fermilab

Proton structure and Parton Showers

- Λ Parameter
- Non-perturbative Physics
- Parton Branching
  - ⋆ Infrared divergences
  - ⋆ Kinematics of parton branching
  - ⋆ Dirac Eqn. Weyl representation
  - ⋆ Branching probabilities
- DGLAP equation
  - ⋆ Quarks and gluons
  - ⋆ Solution by moments

Lambda parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty} \frac{dx}{bx^2(1 + b'x + \ldots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q) \to \infty$ as $Q \to \Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.
- In leading order (LO) keep only first β-function b:

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \qquad \text{(LO)}.$$

- In next-to-leading order (NLO) include also b':

$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b'\alpha_S(Q)}{1+b'\alpha_S(Q)}\right) = b \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

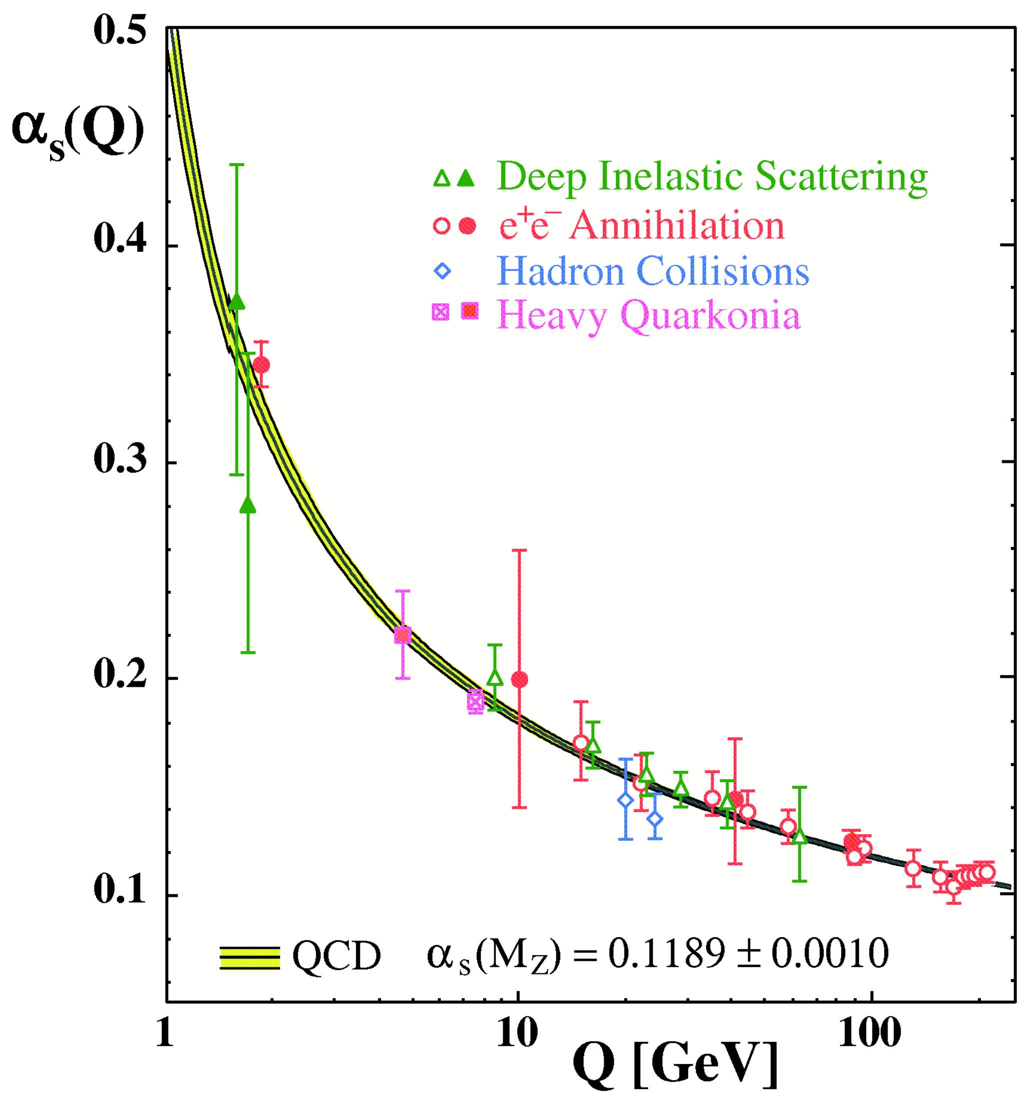
  This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \left[1 - \frac{b'}{b} \frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO)}.$$

  This is Particle Data Group (PDG) definition.

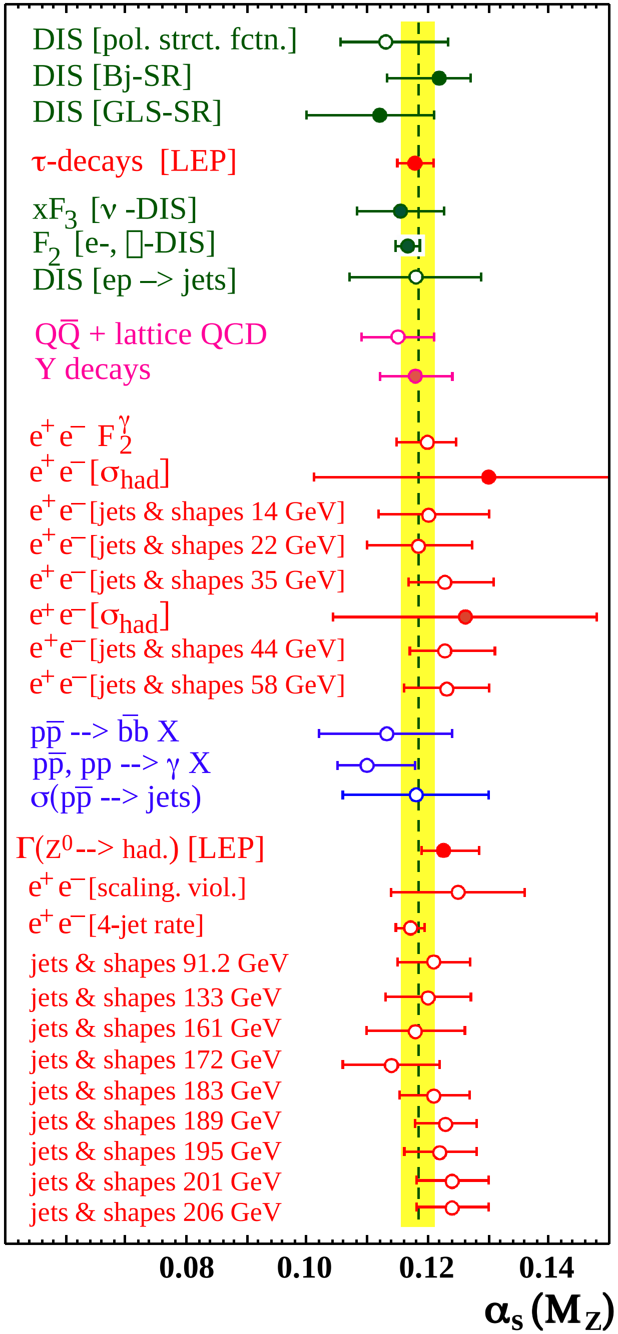
- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$. See ESW for relation between Λ's for different values of N_f.

- The use of Λ is full of pitfalls; however from a parametric point of view it is a fundamental quantity in QCD. The scale of masses of hadrons, protons, neutrons etc are determined primarily by Λ QCD. Thus QCD is responsible for the mass of the observed universe.



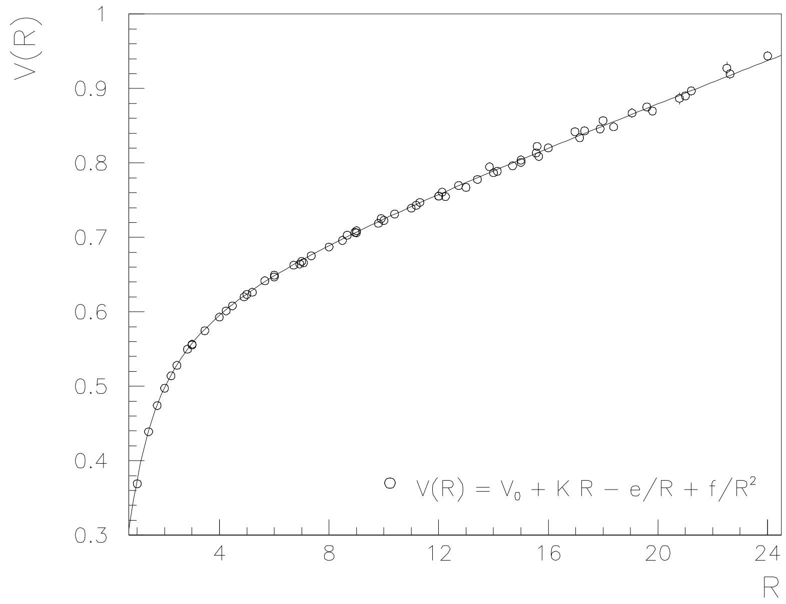
- Measurements of α_S are reviewed in ESW. A more recent compilation from arXiv:hep-ex/0606035 is shown above. Evidence that $\alpha_S(Q)$ has a logarithmic fall-off with Q is persuasive.

α_S at m_Z


DIS [pol. strct. fctn.]
DIS [Bj-SR]
DIS [GLS-SR]
τ-decays [LEP]
xF3 [ν -DIS]
F2 [e-, □-DIS]
DIS [ep –> jets]
QQ̄ + lattice QCD
Υ decays
e+ e− F2γ
e+ e− [σhad]
e+ e− [jets & shapes 14 GeV]
e+ e− [jets & shapes 22 GeV]
e+ e− [jets & shapes 35 GeV]
e+ e− [σhad]
e+ e− [jets & shapes 44 GeV]
e+ e− [jets & shapes 58 GeV]
pp̄ --> bb̄ X
pp̄, pp --> γ X
σ(pp̄ --> jets)
Γ(Z0--> had.) [LEP]
e+ e− [scaling. viol.]
e+ e− [4-jet rate]
jets & shapes 91.2 GeV
jets & shapes 133 GeV
jets & shapes 161 GeV
jets & shapes 172 GeV
jets & shapes 183 GeV
jets & shapes 189 GeV
jets & shapes 195 GeV
jets & shapes 201 GeV
jets & shapes 206 GeV
0.08
0.10
0.12
0.14
αs (MZ)

Non-perturbative QCD

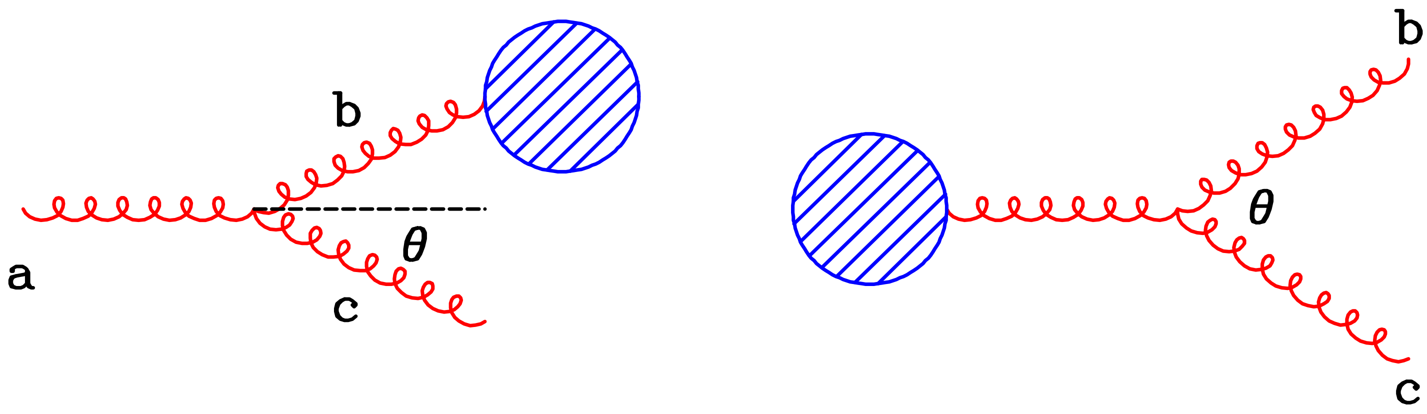
- Corresponding to asymptotic freedom at high momentum scales, we have infra-red slavery: $\alpha_S(Q)$ becomes large a low momenta, (long distances). Perturbation theory is not reliable for large α_S, so non-perturbative methods, (e.g. lattice) must be used.



- Important low momentum scale phenomena
  - ⋆ Confinement: partons (quarks and gluons) found only in colour singlet bound states, hadrons, size $\sim$ 1 fm. If we try top separate them it becomes energetically favourable to create extra partons from the vacuum.
  - ⋆ Hadronization: partons produced in short distance interactions re-organize themselves to make the observed hadrons.

Infrared divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



⋆ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = (p_a - p_c)^2 = -2E_a E_c (1 - \cos\theta) \leq 0 \,.$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity).

If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -2E_a E_c (1 - v_a \cos\theta) \leq 0 ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- Timelike branching: gluon splitting on outgoing line (b)

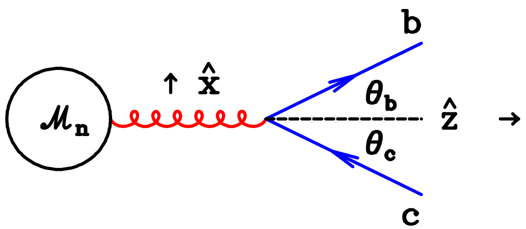
$$p_a^2 = (p_b + p_c)^2 = 2E_b E_c (1 - \cos\theta) \geq 0 .$$

  Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - ⋆ Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - ⋆ Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.
- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - ⋆ Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  - ⋆ Dimensional regularization: analogous to that used for ultraviolet divergences, except we must increase dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$.

Parton branching - kinematics



$$\begin{aligned} p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\ p_b &= (E_b, +E_b \sin\theta_b, 0, +E_b \cos\theta_b) \\ p_c &= (E_c, -E_c \sin\theta_c, 0, +E_c \cos\theta_c) \end{aligned}$$

- the kinematics and notation for the branching of parton a into $b + c$. We assume that

$$p_b^2, p_c^2 \ll p_a^2 \equiv t$$

- a is an outgoing parton, which is called timelike branching since $t > 0$.
- The opening angle is $\theta = \theta_b + \theta_c$. Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a \ ,$$

we have for small angles, $t = 2E_bE_c(1 - \cos\theta) = z(1-z)E_a^2\theta^2$

- using transverse momentum conservation,

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ .$$

Dirac eqn. Massless fermions

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

  The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

  In the Weyl representation upper and lower components have different helicities.
- Both representations satisfy the same commutation relations.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

- in the Weyl representation $\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.

- The massless spinors solns of Dirac eqn are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

In this representation the Dirac conjugate spinors are

$$\overline{u_+}(p) \equiv u_+^\dagger(p)\gamma^0 = \left[0, 0, \sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}\right]$$

$$\overline{u_-}(p) = \left[\sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}, 0, 0\right]$$

- Normalization

$$u_\pm^\dagger u_\pm = 2p^0$$

Branching probabilities

- Consider the case where

$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &\sim (E_b, +E_b\theta_b, 0, +E_b) \\
p_c &\sim (E_c, -E_c\theta_c, 0, +E_c)
\end{aligned}
$$

Thus for example

$$u_+^\dagger(p) = \sqrt{2E_b}\left[1, \frac{\theta_b}{2}, 0, 0\right]$$

and

$$u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c}\begin{bmatrix} 1 \\ -\frac{\theta_c}{2} \\ 0 \\ 0 \end{bmatrix}$$

Hence for polarization vectors $\varepsilon_{in} = (0,1,0,0), \varepsilon_{out} = (0,0,1,0)$

$$g\bar{u}_+^b \; \gamma^0\gamma^1 \; v_-^c = g\sqrt{4E_bE_c}\left(1, \frac{\theta_b}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -g\sqrt{E_bE_c}(\theta_b - \theta_c)$$

$$-g\bar{u}_+^b \gamma_\mu \varepsilon_a^{p_{\text{in}}\,\mu} v_-^c = g\sqrt{E_b E_c}(\theta_b - \theta_c) = g\sqrt{z(1-z)}(1-2z)E_a\theta \ ,$$

$$-g\bar{u}_+^b \gamma_\mu \varepsilon_a^{p_{\text{out}}\,\mu} v_-^c = ig\sqrt{E_b E_c}(\theta_b + \theta_c) = ig\sqrt{z(1-z)}E_a\theta \ ,$$

and the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z;\varepsilon_a,\lambda_b,\lambda_c)|\mathcal{M}_n|^2$$

where the colour factor is now Tr$(t^A t^A)/8 = T_R = 1/2$. The non-vanishing functions $F(z;\varepsilon_a,\lambda_b,\lambda_c)$ for quark and antiquark helicities λ_b and λ_c are

ε_a	λ_b	λ_c	$F(z;\varepsilon_a,\lambda_b,\lambda_c)$
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2\Big[(1-2z)^2 + 1\Big] = 4(z^2 + (1-z)^2).$$

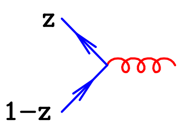
Branching probabilities

$$\int \frac{d\phi}{2\pi} CF = \hat{P}_{ba}(z)$$

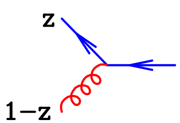
where $\hat{P}_{ba}(z)$ is the appropriate splitting function

$$d\sigma_{n+1} = d\sigma_n \, \frac{dt}{t} \, dz \, \frac{\alpha_S}{2\pi} \, \hat{P}_{ba}(z) \; .$$

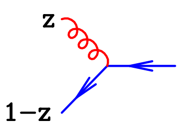
- Including all the color factors we find the results for the unregulated branching probabilities.



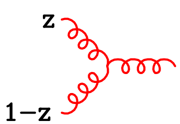
$$\hat{P}_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right], \quad T_R = \frac{1}{2} \; ,$$



$$\hat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)} \right] \; ,$$



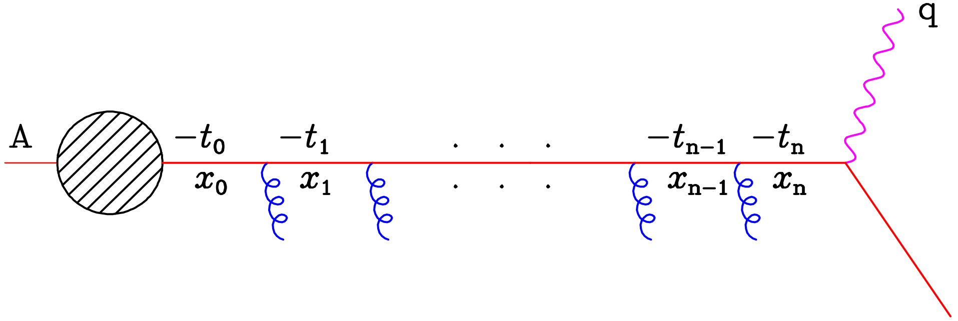
$$\hat{P}_{gq}(z) = C_F \left[\frac{1+(1-z)^2}{z} \right] \; ,$$



$$\hat{P}_{gg}(z) = C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z) \right]$$

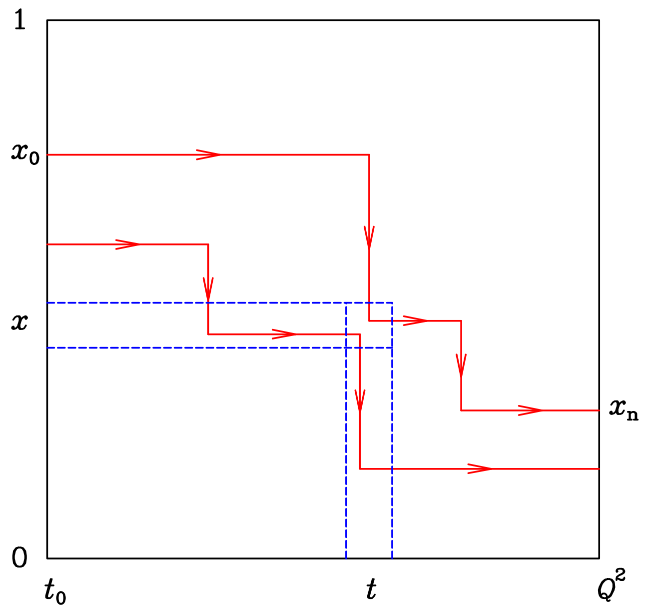
DGLAP equation

- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.
- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.

Change in parton distribution

- Consider change in the parton distribution $D(x,t)$ when t is increased to $t+\delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.
- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned} \delta D_{\text{in}}(x,t) &= \frac{\delta t}{t} \int_x^1 dx'\, dz \frac{\alpha_S}{2\pi} \hat{P}(z) D(x',t)\, \delta(x - zx') \\ &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t) \end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned} \delta D_{\text{out}}(x,t) &= \frac{\delta t}{t} D(x,t) \int_0^x dx'\, dz \frac{\alpha_S}{2\pi} \hat{P}(z)\, \delta(x' - zx) \\ &= \frac{\delta t}{t} D(x,t) \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \end{aligned}$$

Change in parton distribution

- Change in population of element is

$$\begin{aligned}\delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t}\int_0^1 dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\left[\frac{1}{z}D(x/z,t) - D(x,t)\right] .\end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dx\ f(x)\,g(x)_+ \ = \int_0^1 dx\,[f(x) - f(1)]\,g(x) .$$

  Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+ \ ,$$

- Plus-prescription, like the Dirac-delta function, is only defined under integral sign.
- Plus-prescription includes some of the effects of virtual diagrams.

DGLAP

We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t.
- In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quarks and gluons

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t) \; .$$

- Quark ($i = q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2 + (1-z)^2\right] \end{aligned}$$

- Gluon can arrive either from $g \to gg$ (2 contributions) or from $q \to qg$ (or $\bar{q} \to \bar{q}g$). Thus number arriving is

$$\delta D_{g,\text{in}} = \frac{\delta t}{t}\int_0^1 dz \frac{\alpha_S}{2\pi}\left\{\hat{P}_{gg}(z)\left[\frac{D_g(x/z,t)}{z} + \frac{D_g(x/(1-z),t)}{1-z}\right] + \frac{\hat{P}_{qg}(z)}{1-z}\left[D_q\left(\frac{x}{1-z},t\right) + D_{\bar{q}}\left(\frac{x}{1-z},t\right)\right]\right\}$$

$$= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\left\{2\hat{P}_{gg}(z)D_g\left(\frac{x}{z},t\right) + \hat{P}_{qq}(1-z)\left[D_q\left(\frac{x}{z},t\right) + D_{\bar{q}}\left(\frac{x}{z},t\right)\right]\right\},$$

- Gluon can leave by splitting into either gg or $q\bar{q}$, so that

$$\delta D_{g,\text{out}} = \frac{\delta t}{t} D_g(x,t)\int_0^1 dz \frac{\alpha_S}{2\pi}\left[\hat{P}_{gg}(z) + N_f \hat{P}_{qg}(z)\, dz\right] .$$

- After some manipulation we find

$$P_{gg}(z) = 2C_A\left[\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z} + \frac{1}{2}z(1-z)\right] - \frac{2}{3}N_f T_R\,\delta(1-z) ,$$

$$P_{gq}(z) = P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z} .$$

- Using definition of the plus-prescription, can check that

$$\left(\frac{z}{1-z}+\frac{1}{2}z(1-z)\right)_+ = \frac{z}{(1-z)_+}+\frac{1}{2}z(1-z)+\frac{11}{12}\delta(1-z)$$

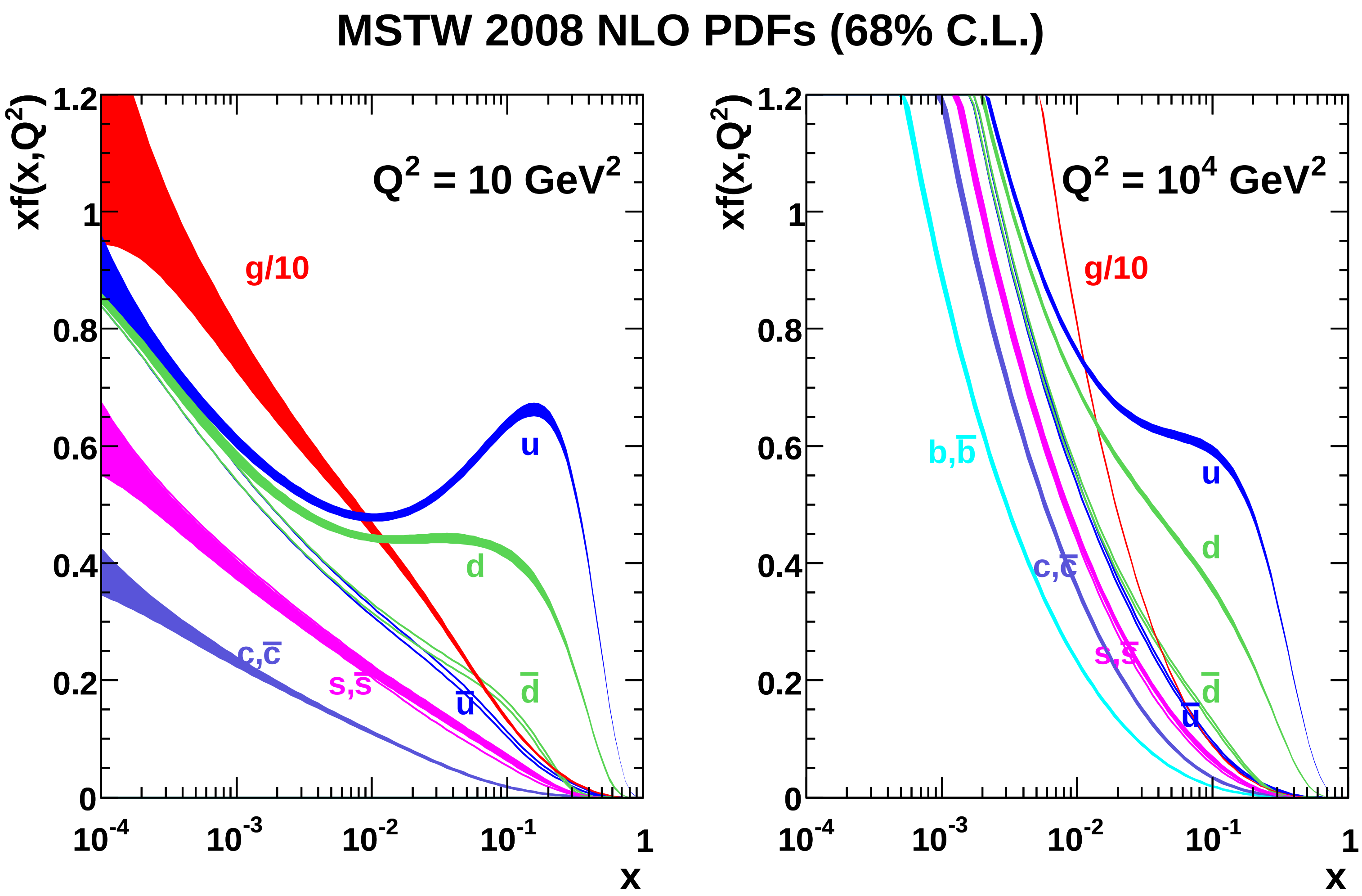
$$\left(\frac{1+z^2}{1-z}\right)_+ = \frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\ ,$$

so P_{qq} and P_{gg} can be written in more common forms

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right]$$

$$P_{gg}(z) = 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\frac{1}{6}(11C_A-4N_fT_R)\,\delta(1-z)\ .$$

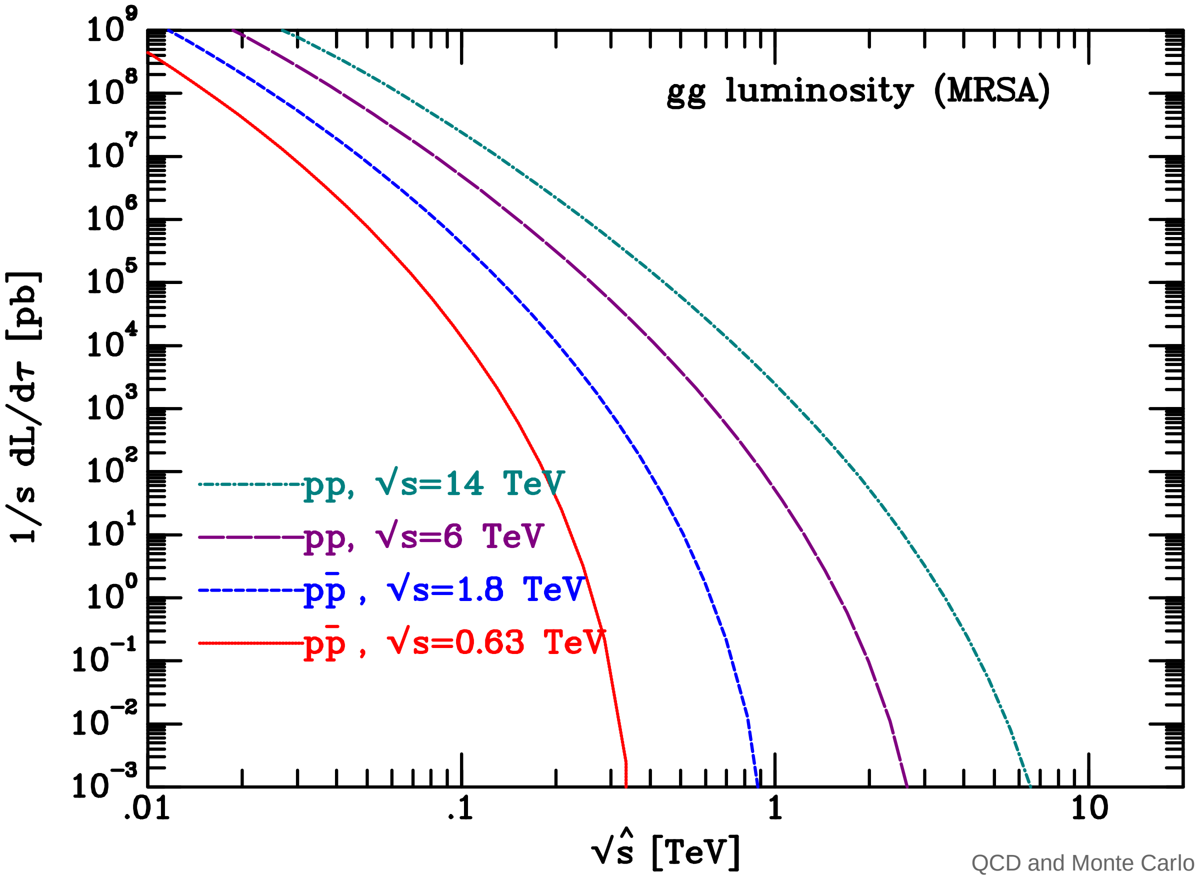
Parton distributions



Parton luminosity

$$\tau \frac{dL_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}} \int_0^1 dx_1 dx_2 \Big[\big(x_1 f_i(x_1,\mu^2)\; x_2 f_j(x_2,\mu^2) \big) + (1 \leftrightarrow 2) \Big] \delta(\tau - x_1 x_2).$$

$$\sigma(s) = \sum_{\{ij\}} \int_{\tau_0}^1 \frac{d\tau}{\tau} \left[\frac{1}{s} \frac{dL_{ij}}{d\tau} \right] \left[\hat{s}\hat{\sigma}_{ij} \right],$$



Solution by moments

- Given $D_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{D}_i(N,t) = \int_0^1 dx\, x^{N-1}\, D_i(x,t)\ .$$

  Inverse Mellin transform is

$$D_i(x,t) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, \tilde{D}_i(N,t)\ ,$$

  where contour C is parallel to imaginary axis to right of all singularities of integrand.
- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{D}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{D}_j(N,t)$$

Anomalous dimensions

- The moments of splitting functions give PT expansion of anomalous dimensions γ_{ij}:

$$\begin{aligned}
\gamma_{ij}(N,\alpha_S) &= \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_S}{2\pi}\right)^{n+1} \\
\gamma_{ij}^{(0)}(N) &= \tilde{P}_{ij}(N) = \int_0^1 dz\ z^{N-1}\ P_{ij}(z)
\end{aligned}$$

- From above expressions for $P_{ij}(z)$ we find

$$\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F\left[-\frac{1}{2}+\frac{1}{N(N+1)}-2\sum_{k=2}^{N}\frac{1}{k}\right] \\
\gamma_{qg}^{(0)}(N) &= T_R\left[\frac{(2+N+N^2)}{N(N+1)(N+2)}\right] \\
\gamma_{gq}^{(0)}(N) &= C_F\left[\frac{(2+N+N^2)}{N(N^2-1)}\right] \\
\gamma_{gg}^{(0)}(N) &= 2C_A\left[-\frac{1}{12}+\frac{1}{N(N-1)}+\frac{1}{(N+1)(N+2)}-\sum_{k=2}^{N}\frac{1}{k}\right]-\frac{2}{3}N_f T_R\ .
\end{aligned}$$

Scaling violation

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

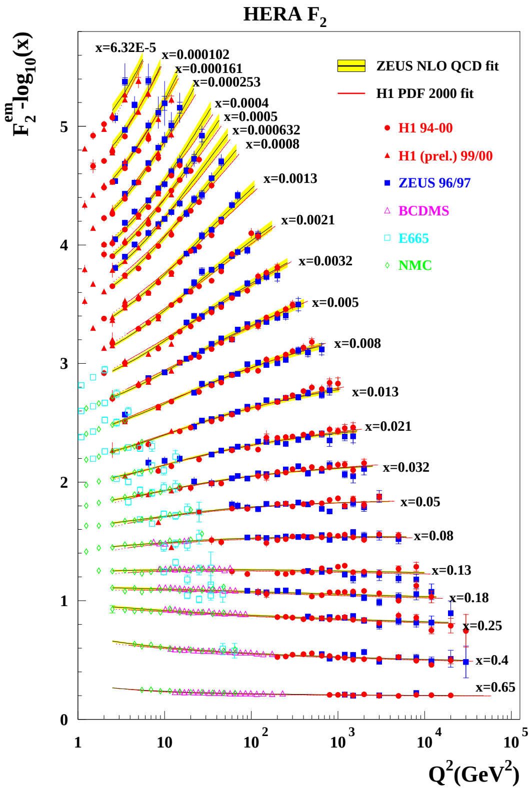
$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} \quad ,$$

- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

  where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.
- Flavour-singlet distribution and quantitative predictions will be discussed later.

Combined data on F2 proton



- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:

Flavour singlet combination

- For flavour-singlet combination, define

$$\Sigma = \sum_i (q_i + \bar{q}_i)\ .$$

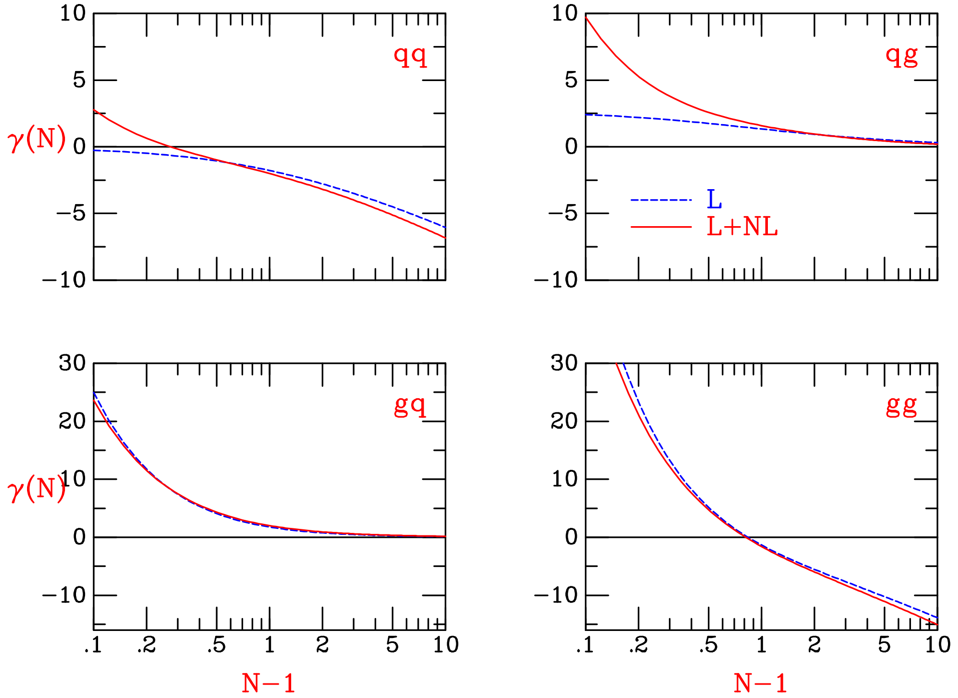
Then we obtain

$$\begin{aligned} t\frac{\partial \Sigma}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes\Sigma + 2N_f P_{qg}\otimes g\right] \\ t\frac{\partial g}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes g\right]\ . \end{aligned}$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution g: evolution equation for moments has matrix form

$$t\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\Sigma}\\ \tilde{g}\end{pmatrix} = \begin{pmatrix}\gamma_{qq} & 2N_f\gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{pmatrix}\begin{pmatrix}\tilde{\Sigma}\\ \tilde{g}\end{pmatrix}$$

Anomalous dimension matrix as a function of N.



- Rapid growth at small N in gq and gg elements at lowest order
- $\ln N$ behaviour at large N in qq and gg elements
- NNLO now known
- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \frac{1}{2}[\gamma_{gg} + \gamma_{qq} \pm \sqrt{(\gamma_{gg} - \gamma_{qq})^2 + 8N_f \gamma_{gq}\gamma_{qg}}] \ .$$

Solution of lowest order DGLAP matrix equation

The reduced DGLAP equation can be written as

$$\frac{d}{du}\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right) = \mathbf{P}\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right)$$

where $u = \frac{1}{2\pi b}\ln\frac{\alpha_S(\mu_0^2)}{\alpha_S(\mu^2)}$

- Define projection operators, $\mathbf{M}_\pm$

$$\mathbf{M}_+ = \frac{1}{\gamma_+ - \gamma_-}\Big[+\mathbf{P} - \gamma_-\mathbf{1}\Big],\ \ \mathbf{M}_- = \frac{1}{\gamma_+ - \gamma_-}\Big[-\mathbf{P} + \gamma_+\mathbf{1}\Big],$$

  where $\mathbf{M}_\pm\mathbf{M}_\pm = \mathbf{M}_\pm, \mathbf{M}_+\mathbf{M}_- = \mathbf{M}_-\mathbf{M}_+ = \mathbf{0}$, $\mathbf{M}_+ + \mathbf{M}_- = \mathbf{1}$ and

$$\mathbf{P} = \gamma_+\mathbf{M}_+ + \gamma_-\mathbf{M}_-$$

- The solution is

$$\left(\begin{array}{c}\tilde{\Sigma}(u)\\ \tilde{g}(u)\end{array}\right) = \Big[\mathbf{M}_+\exp(\gamma_+ u) + \mathbf{M}_-\exp(\gamma_- u)\Big]\left(\begin{array}{c}\tilde{\Sigma}(0)\\ \tilde{g}(0)\end{array}\right)$$

Momentum partition vs Q^2

- For second moment

$$O^{+}(2,t) \quad = \quad \Sigma(2,t) + g(2,t) \quad \text{with eigenvalue } 0 \ ,$$

$$O^{-}(2,t) \quad = \quad \Sigma(2,t) - \frac{n_f}{4C_F} g(2,t) \quad \text{with eigenvalue } -\left(\frac{4}{3}C_F + \frac{n_f}{3}\right) \ .$$

O^{+}, corresponds to the total momentum carried by the quarks and gluons, is independent of t. The eigenvector O^{-} vanishes in the limit $t \to \infty$:

$$O^{-}(2,t) = \left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d^{-}(2)} \to 0, \quad \text{with} \ \ d^{-}(2) = \frac{\gamma_{-}(2)}{2\pi b} = -\frac{\left(\frac{4}{3}C_F + \frac{1}{3}n_f\right)}{2\pi b} \ ,$$

so that asymptotically we have

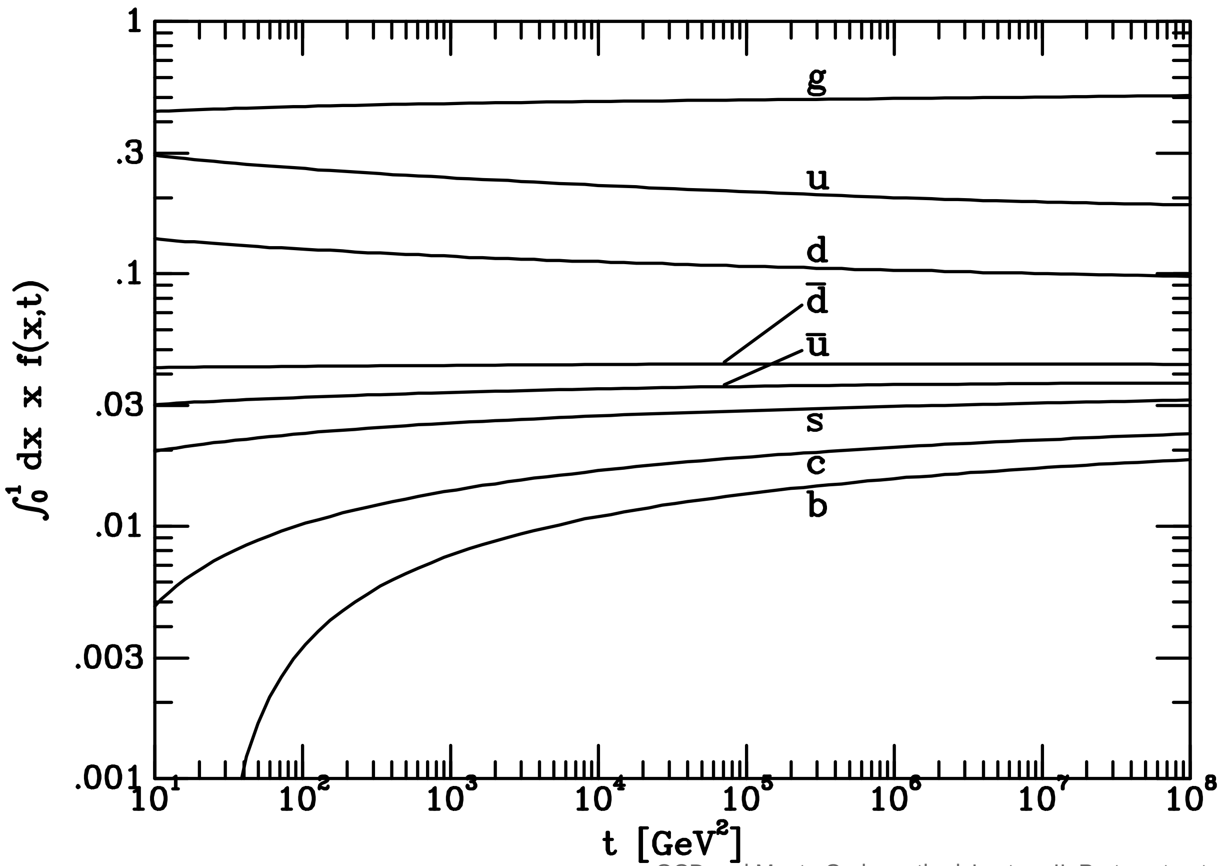
$$\frac{\Sigma(2,t)}{g(2,t)} \to \frac{n_f}{4C_F} = \frac{3}{16} n_f \ .$$

Asymptotia is approached slowly

The momentum fractions f_q and f_g in the $\mu^2 = t \to \infty$ limit are therefore

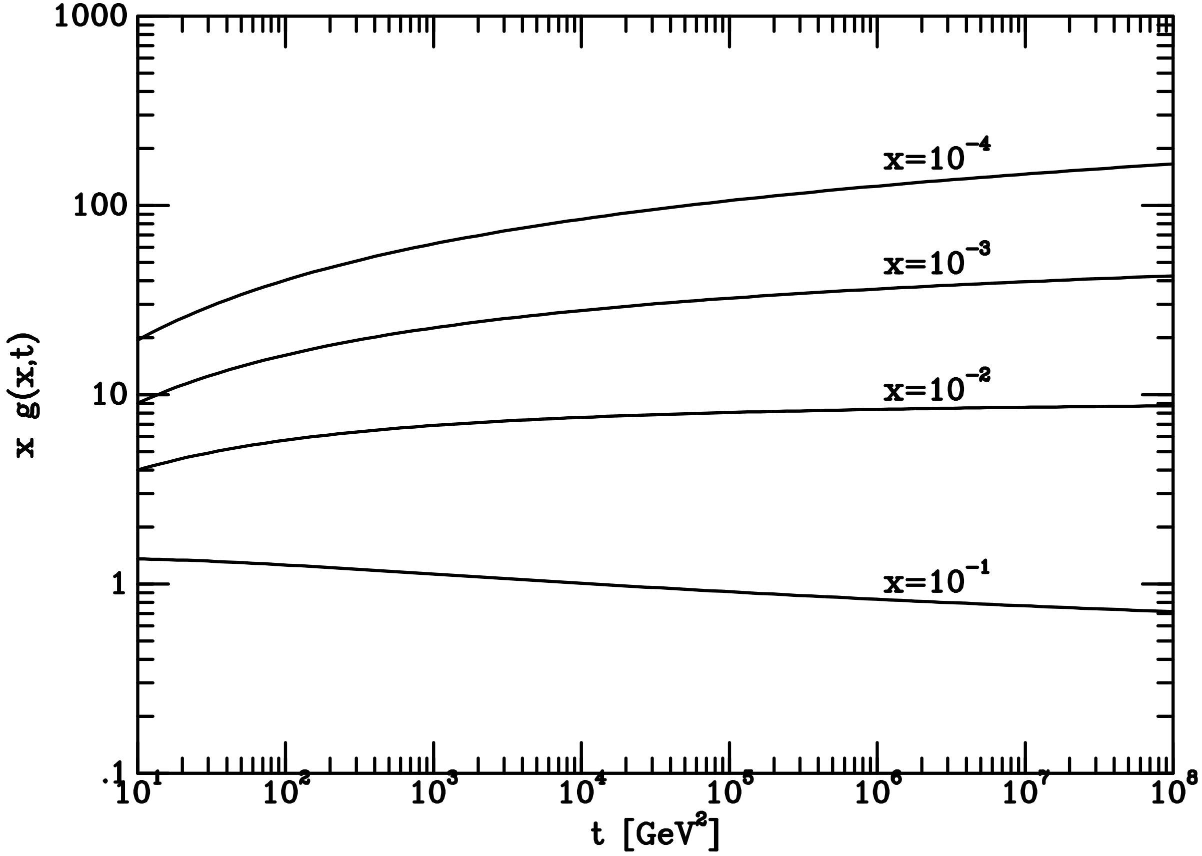
$$f_q = \frac{3n_f}{16 + 3n_f}\ , \quad f_g = \frac{16}{16 + 3n_f}\ .$$

- Scaling violation depends logarithmically on Q^2.
- Large variation at low Q^2



Gluon distribution

- Large number of gluons per unit rapidity
- The LHC is a copious source of gluons



Recap

- The scale Λ is a fundamental parameter of QCD, but its practical use is fraught with difficulties. It is better to $\alpha_S(M_z)$ to specify the size of α_S.
- QCD predicts universal branching probailities. The probability of branching is a property of QCD, and is independent of the process at hand.
- Parton evolution can be represented as a branching process from higher values of x, governed by the DGLAP equation.
- DGLAP equation predicts growth at small x and shrinkage at large x with increasing Q^2.
- Gluon fluxes become very large at the LHC.