QCD and Monte Carlo Methods
Lecture I: QCD, asymptotic freedom and infrared safety

The 2009 Hadron Collider Physics Summer School
CERN, June 2009

Keith Ellis

ellis@fnal.gov

Fermilab

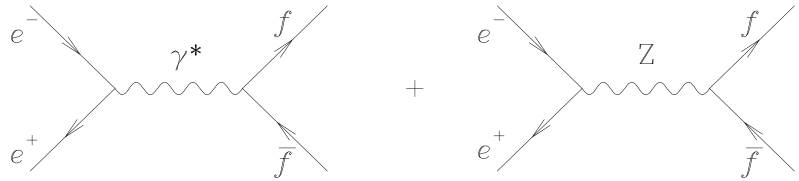
QCD, asymptotic freedom and infrared safety

- Plan for lecture 1
  - ⋆ Motivation for Colour SU(3)
  - ⋆ QCD Lagrangian
  - ⋆ Gauge Invariance
  - ⋆ Feynman rules
  - ⋆ Running coupling
  - ⋆ β-function of QCD
  - ⋆ Measurements of α_S
  - ⋆ Non-perturbative QCD and infra-red divergences.

- Bibliography,
  R. K. Ellis, W.J. Stirling and B.R. Webber,
  QCD and Collider Physics
  (Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology)

Motivation for Colour SU(3)

- Consider the ratio R of the e^+e^- total hadronic cross section to the cross section for the production of a pair of point-like, charge-one objects such as muons.
- The virtual photon excites all electrically charged constituent-anticonstituent pairs from the vacuum.



- At low energy the virtual photon excites only the u, d and s quarks, each of which occurs in three colours.

$$\begin{aligned} R &= N_c \sum_i Q_i^2 \\ &= 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2\,. \end{aligned}$$

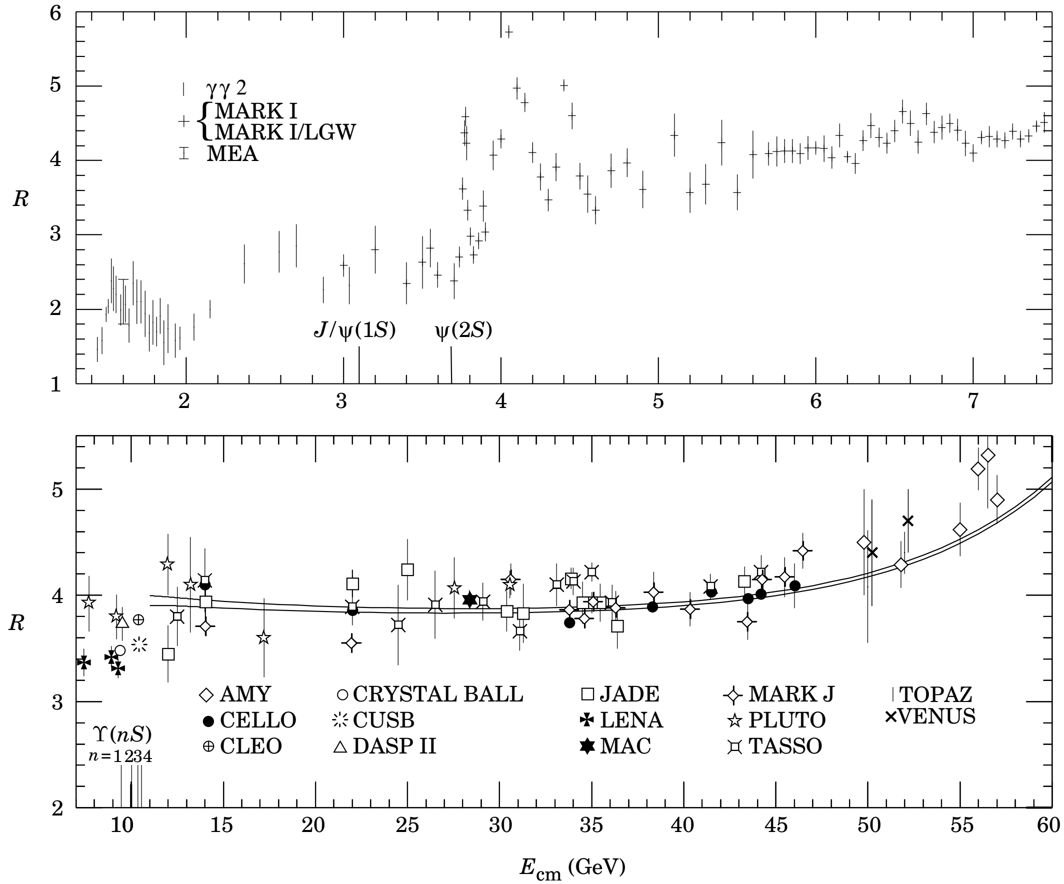
- For centre-of-mass energies $E_{\rm cm} \geq 10$ GeV, one is above the threshold for the production of pairs of c and b quarks, and so

$$R = 3\left[2 \times \left(\frac{2}{3}\right)^2 + 3 \times \left(-\frac{1}{3}\right)^2\right] = \frac{11}{3}\,.$$

Data

The data on R are in reasonable agreement with the prediction of the three colour model.

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



Colour SU(3) and spectroscopy

- The observed baryons are interpreted as three-quark states.
- The quark constituents of the baryons are forced to have half-integral spin in order to account for the spins of the low-mass baryons.
- The quarks in the spin-$\frac{3}{2}$ baryons are then in a symmetrical state of space, spin and SU(3)$_f$ degrees of freedom.
- However the requirements of Fermi-Dirac statistics imply the total antisymmetry of the wave function.
- We introduce the colour degree of freedom: a colour index a with three possible values (usually called red, green, blue for $a = 1, 2, 3$) is carried by each quark.
- The baryon wave functions are totally antisymmetric in this new index.

Quark	Charge	Mass	Baryon Number	Isospin
u	$+\frac{2}{3}$	~ 4 MeV	$\frac{1}{3}$	$+\frac{1}{2}$
d	$-\frac{1}{3}$	~ 7 MeV	$\frac{1}{3}$	$-\frac{1}{2}$
c	$+\frac{2}{3}$	$\sim 1.5\ \mathrm{GeV}$	$\frac{1}{3}$	0
s	$-\frac{1}{3}$	~ 135 MeV	$\frac{1}{3}$	0
t	$+\frac{2}{3}$	$\sim 172\ \mathrm{GeV}$	$\frac{1}{3}$	0
b	$-\frac{1}{3}$	$\sim 5\ \mathrm{GeV}$	$\frac{1}{3}$	0

- The group of colour transformations is SU(3), with the quarks q_a transforming according to the fundamental representation
- Why does this new degree of freedom not lead to a proliferation of states?
- We hypothesize that only colour singlet states can exist in nature.
- For a baryon the colour singlet state is totally antisymmetric
$(|a\rangle|b\rangle|c\rangle + |b\rangle|c\rangle|a\rangle + |c\rangle|a\rangle|b\rangle - |b\rangle|a\rangle|c\rangle - |a\rangle|c\rangle|b\rangle - |c\rangle|b\rangle|a\rangle)/\sqrt{6}$

Argument for SU(3) singlet ground states

- Consider the force between quarks using an (over)simplified model of one gluon exchange.
- In QED: $V \sim \frac{e^2}{r}$ In QCD: $V \sim \frac{\lambda^{(1)}\ \lambda^{(2)}}{r}$
- λ are the eight Gell-Mann matrices, the hermitean and traceless generators of SU(3)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

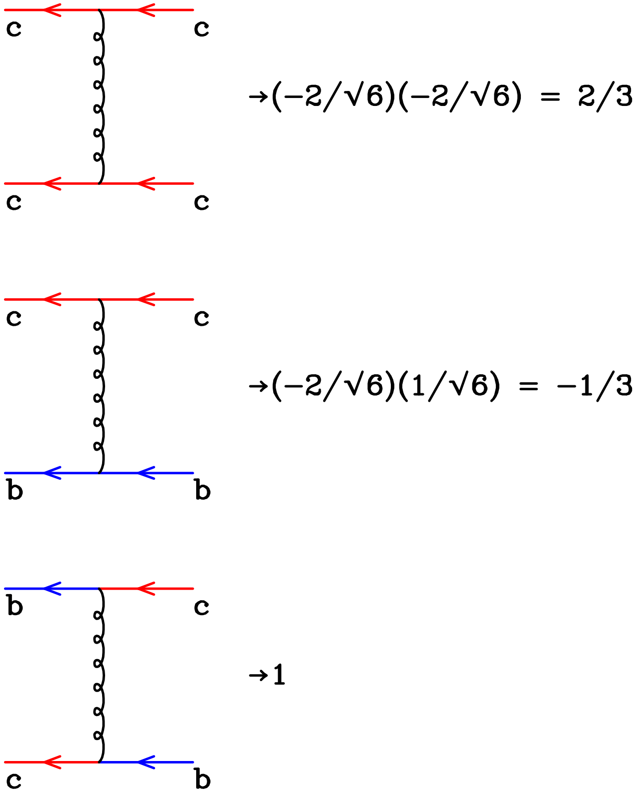
Interquark forces

- We will calculate the weights given by the products of the λ matrices, but using a physical basis

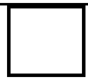
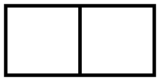
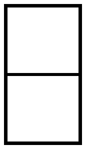
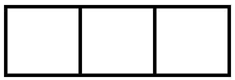
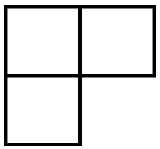
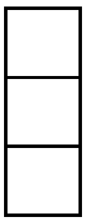
- The eight gluons couple to the colors of the quarks and can be written as $\bar{a}b, \bar{a}c, \bar{b}a, \bar{b}c, \bar{c}a, \bar{c}b$ and $(\bar{a}a - \bar{b}b)/\sqrt{2}, (\bar{a}a + \bar{b}b - 2\bar{c}c))/\sqrt{6}$,
- The last two gluons are orthogonal to the SU(3) singlet gluon, $((\bar{a}a + \bar{b}b + \bar{c}c))/\sqrt{3}$, which is not included.
- We only have to consider two cases, forces between quarks of the same colour and of different colours.

- $\sum_A \lambda^A_{ij}\lambda^A_{kl} = \mathcal{N}\left[\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right]$,

  (Normalization, $\mathcal{N}$)
- Introduce the colour exchange operator P which has eigenvalues, p=+1(-1) for a symmetric (antisymmetric state). Interaction energy can be written as $E \sim (p - \frac{1}{3})$

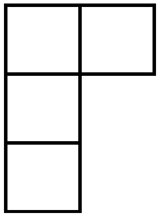
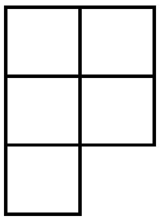
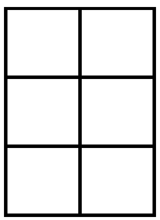


Interaction energies

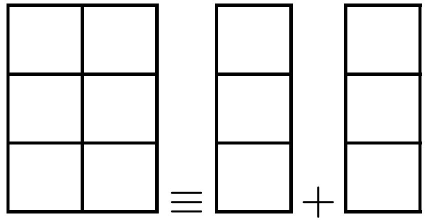
N_q	Young Diagram	Dimensionality	Interaction Energy	Energy $+\frac{4}{3}N_q$
1		3	0	$\frac{4}{3}$
2		6	$1 - \frac{1}{3} = \frac{2}{3}$	$\frac{10}{3}$
2		3	$-1 - \frac{1}{3} = -\frac{4}{3}$	$\frac{4}{3}$
3		10	$3 \times 1 - 3 \times \frac{1}{3} = 2$	6
3		8	$1 + (-1) - 3 \times \frac{1}{3} = -1$	3
3		0	$3 \times (-1 - \frac{1}{3}) = -4$	0

- Add a constant self-energy per quark, $\frac{4}{3}$ in these units, (just a book-keeping device: as long as we only compare states with the same number of quarks)
- 3-quark state, which is totally antisymmetric with respect to colour has the lowest energy: This is the baryon.
- All other three quark states have higher energy.

Higher numbers of quarks

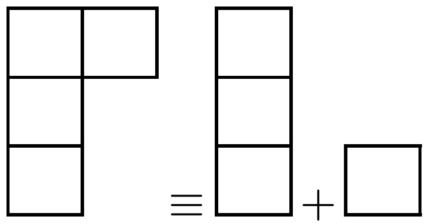
N_q	Young Diagram	Dimensionality	Interaction Energy	Energy $+\frac{4}{3}N_q$
4		3	$1+3\times(-1)-6\times\frac{1}{3}=-4$	$\frac{4}{3}$
5		3	$2\times 1+4\times(-1)-10\times\frac{1}{3}=-\frac{16}{3}$	$\frac{4}{3}$
6		1	$3\times 1+6\times(-1)-15\times\frac{1}{3}=-8$	0

- Six quark state has the same energy as two baryons

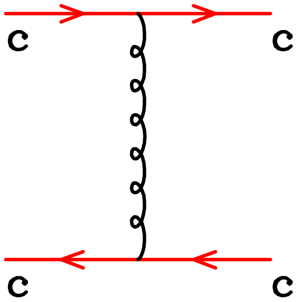


  this crude approximation does not allow us to say whether the two baryon state is bound.

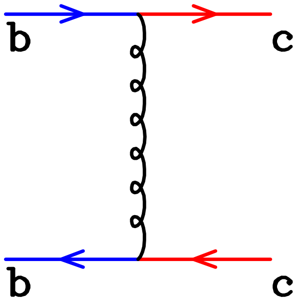
- No strong binding of a quark (or an antiquark) to a baryon



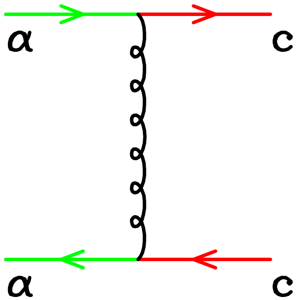
Mesons



$\rightarrow -(-2/\sqrt{6})^2 = -2/3$

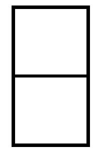
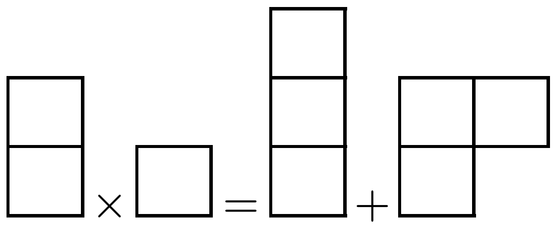


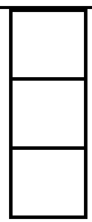
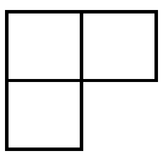
$\rightarrow\ = -1$



$\rightarrow\ = -1$

- For the diagonal interaction we have $E_{c\bar{c}\rightarrow c\bar{c}} = -\frac{2}{3}$.
- Note overall minus sign – just as in QED particle-antiparticle force is attractive.
- We have the off-diagonal interaction $E_{c\bar{c}\rightarrow b\bar{b}} = E_{c\bar{c}\rightarrow a\bar{a}} = -1$

- Colour singlet meson is given by $(c\bar{c} + b\bar{b} + a\bar{a})/\sqrt{3}$
- Overall is $3 \times c\bar{c}/\sqrt{3} \times (c\bar{c} + b\bar{b} + a\bar{a})/\sqrt{3} = -8/3$
- Adding in self-energy of $2 \times \frac{4}{3}$ we get 0
- Coloured meson gives energy of $E_{c\bar{b}\to c\bar{b}} = \frac{1}{3}$. Adding in the self energy of $2 \times \frac{4}{3}$ we get 3.
- Antiquarks are represented by a column of $N_c - 1$ boxes, $\bar{3} =$ 
- $3 \otimes \bar{3} = 8 \oplus 1,$ 

N_q	Young Diagram	Dimensionality	Energy $+\frac{4}{3}N_q$
2		1	0
2		8	3

Lagrangian of QCD

- Feynman rules for perturbative QCD follow from Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

  $F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

  Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_S \equiv g^2/4\pi$. Numbers f^{ABC} $(A, B, C = 1, ..., 8)$ are structure constants of the SU(3) colour group. Quark fields q_a $(a = 1, 2, 3)$ are in triplet colour representation. D is covariant derivative:

$$\begin{aligned} (D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha \right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}^C_\alpha)_{AB} \end{aligned}$$

- t and T are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$t^A = \frac{1}{2}\lambda^A, \quad [t^A, t^B] = if^{ABC}t^C, \quad [T^A, T^B] = if^{ABC}T^C$$

where $(T^A)_{BC} = -if^{ABC}$. We use the metric $g^{\alpha\beta} =$ diag(1,–1,–1,–1) and set $\hbar = c = 1$. $\not{D}$ is symbolic notation for $\gamma^\alpha D_\alpha$. Normalisation of the t matrices is

$$\text{Tr}\ t^A t^B = T_R\,\delta^{AB}, \;\; T_R = \frac{1}{2}.$$

- Colour matrices obey the relations:

$$\begin{aligned}
\sum_A t^A_{ab} t^A_{bc} &= C_F\,\delta_{ac}\,, \quad C_F = \frac{N^2-1}{2N}, \quad \sum_A t^A_{ab} t^A_{cd} = \frac{1}{2}\left[\delta_{ad}\delta_{cb} - \frac{1}{N}\delta_{ab}\delta_{cd}\right] \\
\text{Tr}\ T^C T^D &= \sum_{A,B} f^{ABC} f^{ABD} = C_A\,\delta^{CD}\,, \quad C_A = N
\end{aligned}$$

Thus $C_F = \frac{4}{3}$ and $C_A = 3$ for SU(3).

Gauge invariance

- QCD Lagrangian is invariant under local gauge transformations. That is, one can redefine quark fields independently at every point in space-time,

$$q_a(x) \to q'_a(x) = \exp(it \cdot \theta(x))_{ab} q_b(x) \equiv \Omega(x)_{ab} q_b(x)$$

without changing physical content.

- Covariant derivative is so called because it transforms in same way as field itself:

$$D_\alpha q(x) \to D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) \ .$$

(omitting the colour labels of quark fields from now on). Use this to derive transformation property of gluon field $\mathcal{A}$

$$\begin{aligned} D'_\alpha q'(x) &= \big(\partial_\alpha + igt \cdot \mathcal{A}'_\alpha\big)\Omega(x) q(x) \\ &\equiv (\partial_\alpha \Omega(x)) q(x) + \Omega(x) \partial_\alpha q(x) + igt \cdot \mathcal{A}'_\alpha \Omega(x) q(x) \end{aligned}$$

where $t \cdot \mathcal{A}_\alpha \equiv \sum_A t^A \mathcal{A}^A_\alpha$. Hence

$$t \cdot \mathcal{A}'_\alpha = \Omega(x) t \cdot \mathcal{A}_\alpha \Omega^{-1}(x) + \frac{i}{g}\big(\partial_\alpha \Omega(x)\big)\Omega^{-1}(x) \ .$$

- Transformation property of gluon field strength $F_{\alpha\beta}$ is

$$t \cdot F_{\alpha\beta}(x) \to t \cdot F'_{\alpha\beta}(x) = \Omega(x) F_{\alpha\beta}(x) \Omega^{-1}(x) \ .$$

  Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge invariant because of self-interaction of gluons. Carriers of the colour force are themselves coloured, unlike the electrically neutral photon.

- Note there is no gauge-invariant way of including a gluon mass. A term such as

$$m^2 \mathcal{A}^\alpha \mathcal{A}_\alpha$$

  is not gauge invariant. This is similar to QED result for mass of the photon. On the other hand the quark mass term is gauge invariant.

Feynman rules

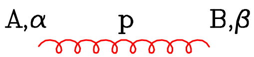
- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
  - ⋆ Quark propagator in momentum space obtained by setting $\partial^\alpha = -ip^\alpha$ for an incoming field. Result is in Table 1. The $i\varepsilon$ prescription for pole of propagator is determined by causality, as in QED.
  - ⋆ Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(\partial^\alpha \mathcal{A}_\alpha^A\right)^2$$

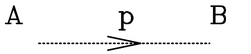
defines *covariant gauges* with gauge parameter λ. Inverse gluon propagator is then

$$\Gamma^{(2)}_{\{AB,\ \alpha\beta\}}(p) = i\delta_{AB}\ \left[p^2 g_{\alpha\beta} - (1-\frac{1}{\lambda})p_\alpha p_\beta\right].$$

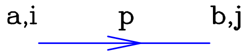
(Check that without gauge-fixing term this function would have no inverse.) Resulting propagator is in the table. $\lambda = 1$ (0) is *Feynman* (*Landau*) gauge.



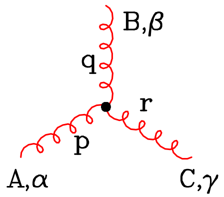
$$\delta^{AB}\ \left[-g^{\alpha\beta}+(1-\lambda)\,\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\ \frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\ \frac{i}{(p^{2}+i\epsilon)}$$

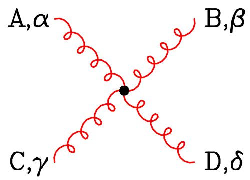


$$\delta^{ab}\ \frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

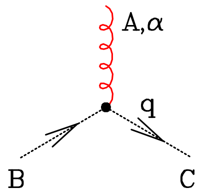


$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

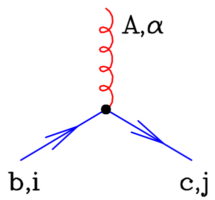
(all momenta incoming)



$$-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$



$$g\ f^{ABC}q^{\alpha}$$



$$-ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience, we usually use Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a *ghost term* which we do not discuss here. Ghost field, shown by dashed lines in the above table, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.
- Propagators determined from $-S$, interactions from S.

Feynman rules – recipe

- Consider a theory which contains only a complex scalar field ϕ and an action (iL) which contains only bilinear terms, $S = \phi^* \left(K + K'\right) \phi$.
- MOE: both K and K' are included in the free Lagrangian, $S_0 = \phi^* \left(K + K'\right) \phi$. Using the above rule the propagator Δ for the ϕ field is given by

$$\Delta = \frac{-1}{K + K'}.$$

- JOE: K is regarded as the free Lagrangian, $S_0 = \phi^* K \phi$, and K' as the interaction Lagrangian, $S_I = \phi^* K' \phi$. Now S_I is included to all orders in perturbation theory by inserting the interaction term an infinite number of times:

$$\begin{aligned}
\Delta &= \frac{-1}{K} + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \cdots \\
&= \frac{-1}{K + K'}
\end{aligned}$$

Alternative choice of gauge

- An alternative choice of gauge fixing is provided by the *axial gauges* which are fixed in terms of another vector which we denote by b

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2\,\lambda}\left(b^{\alpha}\mathcal{A}^{A}_{\alpha}\right)^{2},$$

The advantage of the axial class of gauge is that ghost fields are not required. However one pays for this simplicity because the gluon propagator is more complicated. The inverse propagator is

$$\Gamma^{(2)}_{\{AB,\ \alpha\beta\}}(p) = i\delta_{AB}\ \left[p^{2}g_{\alpha\beta} - p_{\alpha}p_{\beta} + \frac{1}{\lambda}b_{\alpha}b_{\beta}\right].$$

The inverse of this matrix gives the gauge boson propagator,

$$\Delta^{(2)}_{\{BC,\ \beta\gamma\}}(p) = \delta_{BC}\ \frac{i}{p^{2}}\ \left[-g_{\beta\gamma} + \frac{b_{\beta}p_{\gamma} + p_{\beta}b_{\gamma}}{b\cdot p} - \frac{(b^{2} + \lambda p^{2})p_{\beta}p_{\gamma}}{(b\cdot p)^{2}}\right].$$

Notice the new singularities at $b\cdot p = 0$.

- What are the properties of these gauges which make them interesting? Let us specialize to the case $\lambda = 0, b^2 = 0$, (light-cone gauge).

$$\Delta^{(2)}_{\{BC,\beta\gamma\}}(p) = \delta_{BC}\,\frac{i}{p^2}\,d_{\beta\gamma}(p,b)$$

where

$$d_{\beta\gamma} = -g_{\beta\gamma} + \frac{b_\beta p_\gamma + p_\beta b_\gamma}{b \cdot p} \ .$$

In the limit $p^2 \to 0$ we find that

$$b^\beta d_{\beta\gamma}(p,b) = 0,\ p^\beta d_{\beta\gamma}(p,b) = 0 \ .$$

Only two physical polarization states, orthogonal to b and p, propagate. For this reason these classes of gauges are called physical gauges. In the $p^2 \to 0$ limit we may decompose the numerator of the propagator into a sum over two polarizations:

$$d_{\alpha\beta} = \sum_i \varepsilon^{(i)}_\alpha(p)\varepsilon^{(i)}_\beta(p) \ .$$

In addition to the constraint $\varepsilon^{(i)}_\beta(p)p^\beta = 0$, which is always true, in an axial gauge we have the further constraint $\varepsilon^{(i)}_\beta(p)b^\beta = 0$.

Running coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \; .$$

- Introducing

$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} ,$$

we have

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$

This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} , \quad \alpha_S(\mu) \equiv \alpha_S .$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$

and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.

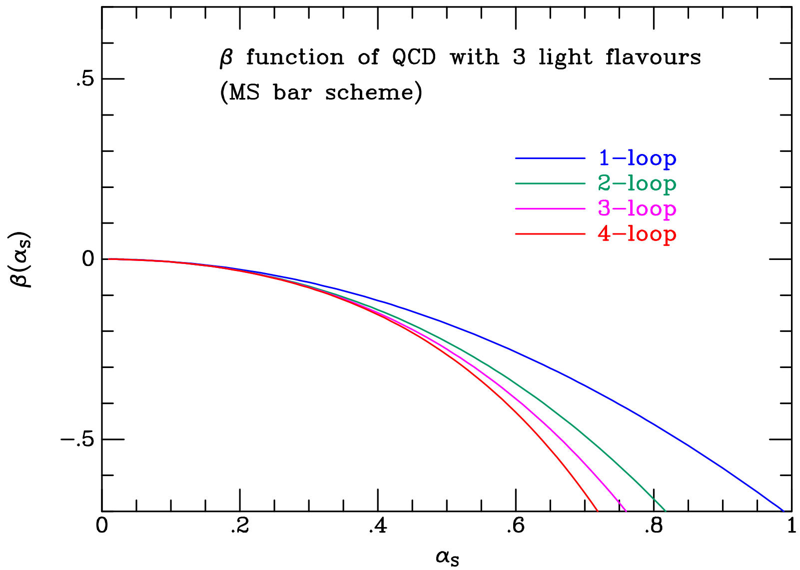
- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

Beta function

- Running of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$
$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)},$$

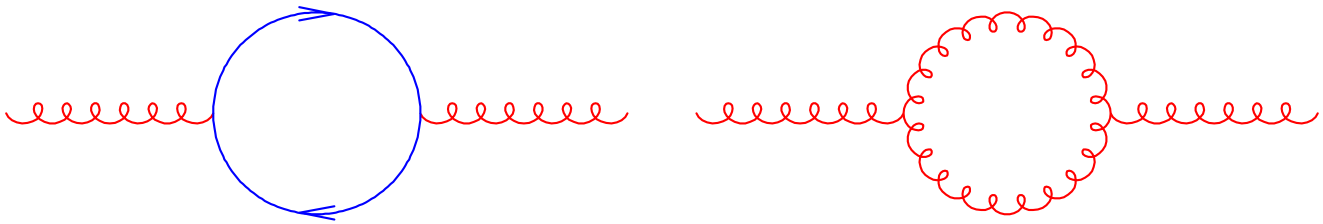
where N_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.



⋆ if $\frac{d\alpha_S}{d\tau} = -b\alpha_S^2(1 + b'\alpha_S)$ and $\alpha_S \to \bar{\alpha}_S(1 + c\bar{\alpha}_S)$, it follows that $\frac{d\bar{\alpha}_S}{d\tau} = -b\bar{\alpha}_S^2(1 + b'\bar{\alpha}_S) + O(\bar{\alpha}_S^4)$

⋆ first two coefficients b, b' are thus invariant under scheme change.

Asymptotic freedom



(a) (b)

- Roughly speaking, quark loop diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.
- QED β function is $\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$
  Thus b coefficients in QED and QCD have opposite signs.
- From earlier slides,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\Big[1 + b'\alpha_S(Q)\Big] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives $\alpha_S(Q) = \frac{\alpha_S(\mu)}{1+\alpha_S(\mu)b\tau}$, $\quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right)$.

Asymptotic freedom

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
- Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

which can be solved numerically, (say using Newton's method).

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity R?
  Assume that R has perturbative expansion

$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \Big[1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu)(b\tau)^2 + \ldots\Big] \end{aligned}$$

Thus there are logarithms of Q^2/μ^2 which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of α_S.

Lambda parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty} \frac{dx}{bx^2(1 + b'x + \ldots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q) \to \infty$ as $Q \to \Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.
- In leading order (LO) keep only first β-function b:

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \qquad \text{(LO).}$$

- In next-to-leading order (NLO) include also b':

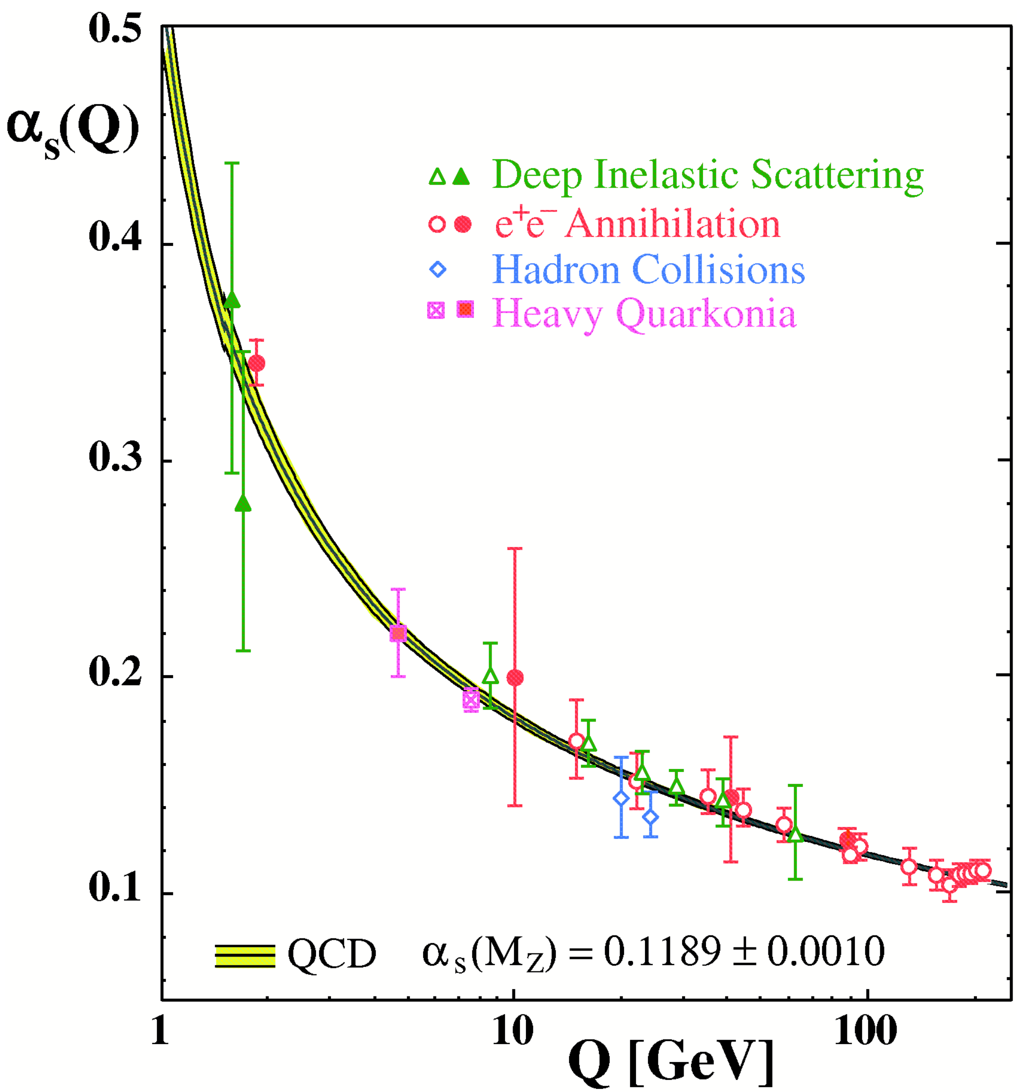
$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b'\alpha_S(Q)}{1+b'\alpha_S(Q)}\right) = b\ln\left(\frac{Q^2}{\Lambda^2}\right).$$

This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_S(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO).}$$

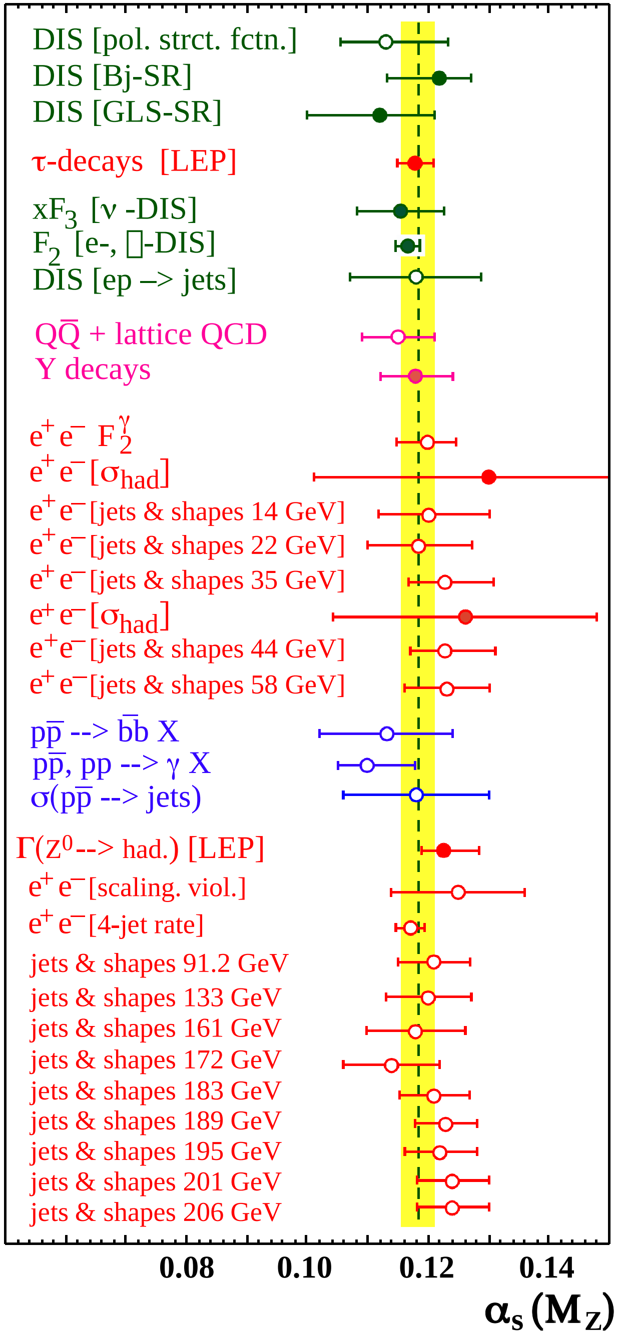
This is Particle Data Group (PDG) definition.

- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$. See ESW for relation between Λ's for different values of N_f.
- The use of Λ is full of pitfalls; however from a parametric point of view it is a fundamental quantity in QCD. The scale of masses of hadrons, protons, neutrons etc are determined primarily by Λ QCD. Thus QCD is responsible for the mass of the observed universe.



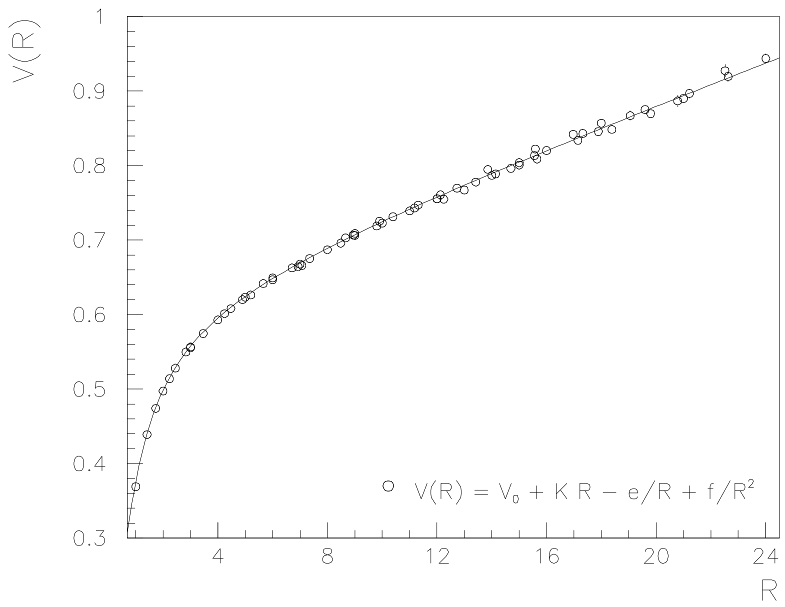
- Measurements of α_S are reviewed in ESW. A more recent compilation from arXiv:hep-ex/0606035 is shown above. Evidence that $\alpha_S(Q)$ has a logarithmic fall-off with Q is persuasive.

α_S at m_Z


DIS [pol. strct. fctn.]
DIS [Bj-SR]
DIS [GLS-SR]
τ-decays [LEP]
xF3 [ν -DIS]
F2 [e-, -DIS]
DIS [ep –> jets]
QQ̄ + lattice QCD
Υ decays
e+e− F2γ
e+e− [σhad]
e+e− [jets & shapes 14 GeV]
e+e− [jets & shapes 22 GeV]
e+e− [jets & shapes 35 GeV]
e+e− [σhad]
e+e− [jets & shapes 44 GeV]
e+e− [jets & shapes 58 GeV]
pp̄ --> bb̄ X
pp̄, pp --> γ X
σ(pp̄ --> jets)
Γ(Z0--> had.) [LEP]
e+e− [scaling. viol.]
e+e− [4-jet rate]
jets & shapes 91.2 GeV
jets & shapes 133 GeV
jets & shapes 161 GeV
jets & shapes 172 GeV
jets & shapes 183 GeV
jets & shapes 189 GeV
jets & shapes 195 GeV
jets & shapes 201 GeV
jets & shapes 206 GeV
0.08
0.10
0.12
0.14
αs (MZ)

Non-perturbative QCD

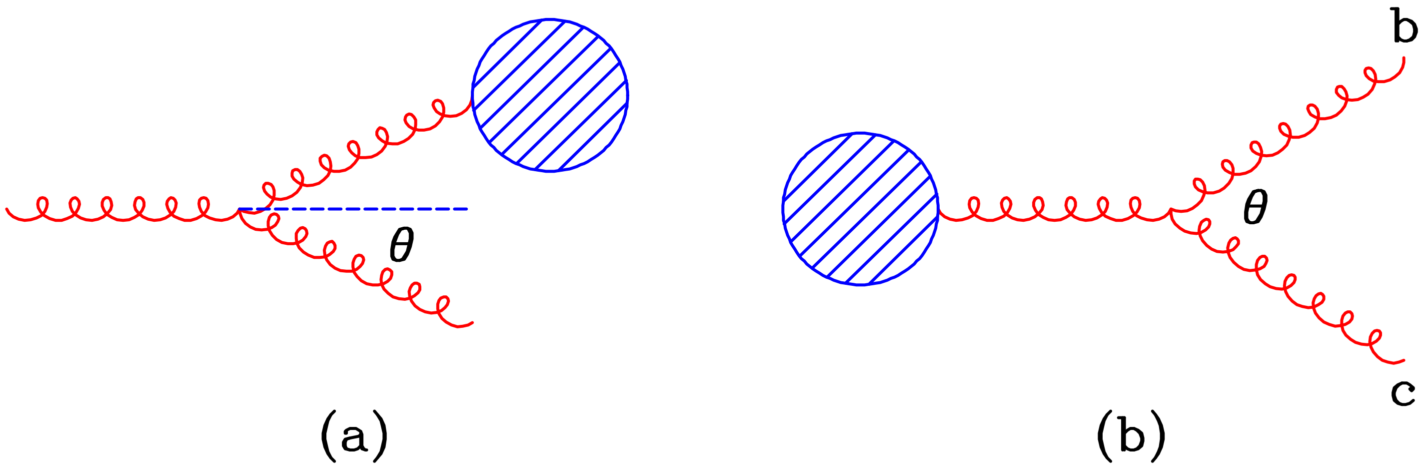
- Corresponding to asymptotic freedom at high momentum scales, we have infra-red slavery: $\alpha_S(Q)$ becomes large a low momenta, (long distances). Perturbation theory is not reliable for large α_S, so non-perturbative methods, (e.g. lattice) must be used.



- Important low momentum scale phenomena
  - ⋆ Confinement: partons (quarks and gluons) found only in colour singlet bound states, hadrons, size $\sim$ 1 fm. If we try top separate them it becomes energetically favourable to create extra partons from the vacuum.
  - ⋆ Hadronization: partons produced in short distance interactions re-organize themselves to make the observed hadrons.

Infrared divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



⋆ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -2E_a E_c (1 - \cos\theta) \leq 0\ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity).

If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -2E_a E_c(1 - v_a \cos\theta) \leq 0 \; ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- Timelike branching: gluon splitting on outgoing line (b)

  $$p_a^2 = 2E_b E_c(1 - \cos\theta) \geq 0 \; .$$

  Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - ⋆ Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - ⋆ Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.
- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - ⋆ Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  - ⋆ Dimensional regularization: analogous to that used for ultraviolet divergences, except we must *increase* dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$.

Recap

- Colour degree of freedom is motivated by $R^{e^+e^-}$, baryon spectroscopy.
- QCD is an SU(3) gauge theory of quarks (3 colours) and gluons (8 colours,self interacting)
- QCD is an SU(3) gauge theory of quarks (3 colours) and gluons (8 colours,self interacting)
- Renormalization of dimensionless observables depending on a single large scale implies that the scale dependence enters through the running coupling.
- Asymptotic freedom implies that IR-safe quantities can be calculated in perturbation theory.
- $\alpha(M_Z) \simeq 0.118$ in five flavour $\overline{MS}$-renormalization scheme.
- Perturbative QCD has infrared singularities due to collinear or soft parton emission. We can calculate infra-red safe or factorizable quantities in perturbation theory.