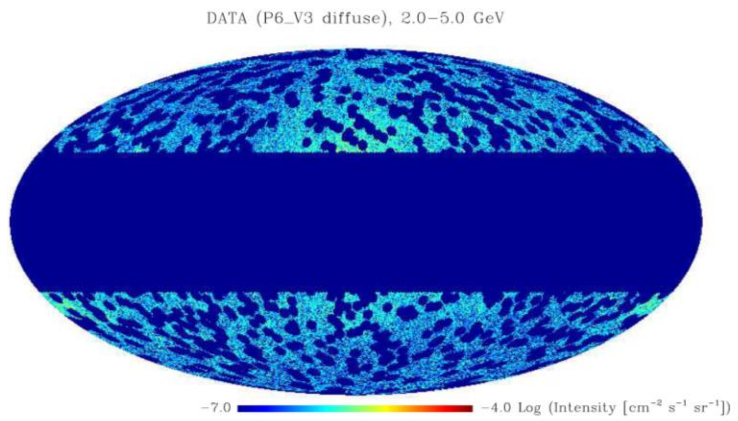

Correlated Statistical Noise in Power Spectrum Measurements

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UCI University of California, Irvine

Motivations

- We want to characterize structure (spatial and spectral) of high energy cosmic radiation.
  - How to quantify structures?
  - How to determine significance of structures/anisotropy?

- Spatial recognition of sources by correlating radiation with known source structures.

- Identify or constrain the presence of exotic or unpredicted sources.

Techniques for Angular Power Spectrum

- The fundamental techniques I describe today can be applied to any function of angular distribution.
  - Kernels like spherical harmonic transforms or wavelet transforms.
  - N-point correlation functions.
  - Power spectrum, bispectrum, etc.

- The power spectrum is a natural first choice to develop.
  - It is a well-studied observable in cosmological applications.
  - Distant extragalactic sources (large-scale-structure) is approximately Gaussian distributed.
    - Then power spectrum components of the sources are approximately statistically independent and contain nearly all spatial information.
  - What about non-Gaussian distributions? Galactic? Local group?

Fundamental Questions

1. How much data (i.e., how many events) is required to make robust spatial measurements?
   - For a given distribution of sources?

2. How much is required to detect particular spectral features?

3. What is an ideal "spatial recognition" instrument?

These require a statistical framework for uncertainty estimation.

- Simulations/mocks provide robust estimations for individual experiments, but are expensive for broad sensitivity studies.
- Is there no **analytic framework** for "distribution sensitivity"?

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Good News–THERE IS!!!

Pioneering Approach: Fermi-LAT (2012)

- Approximate analytic power spectrum uncertainties have long existed (Knox 1995, Hivon+ 2002).
  - They were found accurate enough to be used with the WMAP9 analysis.

    WMAP Collaboration, Astrophys.J.Suppl. 208 (2013) 20 & PTEP 2014 (2014) 6, 06B102

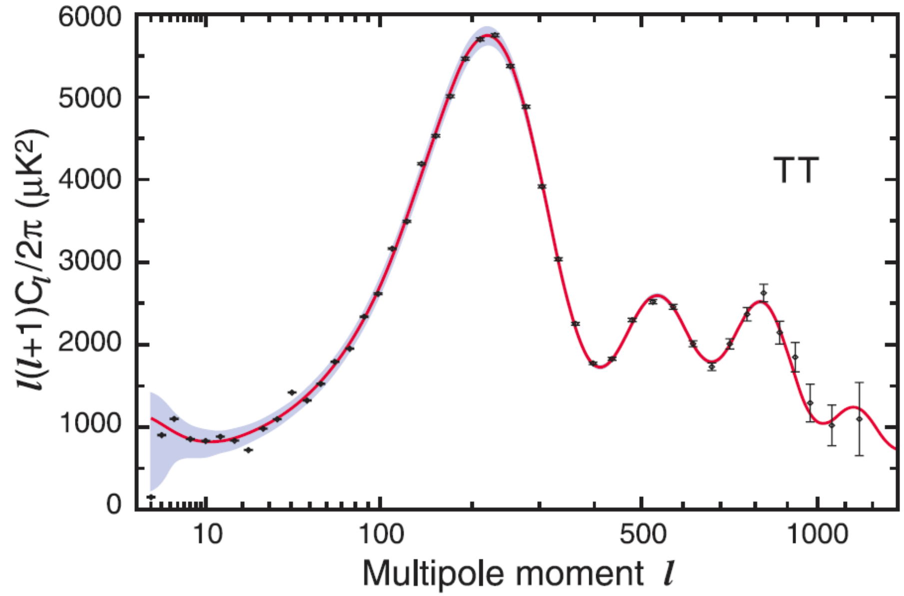

Fig. 5 Nine-year angular power spectrum of the CMB temperature (adapted from [37]). While we measure C_ℓ at each ℓ in $2 \le \ell \le 1200$, the points with error bars show the binned values of C_ℓ for clarity. The error bars show the standard deviation of C_ℓ from instrumental noise, $[2(2C_\ell N_\ell + N_\ell^2)/(2\ell+1)f_{\mathrm{sky},\ell}^2]^{1/2}$. The shaded area shows the standard deviation from the cosmic variance term, $[2C_\ell^2/(2\ell+1)f_{\mathrm{sky},\ell}^2]^{1/2}$ (except at very low ℓ where the 68% CL from the full non-Gaussian posterior probability is shown). The solid line shows the theoretical curve of the best-fit ΛCDM cosmological model.

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$$\mathrm{Cov}[C_\ell, C_{\ell'}] = \underbrace{\frac{2\delta_{\ell\ell'}}{(2\ell+1)f_{\mathrm{sky}}^2}}_{\substack{\text{Information}\\ \text{Prefactor:}\\ \text{Inverse \# of}\\ \text{Angular Modes}}} \Big[\underbrace{N_\ell^2 + 2N_\ell C_\ell}_{\substack{\text{Uncorrelated}\\ \text{Noise}}} + \underbrace{C_\ell^2}_{\substack{\text{Gaussian}\\ \text{Cosmic}\\ \text{Variance}\\ \text{Term}}}\Big]$$

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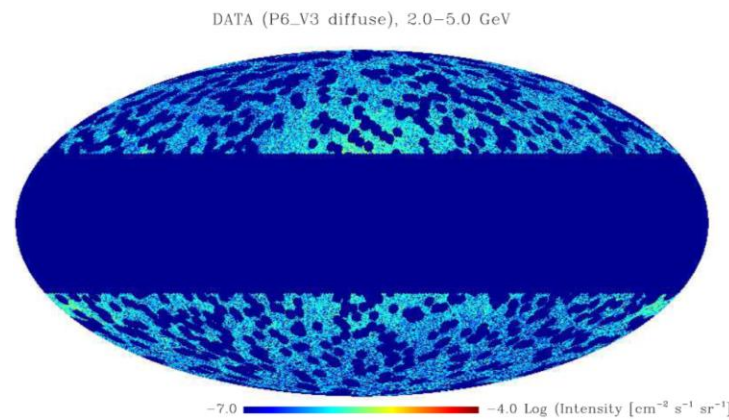
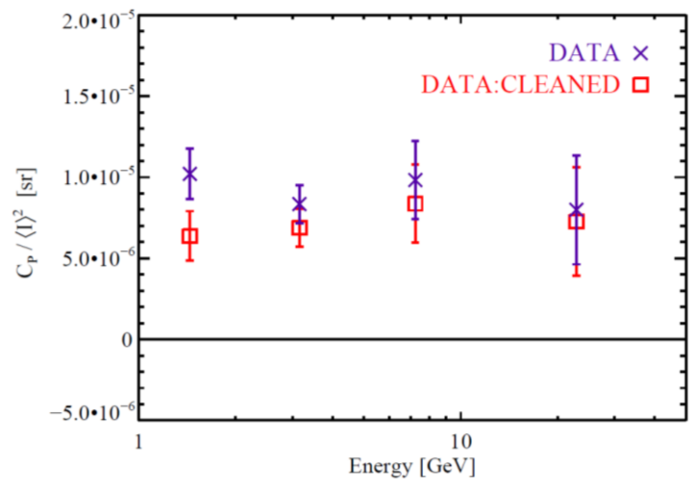
Pioneering Approach: Fermi-LAT (2012)

- Application to γ rays:
  - Photon shot noise applies as uncorrelated noise.
  - Add effects to account for the instrument's angular resolution.
  - Increase signal-to-noise with:
    - large foreground mask,
    - wide energy bars,
    - average over multipole range.
      - Weighted average in each energy bin shows significant power consistent with no energy modulation.



Fermi-LAT Collaboration, PRD85 (2012) 083007

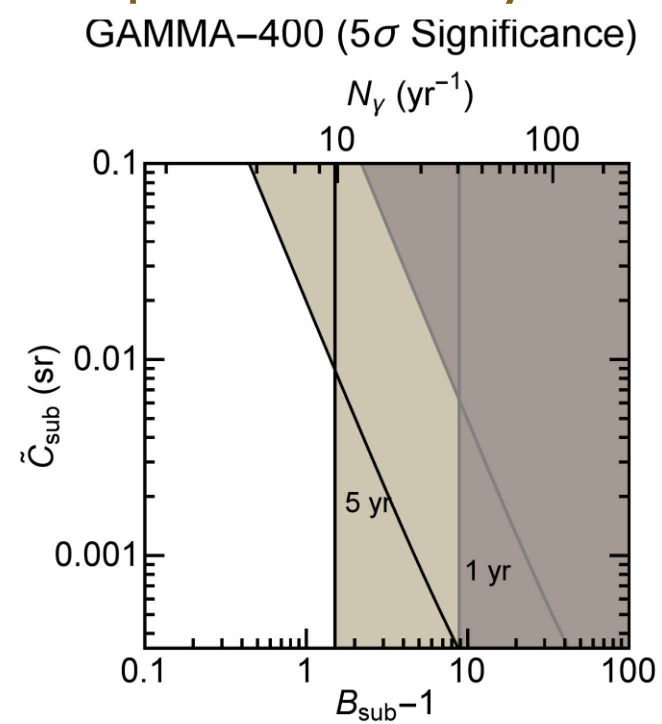


Room for Improvements

1. The analytic analysis is valid for a Gaussian source distribution.
   - What are the effects on the power spectrum measurement of non-Gaussianities that are in the data?
   - How can we estimate the non-Gaussianity?
2. Cosmic variance is not present for distribution measurement, but is necessary for parameter estimation of source modeling.
   - This is only an academic point because cosmic variance is negligible for these shot noise dominated measurements.
   - My results will not contain cosmic variance, though there is consistent methodology for it when appropriate.
3. There **must** be statistical dependence between the C_ℓ.
   - They are all estimated with the same finite point data.
   - How can this be estimated? What are the effects?

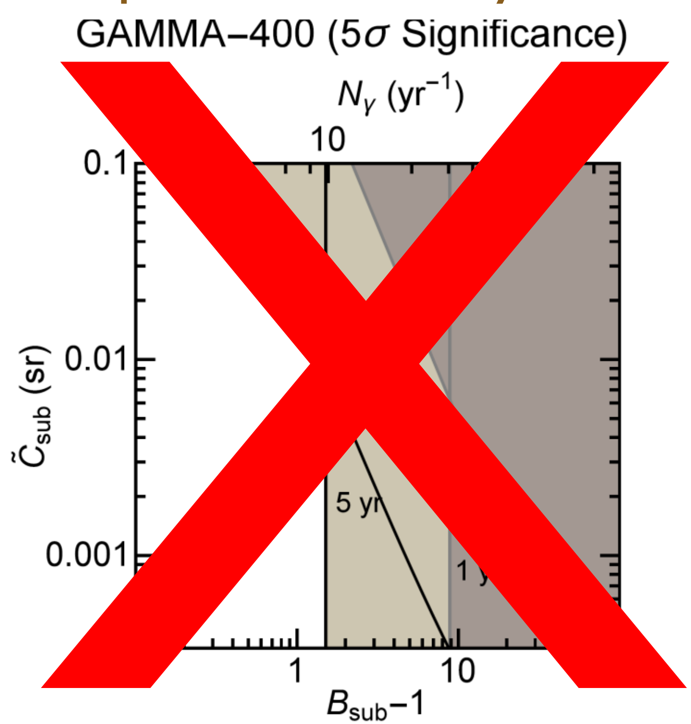
Aren't All These Effect Negligible?

- Alas, early sensitivity studies on signals with planned future missions using this analytic error analysis give impossible results.
  - Predictions of sensitivity to large power spectra with only few events detected.



Aren't All These Effect Negligible?

- Alas, early sensitivity studies on signals with planned future missions using this analytic error analysis give impossible results.
  - Predictions of sensitivity to large power spectra with only few events detected.
  - Catastrophic failure with parameters within an order-of-magnitude of Fermi-LAT's. We had better estimate the effects on the Fermi-LAT measurement.



Outline to Solution

1) A statistical framework for high-energy cosmic observation–the spatial PDF.

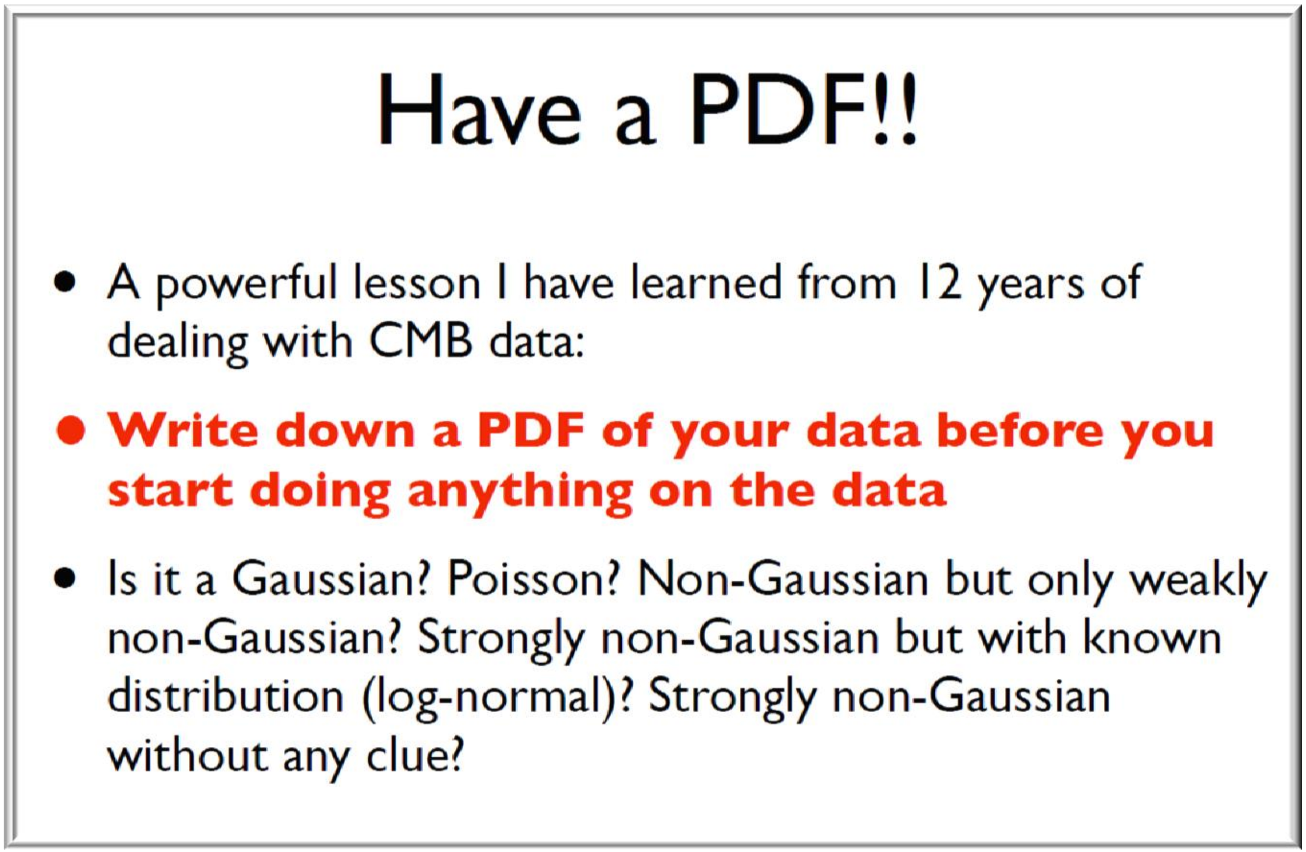
2) Analytic power spectrum estimation and covariance.

3) Detectability/sensitivity analysis and correlated noise effects.

4) Sensitivity to spectral effects–a power spectrum line search.

Statistical Framework for Cosmic γ rays

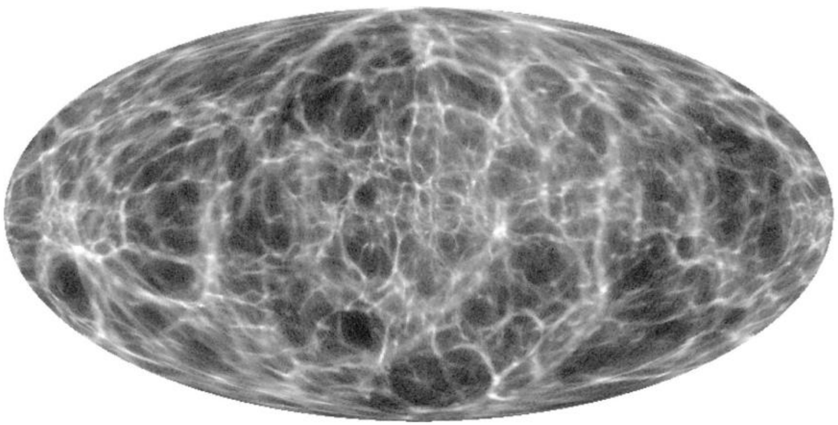
- A slide from Eiichiro Komatu's talk at the 1st Anisotropic Universe Workshop 2.5 years ago.



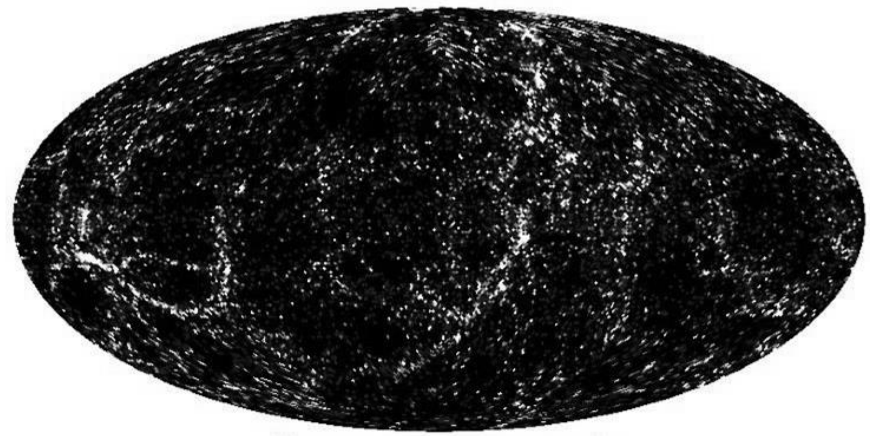
The Spatial PDF of γ rays

- It is determined by the distribution of the sources.
  - Radiation events are a sampling of sources.

Francisco-Shu Kitaura et al., MNRAS 427, L35 (2012)



A skymap (catalog) of sources.

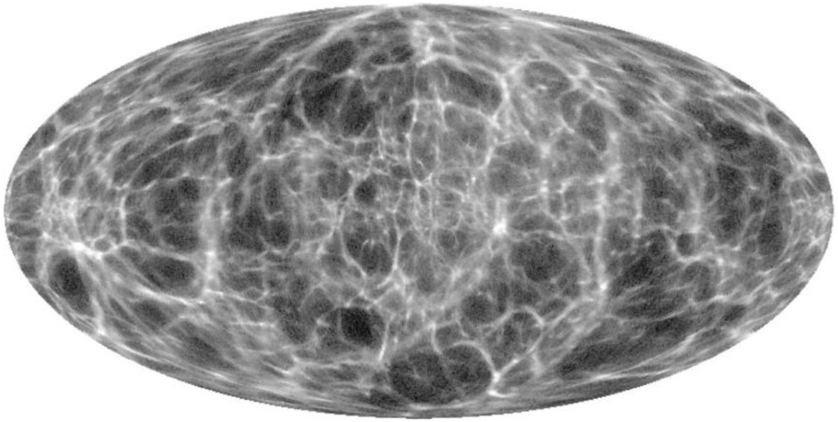


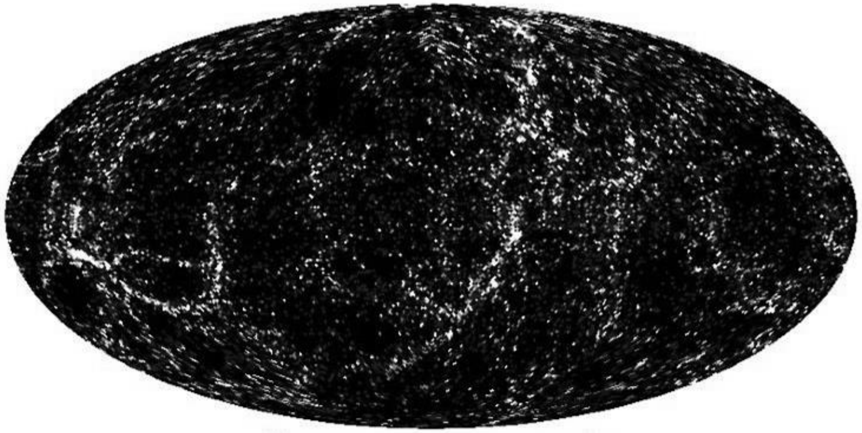
Sample gamma-ray events observed from those sources.

Given N events, what can we infer about the full **skymap**?

γ-ray Observations as a Statistical Point Process

- Only two reasonable assumptions used:





1. The intensity skymap of sources is **stationary**.

Requires methods to identify and remove transient signals.

2. The position of each event is **statistically independent**.

Event position PDF sourced by intensity $I(\boldsymbol{n})$ and exposure map $d\varepsilon/d\Omega$.

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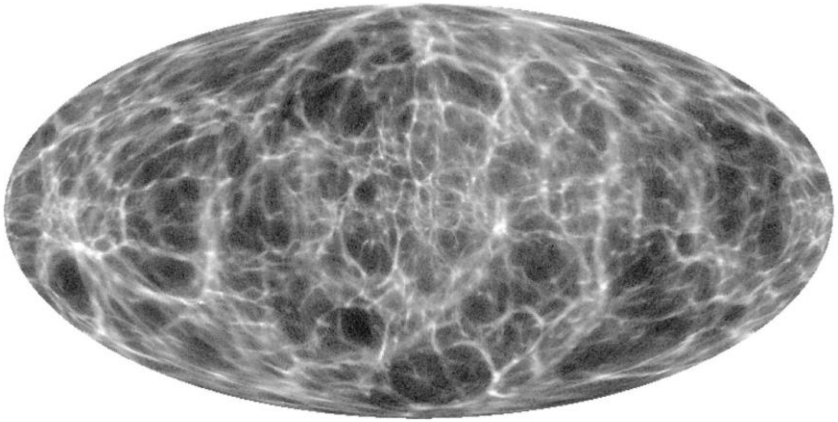

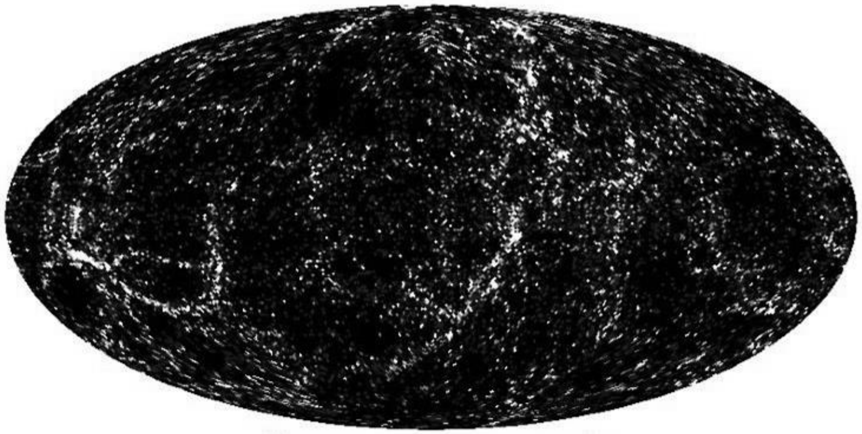

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$$P(\boldsymbol{n}) = \frac{1}{N_{\text{exp}}} I(\boldsymbol{n}) \frac{d\varepsilon}{d\Omega}(\boldsymbol{n})$$

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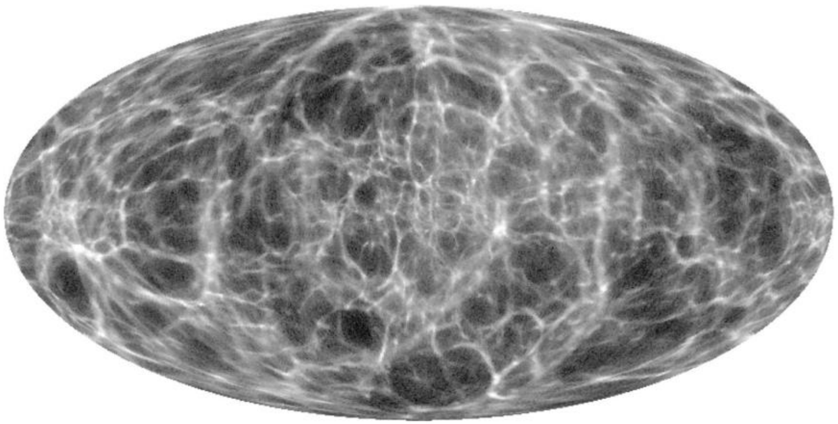

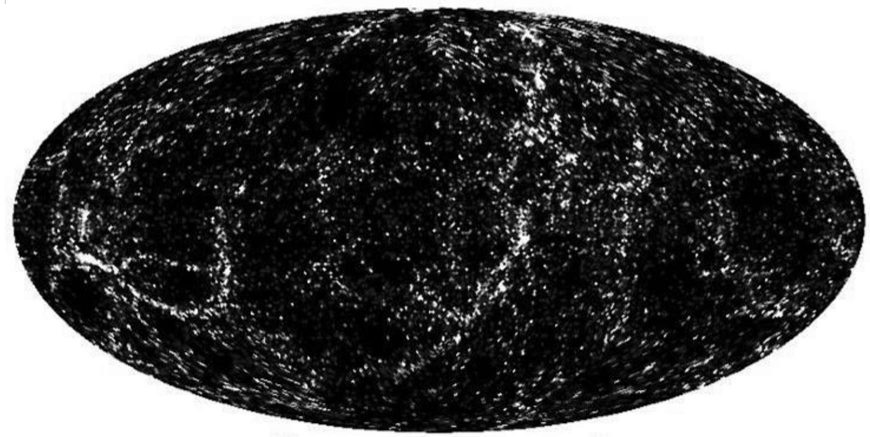

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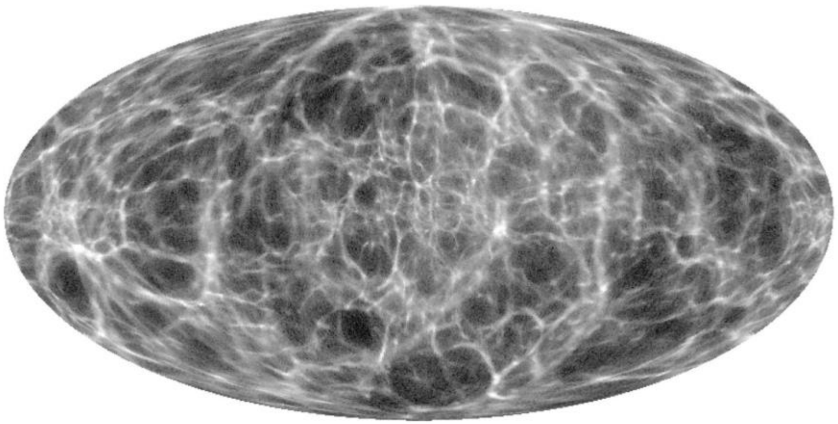
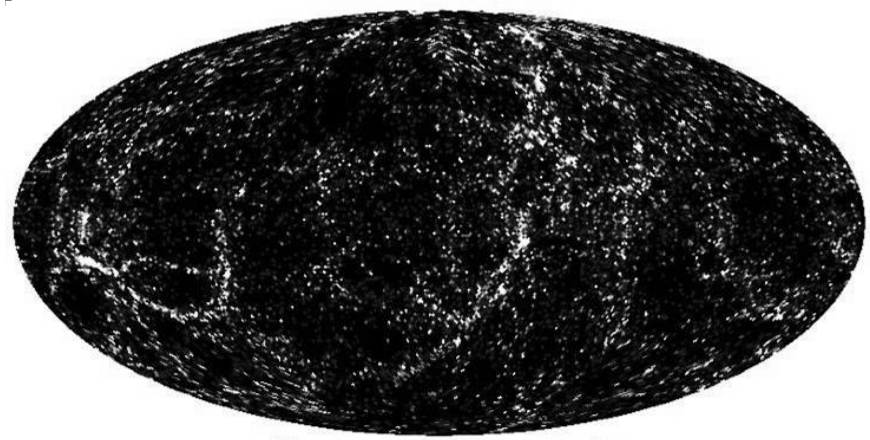
$$P(\boldsymbol{n}) = \frac{1}{N_{\text{exp}}} I(\boldsymbol{n}) \frac{d\varepsilon}{d\Omega}(\boldsymbol{n}) \xrightarrow[\frac{d\varepsilon}{d\Omega}\text{ constant}]{} \tilde{I}(\boldsymbol{n}) = \frac{I(\boldsymbol{n})}{\langle I \rangle}$$

γ-ray Observations as a Statistical Point Process

- Only two reasonable assumptions used:

1. The intensity skymap of sources is **stationary**.

Requires methods to identify and remove transient signals.

2. The position of each event is **statistically independent**.

Event position PDF sourced by intensity $I(\boldsymbol{n})$ and exposure map $d\varepsilon/d\Omega$.

$$P(\boldsymbol{n}) = \tilde{I}(\boldsymbol{n})$$

Now any function of event positions can be statistically analyzed.

Ensemble Marginalization over Data

- Data is simply a list of angular positions $\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_N$.

- Marginalize any function $f(\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_N)$ over the data:

1. Fixed-exposure statistics:
   - The number of events is a random statistic (typically Poisson distributed) with PDF $P_P(N)$.

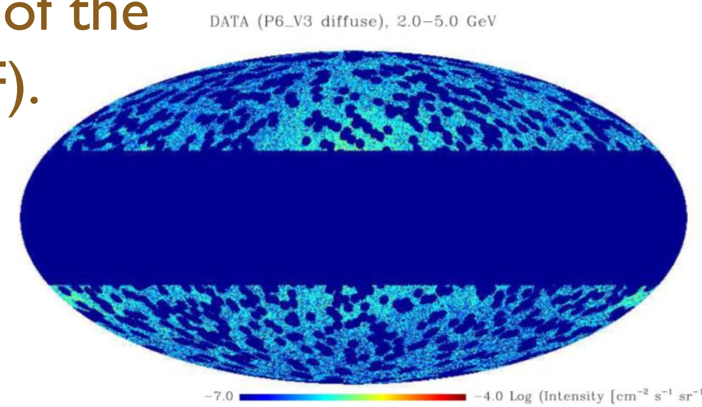
$$\langle f\rangle(\mathcal{E}) = \int dN d\boldsymbol{n}_1 d\boldsymbol{n}_2 \cdots d\boldsymbol{n}_N \, P_P(N) P(\boldsymbol{n}_1) P(\boldsymbol{n}_2) \cdots P(\boldsymbol{n}_N) f(\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_N, \mathcal{E})$$

2. Fixed-count statistics:
   - Useful if observable is independent of exposure
     - e.g., Dimensionless Power Spectrum $\tilde{C}_\ell$ with uniform exposure data.
   - More convenient for sensitivity analyses.

$$\langle f\rangle(N) = \int d\boldsymbol{n}_1 d\boldsymbol{n}_2 \cdots d\boldsymbol{n}_N \, P(\boldsymbol{n}_1) P(\boldsymbol{n}_2) \cdots P(\boldsymbol{n}_N) f(\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_N)$$

Power Spectrum of a Masked Sky

- Instrument Response:
  - W_ℓ is Legendre polynomial transform of the instrument point-spread-function (PSF).
  - σ_b is angular diameter of PSF.
- Analysis:
  - f_{sky} is the fraction of unmasked sky.
  - Probe multipoles over the range $\ell_{\min} \le \ell \le \ell_{\max}$.



- An unbiased estimator $\hat{\tilde{C}}_{\ell,N}$ with $\left\langle \hat{\tilde{C}}_{\ell,N} \right\rangle = \tilde{C}_\ell$:

$$\hat{\tilde{C}}_{\ell,N} = \frac{1}{1-\frac{1}{N}} \frac{f_{\text{sky}}}{W_\ell^2}\left[\tilde{C}_{\ell,N,\text{raw}} - \frac{4\pi}{N}\right] = \frac{4\pi f_{\text{sky}}}{N(N-1)W_\ell^2} \sum_i \sum_{j\neq i} P_\ell(\boldsymbol{n}_i \cdot \boldsymbol{n}_j)$$

Exact Statistical Covariance of $\hat{\tilde{C}}_{\ell,N}$

$$\mathrm{Cov}\left[\hat{\tilde{C}}_{\ell_1,N}, \hat{\tilde{C}}_{\ell_2,N}\right] = \frac{(4\pi)^2}{N(N-1)}\left\{2\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\right] + 4(N-2)\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi}\right]\right\}$$

Neglecting mask and PSF for now.

Diagonal terms in agreement with predecessor analytic methods.

Exact Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$

$$\mathrm{Cov}\left[\hat{\hat{C}}_{\ell_1,N}, \hat{\hat{C}}_{\ell_2,N}\right] = \frac{(4\pi)^2}{N(N-1)}\left\{2\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1} + \tilde{C}^{(2)}_{\ell_1\ell_2}\right] + 4(N-2)\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi}\right]\right\}$$

Neglecting mask and PSF for now.

Shot term corrected by neighboring multipoles.

New Spherical Tensors

Composite Power Spectrum

$$\tilde{C}^{(2)}_{\ell_1\ell_2} = \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{2\ell'+1}{4\pi}\begin{pmatrix}\ell_1 & \ell_2 & \ell'\\ 0 & 0 & 0\end{pmatrix}^2 \tilde{C}_{\ell'}$$

Exact Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$

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Neglecting mask and PSF for now.

Non-Gaussianities modify the signal term.

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Open Bispectrum

$$\tilde{C}^{(3)}_{\ell_1\ell_2} = \frac{1}{\sqrt{(2\ell_1+1)(2\ell_2+1)}} \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sqrt{\frac{2\ell'+1}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell' \\ 0 & 0 & 0\end{pmatrix} \tilde{B}_{\ell_1\ell_2\ell'}$$

Exact Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$

$$\mathrm{Cov}\left[\hat{\hat{C}}_{\ell_1,N}, \hat{\hat{C}}_{\ell_2,N}\right] = \frac{(4\pi)^2}{N(N-1)}\left\{2\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1} + \tilde{C}^{(2)}_{\ell_1\ell_2} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right] + 4(N-2)\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi} + \frac{\tilde{C}^{(3)}_{\ell_1\ell_2}}{4\pi} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right]\right\}$$

Neglecting mask and PSF for now.

Unconnected part of the trispectrum relevant only for large power.

New Spherical Tensors

Composite Power Spectrum

$$\tilde{C}^{(2)}_{\ell_1\ell_2} = \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{2\ell'+1}{4\pi}\begin{pmatrix}\ell_1 & \ell_2 & \ell' \\ 0 & 0 & 0\end{pmatrix}^2 \tilde{C}_{\ell'}$$

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Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$ $(N \gg 1)$

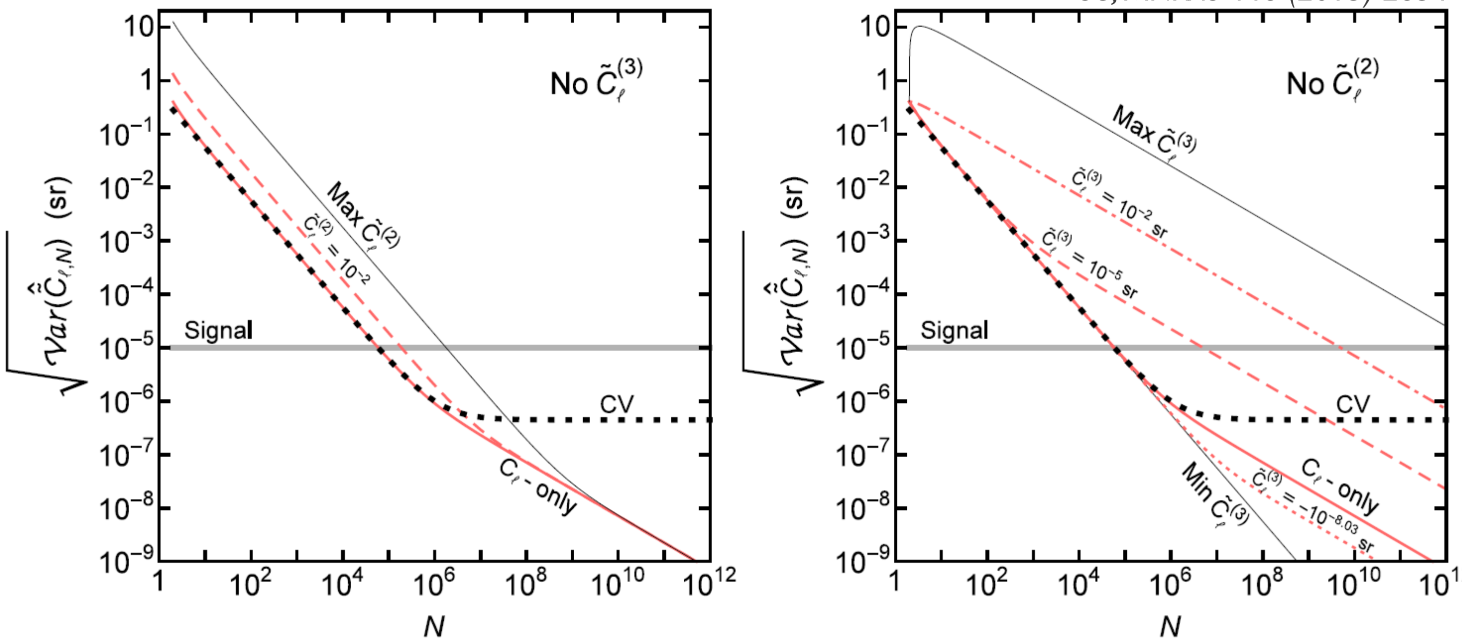
$$\mathrm{Cov}\left[\hat{\hat{C}}_{\ell_1,N}, \hat{\hat{C}}_{\ell_2,N}\right] = (4\pi)^2 \left\{ \frac{2}{N^2}\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1} + \tilde{C}^{(2)}_{\ell_1\ell_2} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right] + \frac{4}{N}\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi} + \frac{\tilde{C}^{(3)}_{\ell_1\ell_2}}{4\pi} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right]\right\}$$

- New terms
  - add corrections to the diagonal part
  - provide the previously missing non-diagonal components

- Unbiased estimators for all these terms from the data have been determined.
- This provides a non-parametric method for measuring power spectra.

Diagonal Corrections Can Be Important

SC, MNRAS 448 (2015) 2854



- Example uncertainty evolution at $\ell = 500$ with $\tilde{C}_\ell = 10^{-5}$ sr.

- Unbiased estimators of these new spectra allow for unparametric uncertainty estimation from the data without any source model.

Instrument Response and Masking

- For instrument sensitivity analysis, we don't have data, so it is helpful to use source models.

- To improve intuition, consider the simplest scenario:
  - $\tilde{C}$ is a white spectrum,
  - $\tilde{C} \ll \frac{4\pi}{2\ell+1} \ll 1$, (okay for $1 \ll \ell \lesssim 10^5$)
  - the bispectra are negligible for the $\tilde{C}_\ell$ measurement.
    - assumes N is small enough that the shot term dominates the non-diagonal.

- These assumptions are consistent with current measurements, and are predicted by some source scenarios.

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    - assumes N is small enough that the shot term dominates the non-diagonal.
- These assumptions are consistent with current measurements, and are predicted by some source scenarios.

$$\tilde{C}^{(2)}_{\ell\ell'} = \left(W_\ell^2 W_{\ell'}^2 - \frac{\delta_{\ell\ell'}}{2\ell+1}\right)\frac{\tilde{C}}{4\pi f_{\text{sky}}} - \frac{\delta_{\ell\ell'}}{2\ell+1}\left(1 - \frac{1}{f_{\text{sky}}}\right)$$

$$\tilde{C}^{(3)}_{\ell\ell'} = -\frac{\delta_{\ell\ell'}}{2\ell+1}\left(1 - \frac{1}{f_{\text{sky}}}\right)\frac{W_\ell^2}{f_{\text{sky}}}\tilde{C}$$

$\tilde{C}$ With Negligible Bispectrum

- The multipole covariance of white power spectrum measurements:

$$\mathrm{Cov}\left[\widehat{\widetilde{C}}_{\ell_1,N}, \widehat{\widetilde{C}}_{\ell_2,N}\right] \simeq \frac{2}{f_{\mathrm{sky}}}\left[\frac{\delta_{\ell\ell'}}{2\ell+1}\frac{4\pi f_{\mathrm{sky}}}{NW_\ell^2}\left(\frac{4\pi f_{\mathrm{sky}}}{NW_\ell^2}+2\widetilde{C}\right)\right.$$

$$\left.+\left(\frac{4\pi f_{\mathrm{sky}}}{N}\right)^2\frac{\tilde{C}}{4\pi}\right]$$

Old analytic formula

New correlated noise of a white Gaussian spectrum.

Note lack of ℓ dependence.

Consequences of Correlated Noise

- Explore sensitivity of experiments to $\tilde{C}$.
  Take variance-weighted mean over a range of ℓ.

$$\bar{\tilde{C}} = \frac{1}{A} \sum_{\ell=\ell_{\min}}^{\ell_{\max}} \frac{\tilde{C}_\ell}{\mathcal{V}\mathrm{ar}[\tilde{C}_\ell]},$$

$$A = \sum_{\ell=\ell_{\min}}^{\ell_{\max}} \frac{1}{\mathcal{V}\mathrm{ar}[\tilde{C}_\ell]}.$$

- For a white spectrum, $\bar{\tilde{C}} = \tilde{C}$.

Uncertainty of $\bar{\tilde{C}}$ neglecting PSF and mask

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] = \frac{1}{A^2} \sum_{\ell=\ell_{\min}}^{\ell_{\max}} \sum_{\ell'=\ell_{\min}}^{\ell_{\max}} \frac{\mathrm{Cov}[\tilde{C}_\ell, \tilde{C}_{\ell'}]}{\mathcal{V}\mathrm{ar}[\tilde{C}_\ell]\mathcal{V}\mathrm{ar}[\tilde{C}_{\ell'}]}$$

For our modelled scenario:

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] = \frac{2}{\ell_{\max}^2 - \ell_{\min}^2}\left[\left(\frac{4\pi}{N}\right)^2 + 2\left(\frac{4\pi}{N}\right)\tilde{C}\right] + \frac{8\pi\tilde{C}}{N^2}$$

The extra term from the non-diagonal covariance.

Uncertainty of $\bar{\tilde{C}}$: Mask and PSF Effects

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] = \frac{2}{\ell_{\max}^2 - \ell_{\min}^2}\left[\left(\frac{4\pi}{N}\right)^2\left(1 + \Lambda^2 \tilde{C}\right) + 2\left(\frac{4\pi}{N}\right)\tilde{C}\right]$$

The result has a simple prescription:

Uncertainty of $\bar{\tilde{C}}$: Mask and PSF Effects

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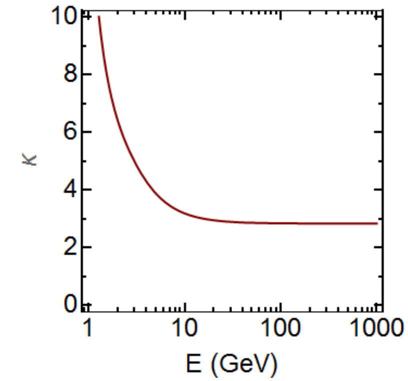


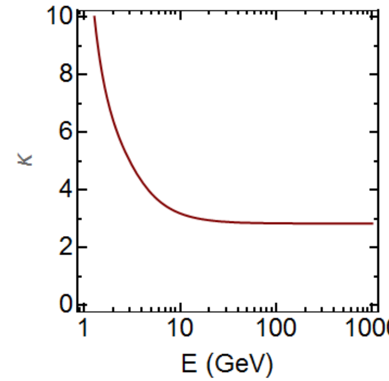
The result has a simple prescription:

1. Replace 4π with the *shot parameter*.

$$\kappa = 4\pi f_{\mathrm{sky}} \frac{2}{W_{\ell_{\mathrm{min}}}^2 + W_{\ell_{\mathrm{max}}}^2}$$

Uncertainty of $\bar{\tilde{C}}$: Mask and PSF Effects

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] = \frac{2}{\ell_{\max}^2 - \ell_{\min}^2}\left[\left(\frac{4\pi}{N}\right)^2 \left(1 + \Lambda^2 \tilde{C}\right) + 2\left(\frac{4\pi}{N}\right)\tilde{C}\right]$$



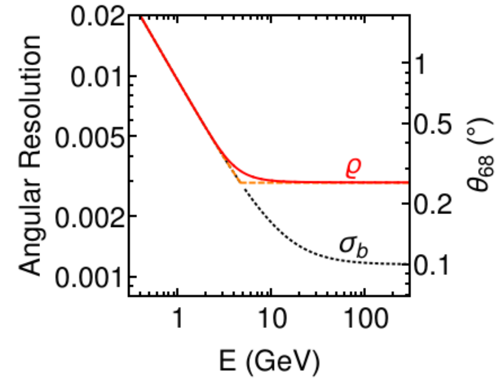
The result has a simple prescription:

1. Replace 4π with the *shot parameter.*

$$\kappa = 4\pi f_{\mathrm{sky}} \frac{2}{W_{\ell_{\min}}^2 + W_{\ell_{\max}}^2}$$

2. Replace the coefficient with the *resolution parameter* ϱ.

$$\frac{\varrho^2}{f_{\mathrm{sky}}} = \frac{\sigma_b^2}{f_{\mathrm{sky}}} \frac{W_{\ell_{\min}}^2 + W_{\ell_{\max}}^2}{W_{\ell_{\min}}^2 - W_{\ell_{\max}}^2} \approx \frac{1}{f_{\mathrm{sky}}} \max\left[\sigma_b^2, \frac{2}{\ell_{\max}^2 - \ell_{\min}^2}\right]$$



Uncertainty of $\bar{\tilde{C}}$: Mask and PSF Effects

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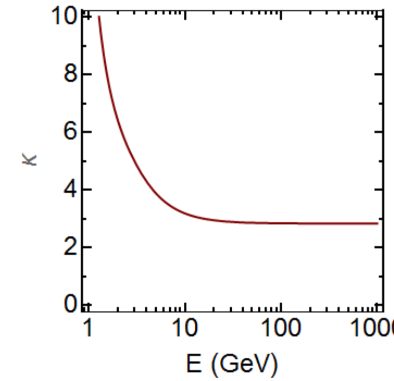
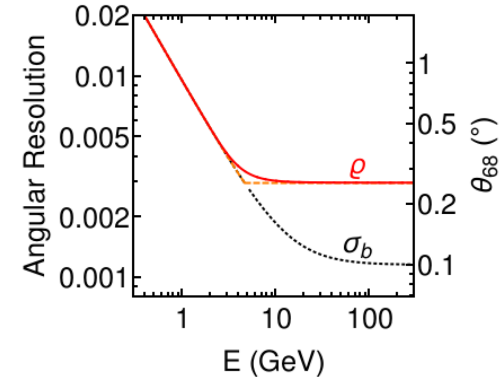
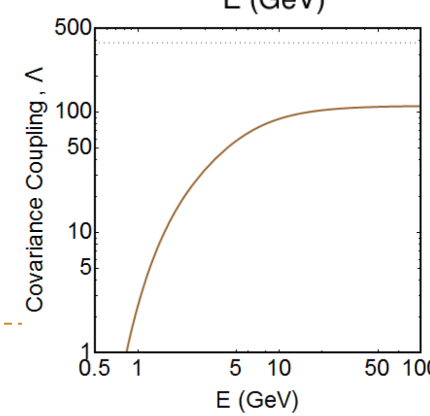
3. We introduce the covariance coupling.

$$\Lambda = \frac{W_{\ell_{\min}}^2 + W_{\ell_{\max}}^2}{\sqrt{8\pi}\varrho}$$







Instrument Design Lessons

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] = \frac{\varrho^2}{f_{\mathrm{sky}}}\left[\left(\frac{\kappa}{N}\right)^2\left(1+\Lambda^2\tilde{C}\right)+2\left(\frac{\kappa}{N}\right)\tilde{C}\right]$$

In shot-dominated regime, 3 scales determine statistical reach:

$$\mathcal{V}\mathrm{ar}\left[\bar{\tilde{C}}\right] \propto \varrho^2 + \varrho^2\Lambda^2\tilde{C} \approx \max\left[\sigma_b^2, \frac{2}{\ell_{\mathrm{max}}^2 - \ell_{\mathrm{min}}^2}, 2\left(\frac{\mathrm{W}_{\ell_{\mathrm{min}}}^2 + \mathrm{W}_{\ell_{\mathrm{max}}}^2}{2}\right)^2 \frac{\tilde{C}}{4\pi}\right]$$

Resolution Limited

Gains from improving angular resolution.

Angular Band Limited

Gains from increasing multipole range.

Signal Limited

No further gains from access to additional angular modes.

Instrument Design Lessons

Number of events to detect $\tilde{C}$ to N_σ significance:

$$N_{\rm det} = \frac{\kappa}{\tilde{C}} \frac{1+\Lambda^2\tilde{C}}{\sqrt{\frac{1}{\mathcal{R}_{\rm exp}}\left(1+\Lambda^2\tilde{C}\right)+1}-1}$$

Fundamental Experiment Resolution Scale

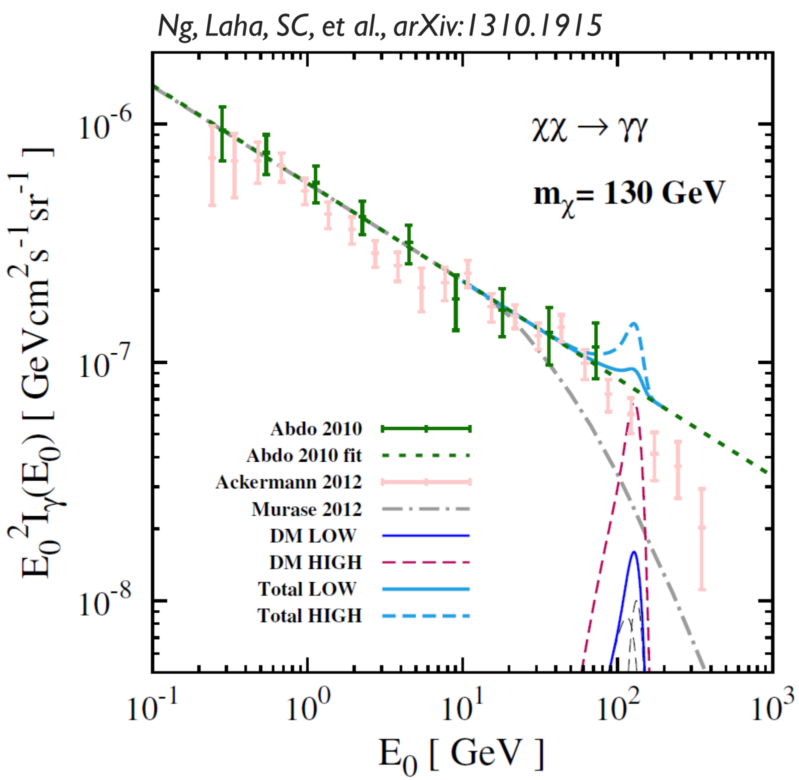
$$\mathcal{R}_{\rm exp} \equiv \frac{N_\sigma \varrho}{\sqrt{f_{\rm sky}}}$$

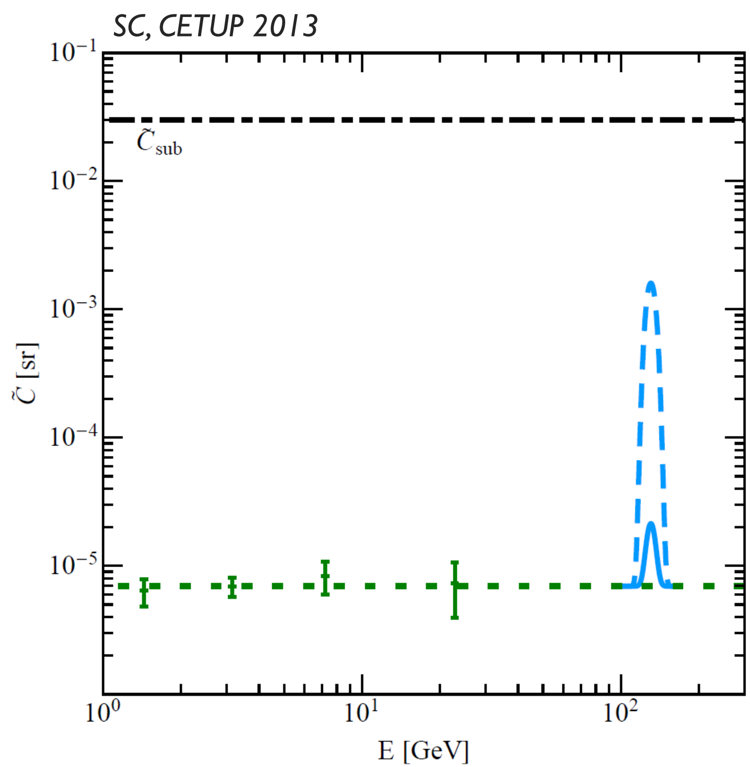
Example back-of-envelope estimate: in diagonal covariance case $\Lambda^2\tilde{C} \ll 1$,

$$N_{\rm det} \approx \frac{\kappa}{\tilde{C}} \times \begin{cases} \mathcal{R}_{\rm exp}, & \mathcal{R}_{\rm exp} \ll 1 \\ 2\mathcal{R}_{\rm exp}^2, & \mathcal{R}_{\rm exp} \gg 1 \end{cases}$$

Design the resolution so that $\mathcal{R}_{\rm exp} \ll 1$, then design exposure so that $N \gtrsim N_{\rm det}$.

Spectral Signals: Line in $I(E)$ vs. $\tilde{C}(E)$

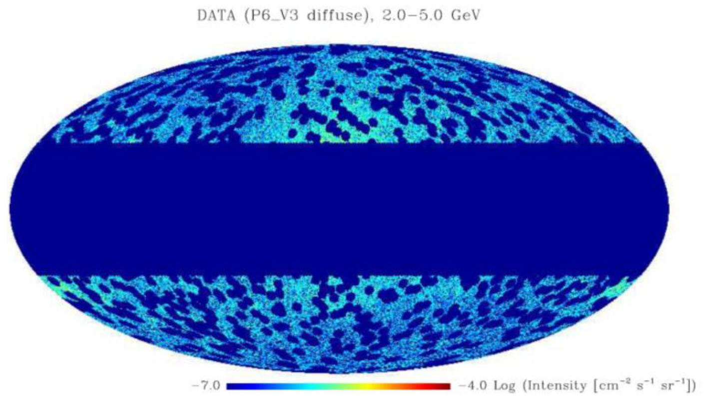


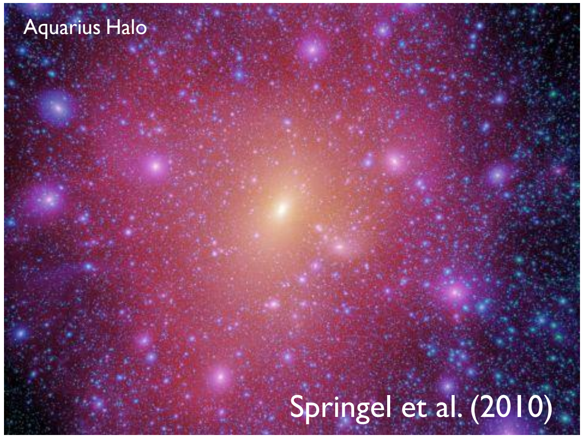


- If relative brightness of line sources is low, but structure is significant, they power spectrum can be more sensitive.
- Uncertainty of $\tilde{C}_\ell$ measurements is crucial to understand sensitivity.

Diffuse Background & Line Signal Models

- Dark Matter Model
  - Consider a signal dominated by annihilation in the Galactic halo.
  - Anisotropy of the signal is produced by halo substructure.
  - Smooth component of Galactic dark matter halo provides *flux* Φ_{sm} and no small scale structure at high latitudes.
  - Substructure provides a *flux boost* B_{sub} and distribution *power spectrum* $\tilde{C}_{\mathrm{sub}}$.





$$\Phi = \Phi_{\mathrm{bkg}} + B_{\mathrm{sub}}\Phi_{\mathrm{sm}}$$

$$\tilde{C} = \left(\frac{\Phi_{\mathrm{bkg}}}{\Phi}\right)^2 \tilde{C}_{\mathrm{bkg}} + \left(\frac{(B_{\mathrm{sub}}-1)\Phi_{\mathrm{sm}}}{\Phi}\right)^2 \tilde{C}_{\mathrm{sub}}$$

Line Strengths

- Recall that $\tilde{C}$ has a complicated dependence on B_{sub} and $\tilde{C}_{\text{sub}}$. Simpler in terms of line strengths.

$$\Phi = \Phi_{\text{bkg}} + B_{\text{sub}}\Phi_{\text{sm}}$$

$$\tilde{C} = \left(\frac{\Phi_{\text{bkg}}}{\Phi}\right)^2 \tilde{C}_{\text{bkg}} + \left(\frac{(B_{\text{sub}} - 1)\Phi_{\text{sm}}}{\Phi}\right)^2 \tilde{C}_{\text{sub}}$$

- Flux Strength: $S_\Phi = \dfrac{\Phi_{\text{DM}}}{\Phi_{\text{bkg}}} = B_{\text{sub}} \dfrac{\Phi_{\text{sm}}}{\Phi_{\text{bkg}}}$.

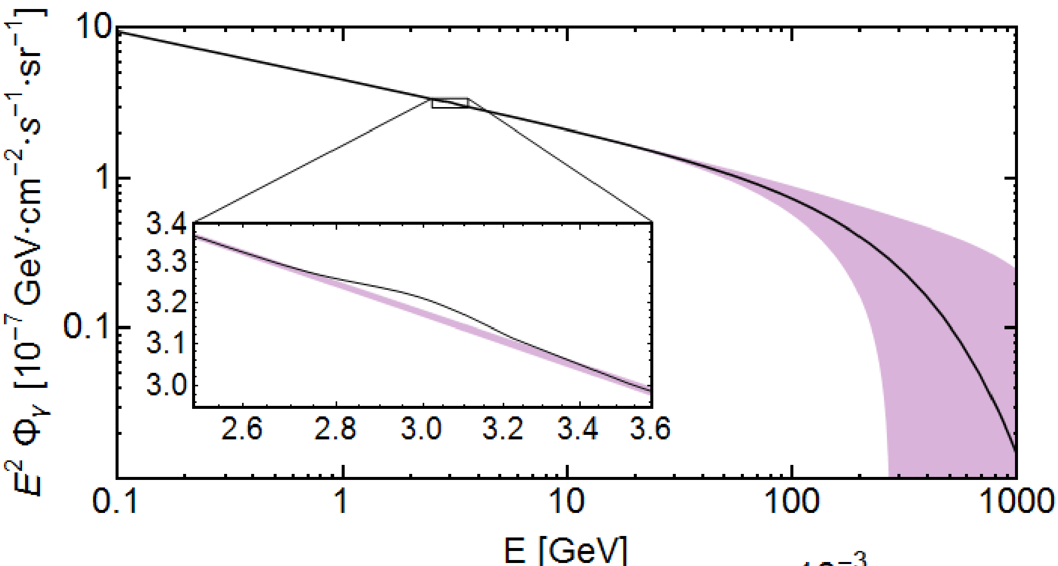
- Anisotropy Strength: $S_C = (B_{\text{sub}} - 1)^2 \left(\dfrac{\Phi_{\text{sm}}}{\Phi_{\text{bkg}}}\right)^2 \dfrac{\tilde{C}_{\text{sub}}}{\tilde{C}_{\text{bkg}}}$.

- In terms of these observables,

$$\Phi = \Phi_{\text{bkg}}(1 + S_\Phi), \;\; \tilde{C} = \tilde{C}_{\text{bkg}} \frac{1+S_C}{(1+S_\Phi)^2}.$$

Highly Clustered Dim Lines Probed by $\tilde{C}$

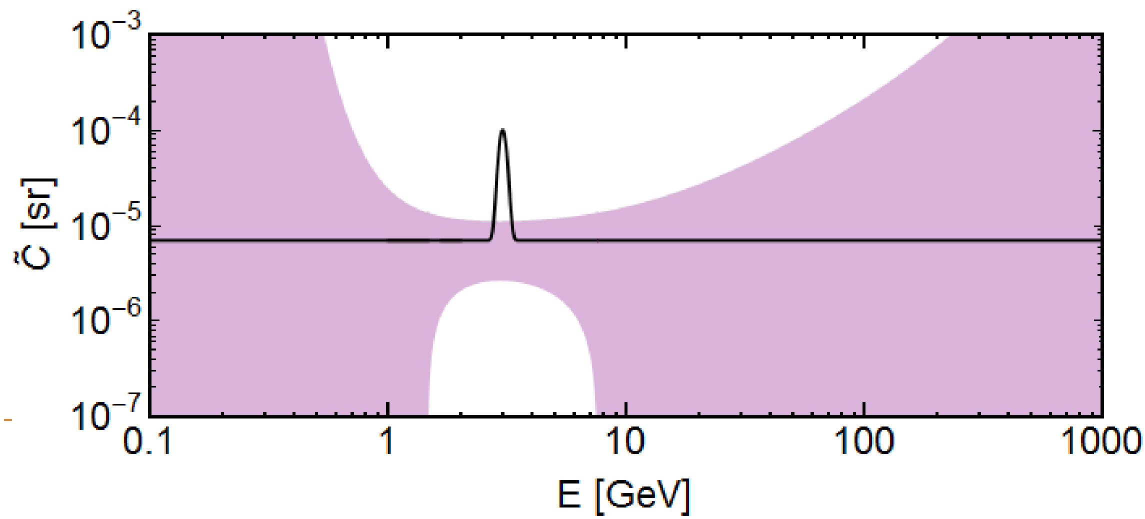
- Example: 3 GeV line with $S_{\Phi} = 0.01, S_C = 10$.



- Expected background statistical uncertainty for 10 years livetime.
- Line height is expected height of energy bin log-centered at that energy with width of 68% line containment.

Expected flux line significance is 1.6σ.

Expected anisotropy line significance is 13.8σ.
(But for $S_C = 1$ only 1.5σ.)



Condition to Observe Anisotropy Line

- The line is observed to N_σ sensitivity if:

$$\left|\tilde{C} - \tilde{C}_{\mathrm{bkg}}\right| > N_\sigma \sigma_{\tilde{\bar{C}}}.$$

- The number of events needed to observe a line of given strength with anisotropy is:

$$N > \frac{\kappa}{\tilde{C}} \frac{1 + \Lambda^2 \tilde{C}}{\sqrt{\left(\frac{\mathcal{R}_{\mathrm{sig}}}{\mathcal{R}_{\mathrm{exp}}}\right)^2 \left(1 + \Lambda^2 \tilde{C}\right) + 1} - 1}$$

Fundamental **Signal** Resolution Scale

$$\mathcal{R}_{\mathrm{sig}} \equiv \frac{S_C - S_\Phi(2 + S_\Phi)}{1 + S_C}$$

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Fundamental **Signal** Resolution Scale

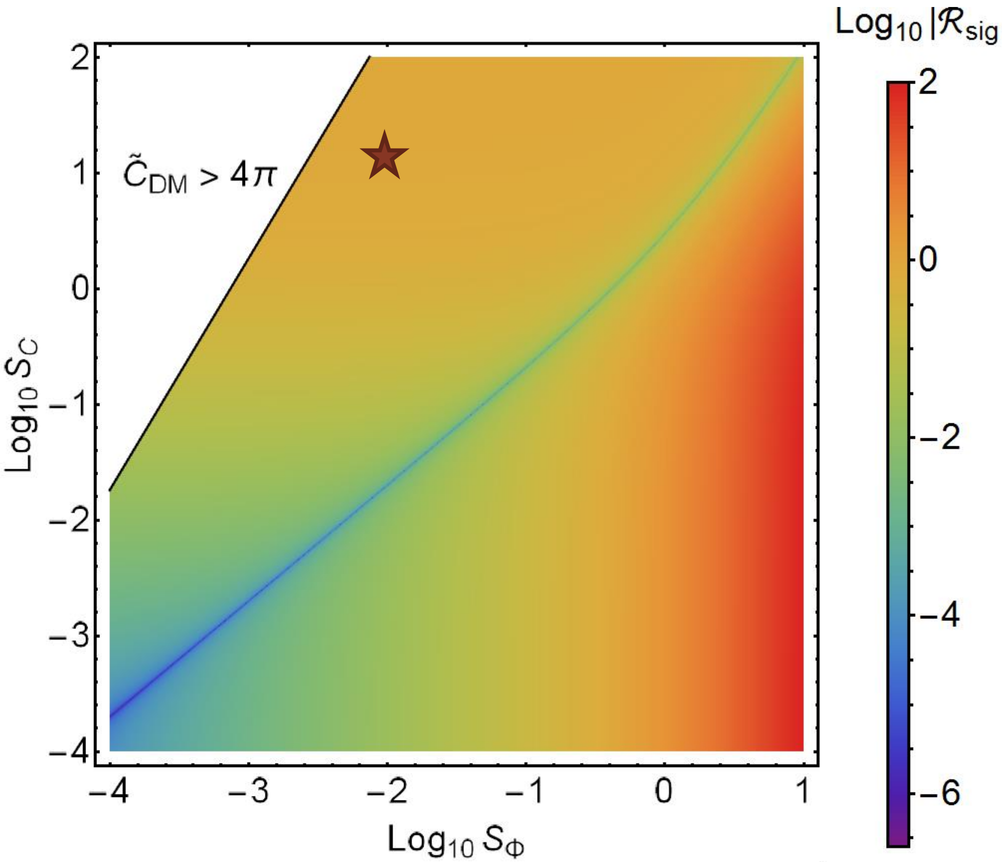
$$\mathcal{R}_{\text{sig}} \equiv \frac{S_C - S_\Phi(2 + S_\Phi)}{1 + S_C}$$

When $S_C = S_\Phi(2 + S_\Phi)$, the excess line radiation does not modify the power spectrum

$\Longrightarrow$ No line anisotropy condition.

When $S_C < S_\Phi(2 + S_\Phi)$, the line washes out structure $\Rightarrow$ Line dip feature.

Signal Resolution Parameter Space



- Narrow blue region: line is unobservable in power spectrum.
- Above blue region: power spectrum bump signals.
- Below blue region: power spectrum dip signals.
- The red star indicates our earlier line example.

Condition to Observe Anisotropy Line

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- The number of events needed to observe a line of given strength with anisotropy is:

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Fundamental **Signal** Resolution Scale

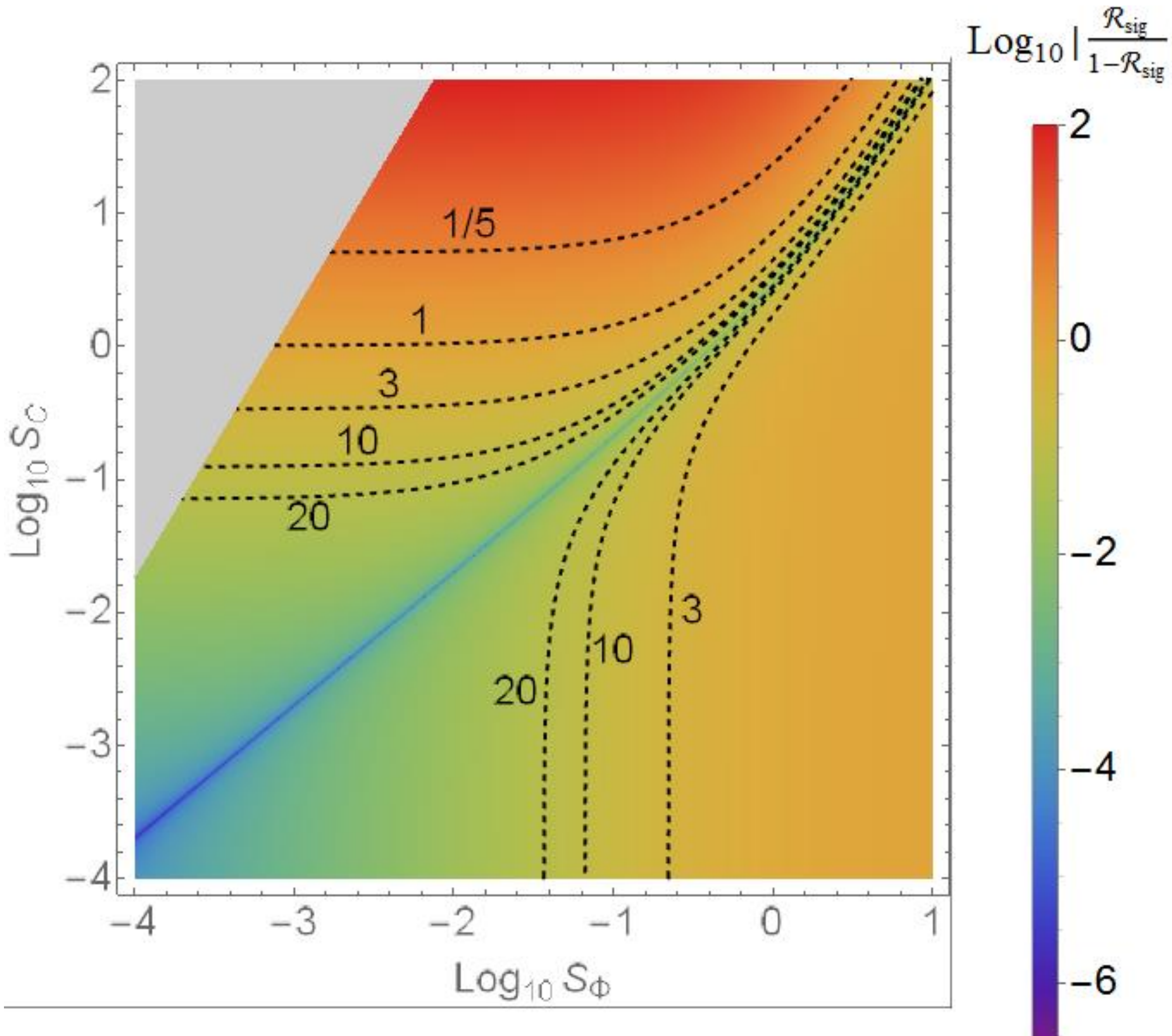
$$\mathcal{R}_{\text{sig}} \equiv \frac{S_C - S_\Phi(2 + S_\Phi)}{1 + S_C}$$

Example back-of-envelope estimate: in diagonal covariance case $\Lambda^2 \tilde{C} \ll 1$ and shot-dominated, **signal parameter space that is probed is:**

$$\frac{\left|\mathcal{R}_{\text{sig}}\right|}{1 - \mathcal{R}_{\text{sig}}} > \frac{\kappa \mathcal{R}_{\text{exp}}}{\tilde{C}_{\text{bkg}}} \frac{1}{N}$$

Joint Constraint on $B_{\text{sub}}, \tilde{C}_{\text{sub}}, \sigma v, m.$

Constraining Signal Strengths



$$\frac{|\mathcal{R}_{\mathrm{sig}}|}{1-\mathcal{R}_{\mathrm{sig}}} > \frac{N_{\mathrm{crit}}}{N}\sqrt{1+\frac{N}{N_{\mathrm{tr}}}}$$

$$N_{\mathrm{crit}} = \frac{\kappa}{\tilde{C}}\mathcal{R}_{\mathrm{exp}}\sqrt{1+\Lambda^2\tilde{C}}$$

$$N_{\mathrm{tr}} = \frac{1}{2}\frac{\kappa}{\tilde{C}}(1+\Lambda^2\tilde{C})$$

Contours show constraint after $\frac{N}{N_{\mathrm{crit}}}$ observations.

Note the super-Poissonian $1/N$ improvement in constraints when $N \ll N_{\mathrm{tr}}$.

Concluding Remarks

1. This demonstrates simple sensitivity estimates for anisotropy detection, and detection of anisotropy spectral features.

2. Lack of statistical independence of $\tilde{C}_\ell$ multipoles is important for large $\tilde{C}/\varrho$.

3. Sensitivity improvement as $1/N$ is strong scientific motivation for experiment extensions like Fermi-LAT.

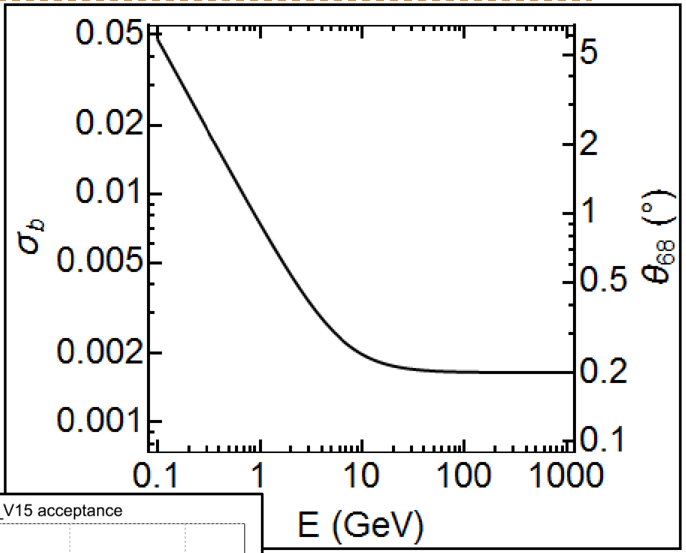
4. If Fermi-LAT's angular resolution is already near the signal-dominated error regime, more data with better-resolution instruments will not improve statistics at a faster rate.
   Do we require new technology to launch larger effective areas?

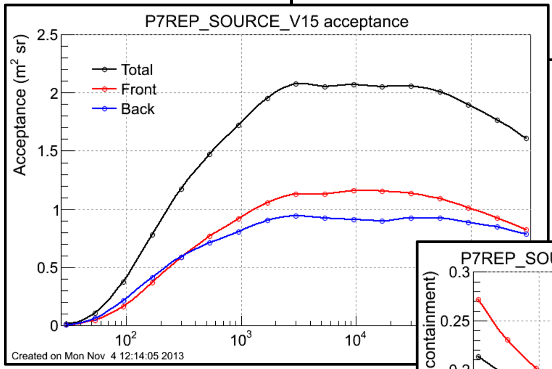
5. We saw that weak spectral signatures with highly clustered sources may be readily detected with the angular power spectrum.

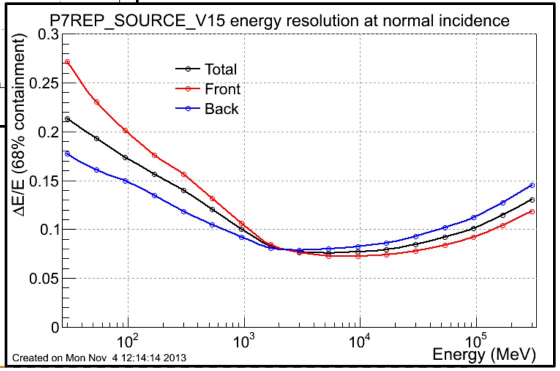
Extra Slides

Instrument Response Functions

1. Angular Resolution
   - Modelled as a Gaussian beam of width $\sigma_b(E)$ in radians.
2. Energy Resolution
   - Approximated as 10% for Fermi-LAT over energy range of interest.
3. Acceptance
   - Effective area integrated over field of view, $A(E)$.
4. Exposure
   - take to be uniform over unmasked sky for this analysis.
5. Exposed Acceptance, $A_{\text{exp}}(E)$.
   - $A_{\text{exp}}(E) = f_{\text{sky}}\, A(E) \simeq 6500\ \text{cm}^2\ \text{sr}$

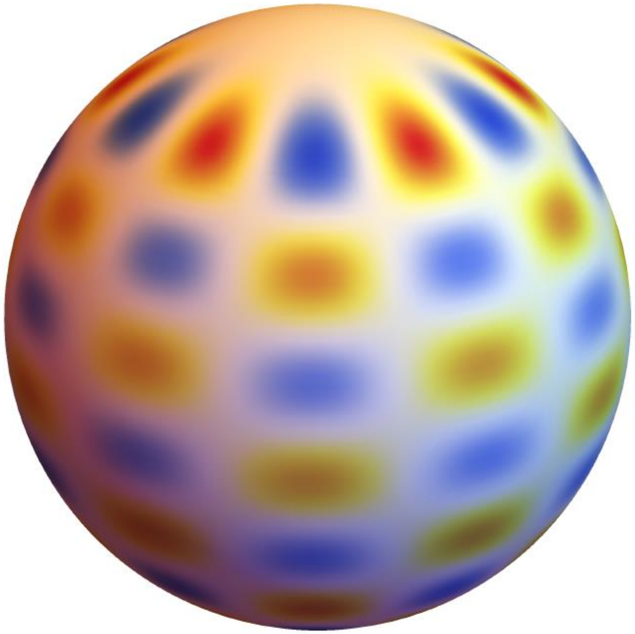




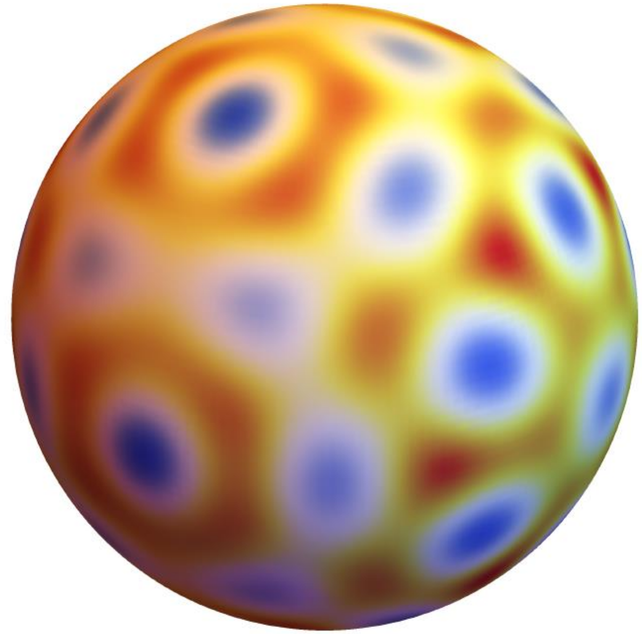


Test with Monte-Carlo Sampling

SC, MNRAS 448 (2015) 2854



(a) $\tilde{S}_{NB}(\mathbf{n})$

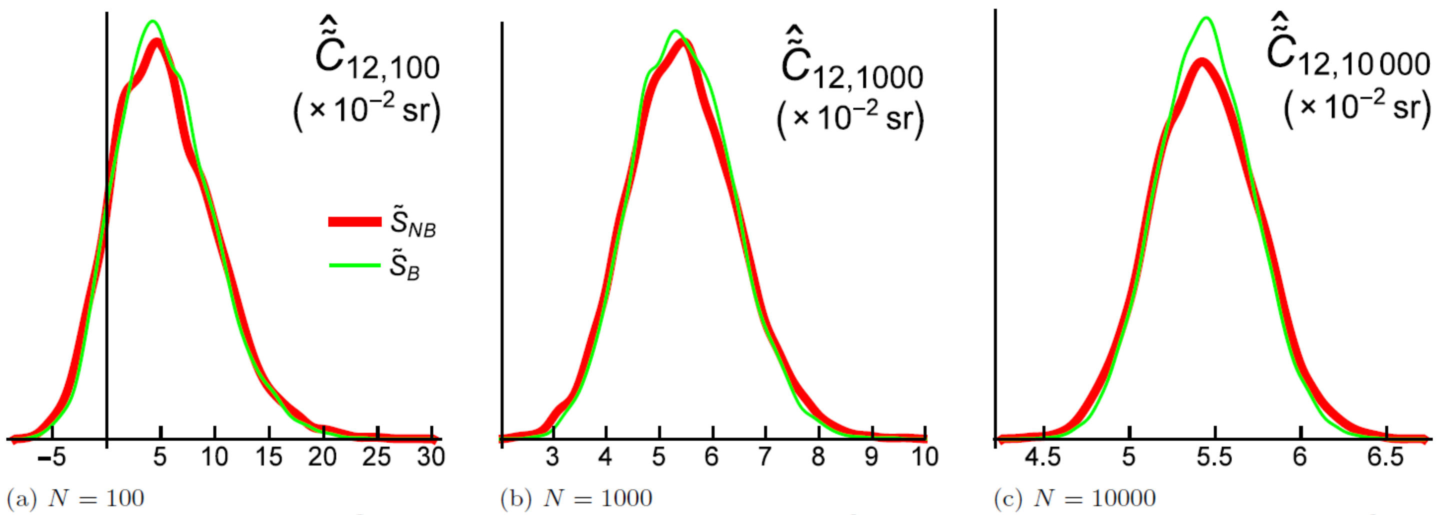


(b) $\tilde{S}_{B}(\mathbf{n})$

$\tilde{C}_\ell = (0.0544\text{ sr})\ \delta_{\ell,12}$	$\tilde{C}_\ell = (0.0544\text{ sr})\ \delta_{\ell,12}$
$\tilde{C}^{(3)}_{\ell\ell} = 0$	$\tilde{C}^{(3)}_{\ell\ell} = (-0.000413\text{ sr})\ \delta_{\ell,12}$

$\hat{\tilde{C}}_{\ell,N}$ Distribution of 10 000 Samplings

SC, MNRAS 448 (2015) 2854



- Low counts gives very wide distribution. Shot noise subtraction can give negative power spectrum estimates.
- At high counts, the distribution becomes narrow, and the distribution with negative bispectrum is visibly narrower.

$\sigma_{\hat{\hat{C}}_{\ell,N}}$ Distribution of 10 000 Samplings

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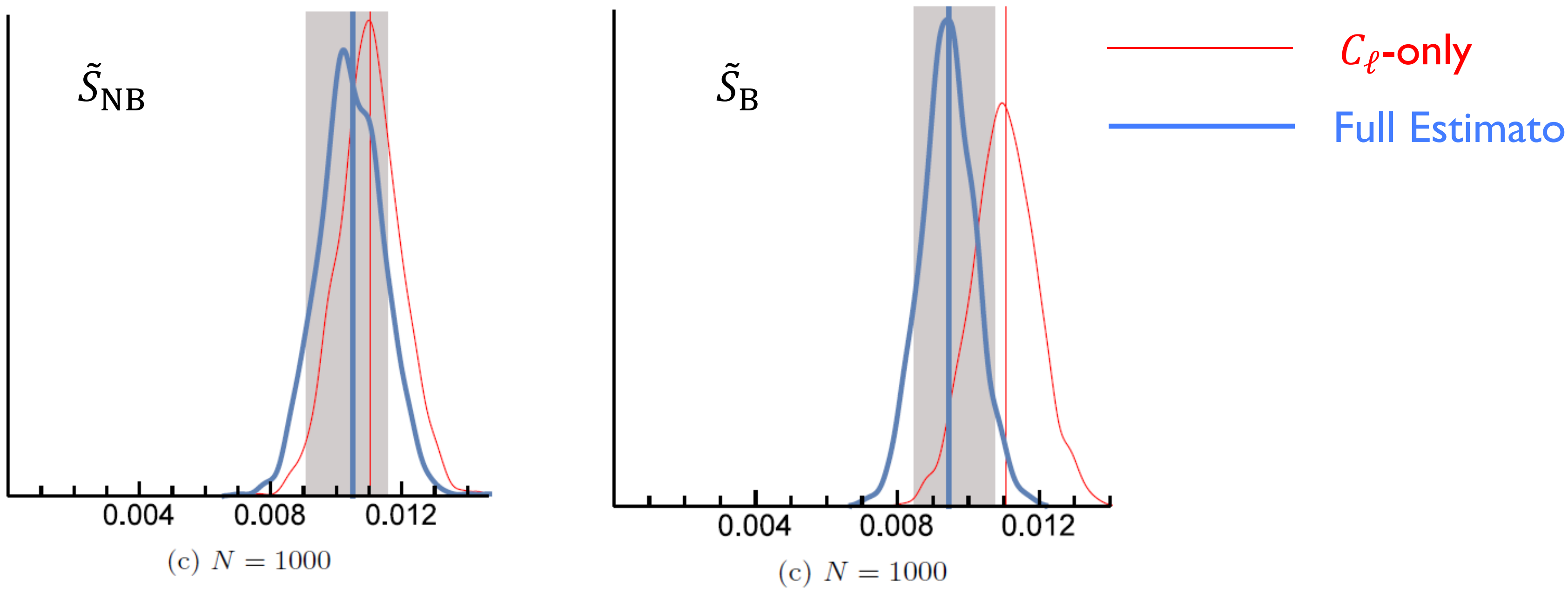

- The negative bispectrum does indeed appear to lower the variance of the power spectrum measurement.
- Even the distribution without bispectrum is affected by the other higher-order spectra, but those effects are small and unresolved in this example.