Causality Constraints in Conformal Field Theory

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Causality

Places highly nontrivial bounds on EFT:

- $\lambda(\partial\phi)^4 \quad \Rightarrow \quad \lambda > 0$ [Adams, Arkani-Hamed, Dubovsky, Nicolas, Rattazzi '06]
- Proof of the *a*-theorem [Komargodski, Schwimmer '11]
- coefficients of R^2 gravity [Brigante et al '07; Hofman '09; Camanho et al '14]

These constraints seem to rely inherently on Lorentzian signature.

On the other hand:

Theorem (60's, 70's)

(Schwinger, Wightman, Osterwalder & Schrader, etc.)

Good Lorentzian QFTs:

- Unitary
- Poincare covariant
- Causal

Are in one-to-one correspondence with a class of "good" Euclidean QFTs:

- Reflection-positive
- $SO(d)$ covariant
- Permutation symmetric (ie, crossing invariant)

Conformal Bootstrap

Places highly nontrivial constraints on CFT.

These constraints are mostly either:

- Euclidean signature (*eg:* numerical bootstrap)

  [Rattazzi, Rychkov, Tonni and Vichi; and refs thereof.]

- Lorentzian signature, but at spacelike separation (*eg:* lightcone bootstrap)

  [Komargodski, Zhiboedov; Fitzpatrick et al; Alday et al; etc]

This talk:

Bootstrap at timelike separation <-> Causality constraints in CFT

This can be viewed as:

- A statement entirely about *conformal* theories
- A statement about non-conformal theories in AdS, with or without gravity
- A statement about non-conformal theories in flat space

[Penedones; Mack; etc]

Hidden agenda:

- Derive gravity from CFT ???

[eg: Heemskerk, Penedones, Polchinski, Sully]

- For example,

$$a \approx c + \frac{1}{\Delta_{gap}^{\#}}$$

- Causality seems likely to play a central role

[Camanho, Edelstein, Maldacena, Zhiboedov '14]

Outline

Causality in quantum field theory

OPE at timelike separation

Shockwaves in CFT —> main theorem (causality sum rule)

Application: holographic derivation of $(\partial\phi)^4$ constraint

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***NB:* No large-*N* until the $(\partial\phi)^4$ application.**

I will eventually assume that the stress tensor is the lowest-twist operator appearing in an OPE. This is the *only* assumption about the CFT beyond the usual axioms.

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Causality Review

Point of view in this talk:

QFT is defined by a set of Euclidean *n*-point correlators.

How is causality encoded in these correlators?

Refs: Haag; Streater and Wightman.

Euclidean correlators

$$G(x_1, x_2, \ldots) \equiv \langle O(x_1) O(x_2) \ldots \rangle$$

are:

- Permutation invariant $\quad G(x_1, x_2, \ldots) = G(x_2, x_1, \ldots)$
- With singularities only at coincident points
- and no branch cuts (ie, single-valued).

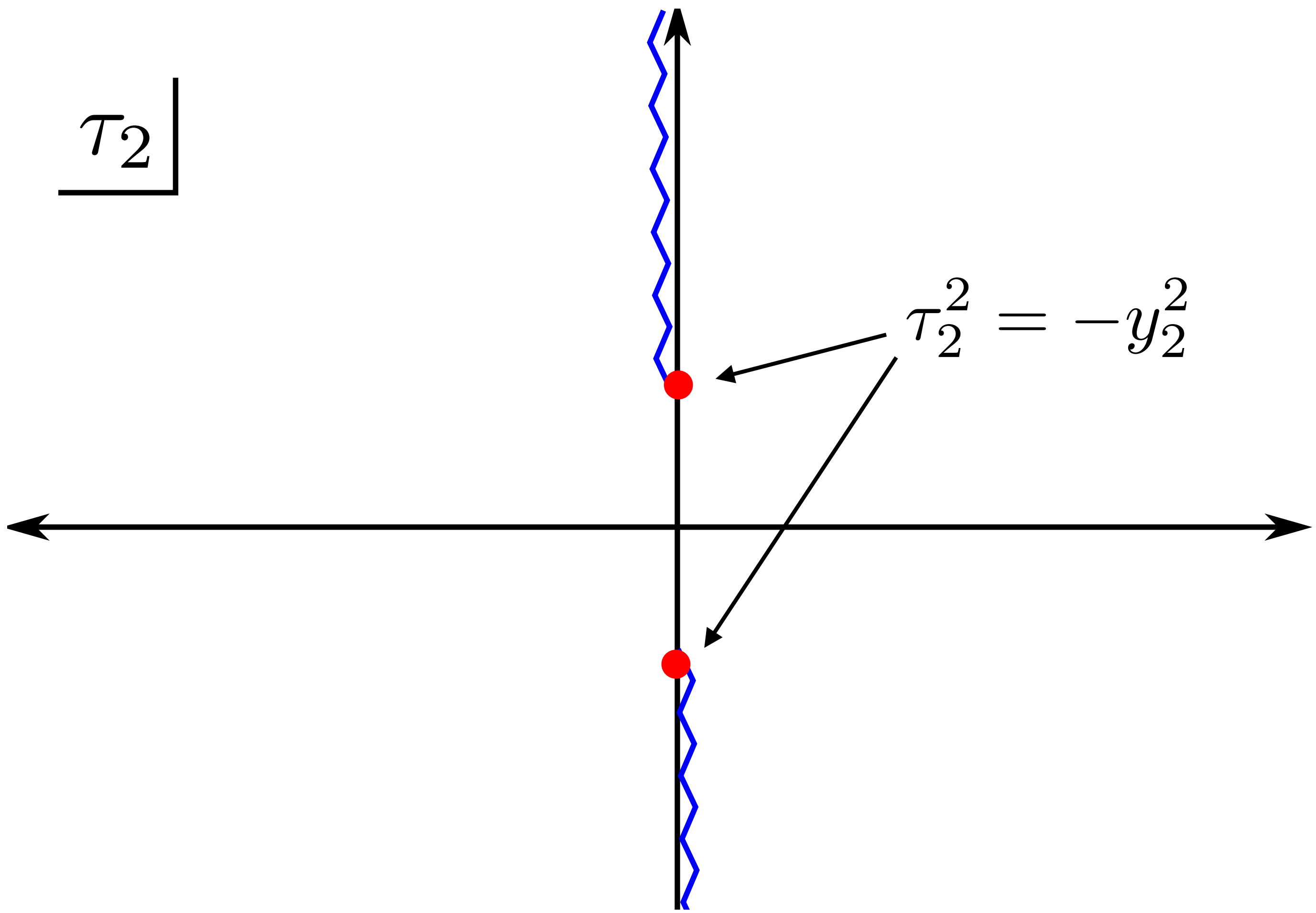
Ex: conformal scalar

$$\langle O(0,0) O(\tau_2, y_2) \rangle = (\tau_2^2 + y_2^2)^{-2\Delta}$$

But if we analytically continue to complex time: $\tau_i \in \mathbf{C}$

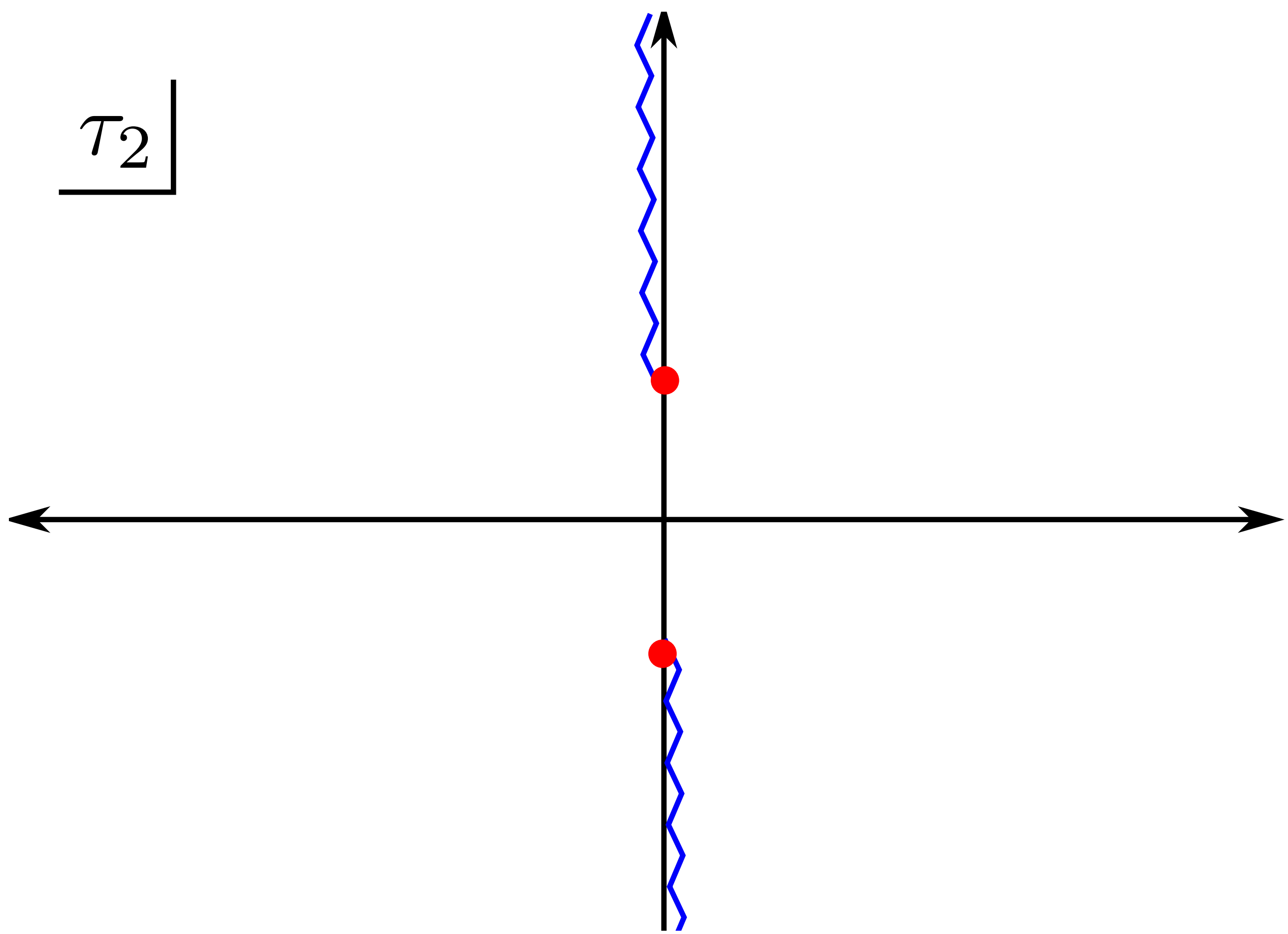
then there is an intricate structure of singularities and branch cuts.

Ex: conformal scalar 2pt function $G = (\tau_2^2 + y_2^2)^{-2\Delta}$



Therefore the analytic continuation to Lorentzian signature is ambiguous.

This ambiguity is why operators do not commute in Lorentzian QFT.



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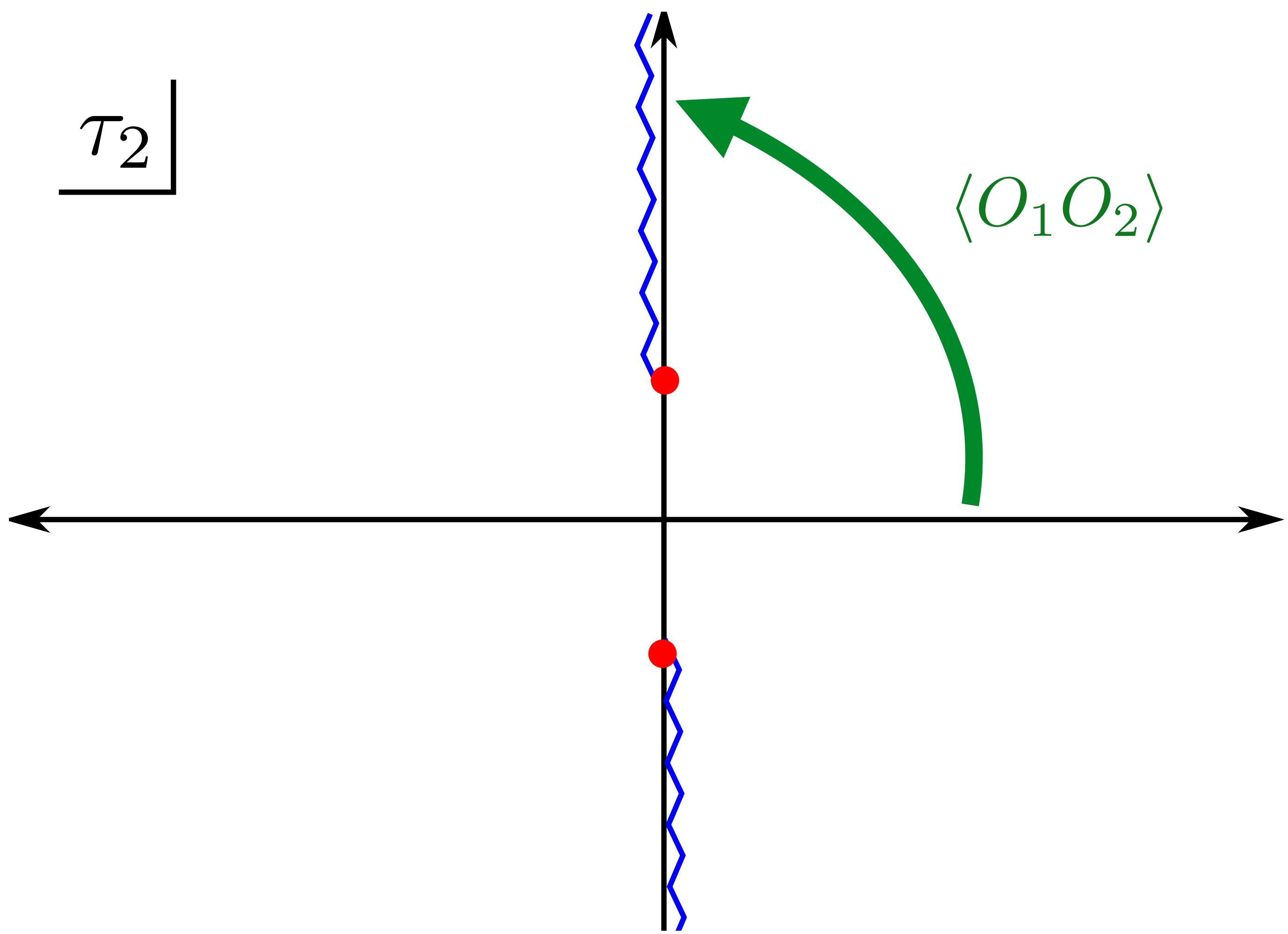

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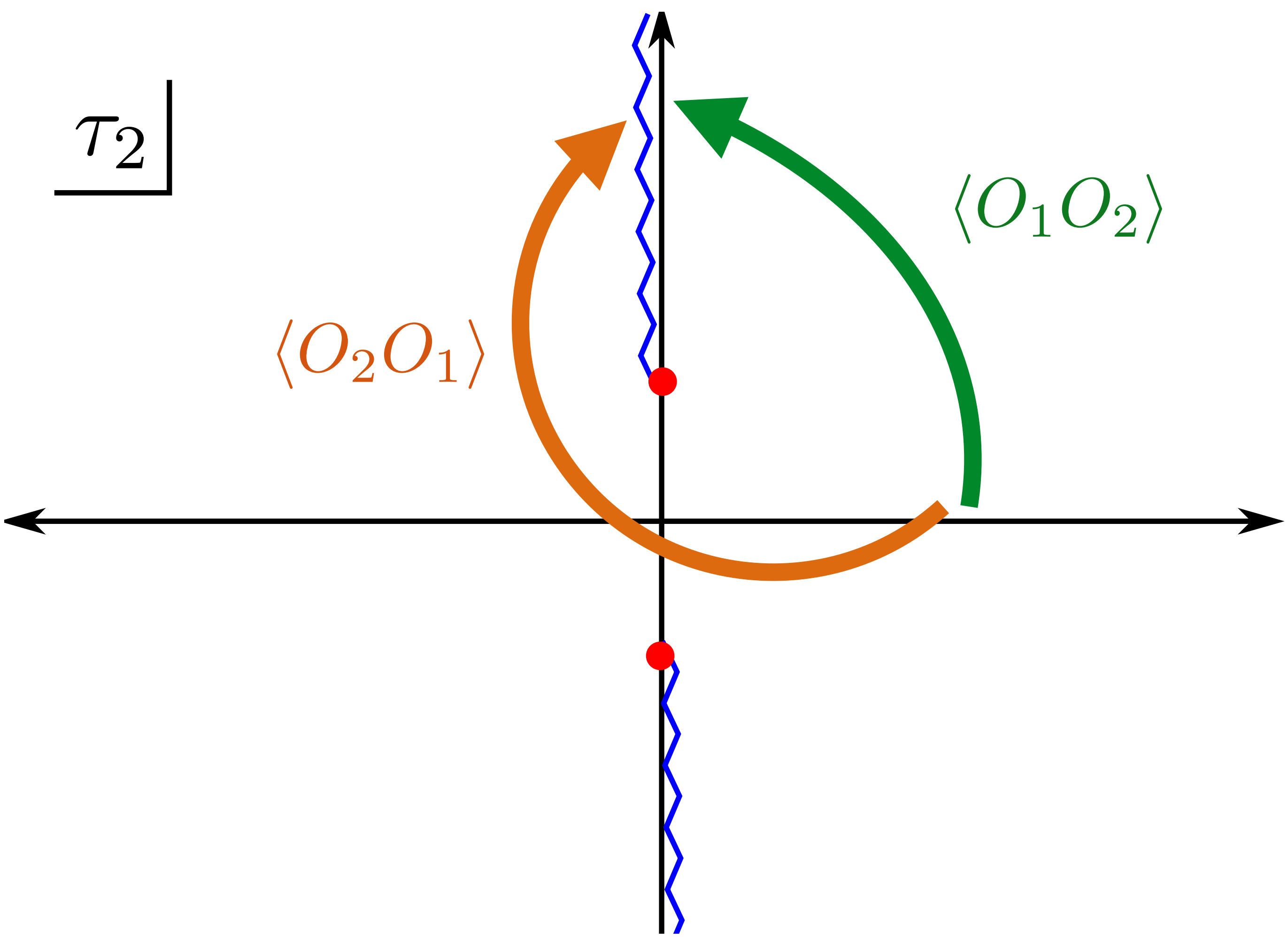

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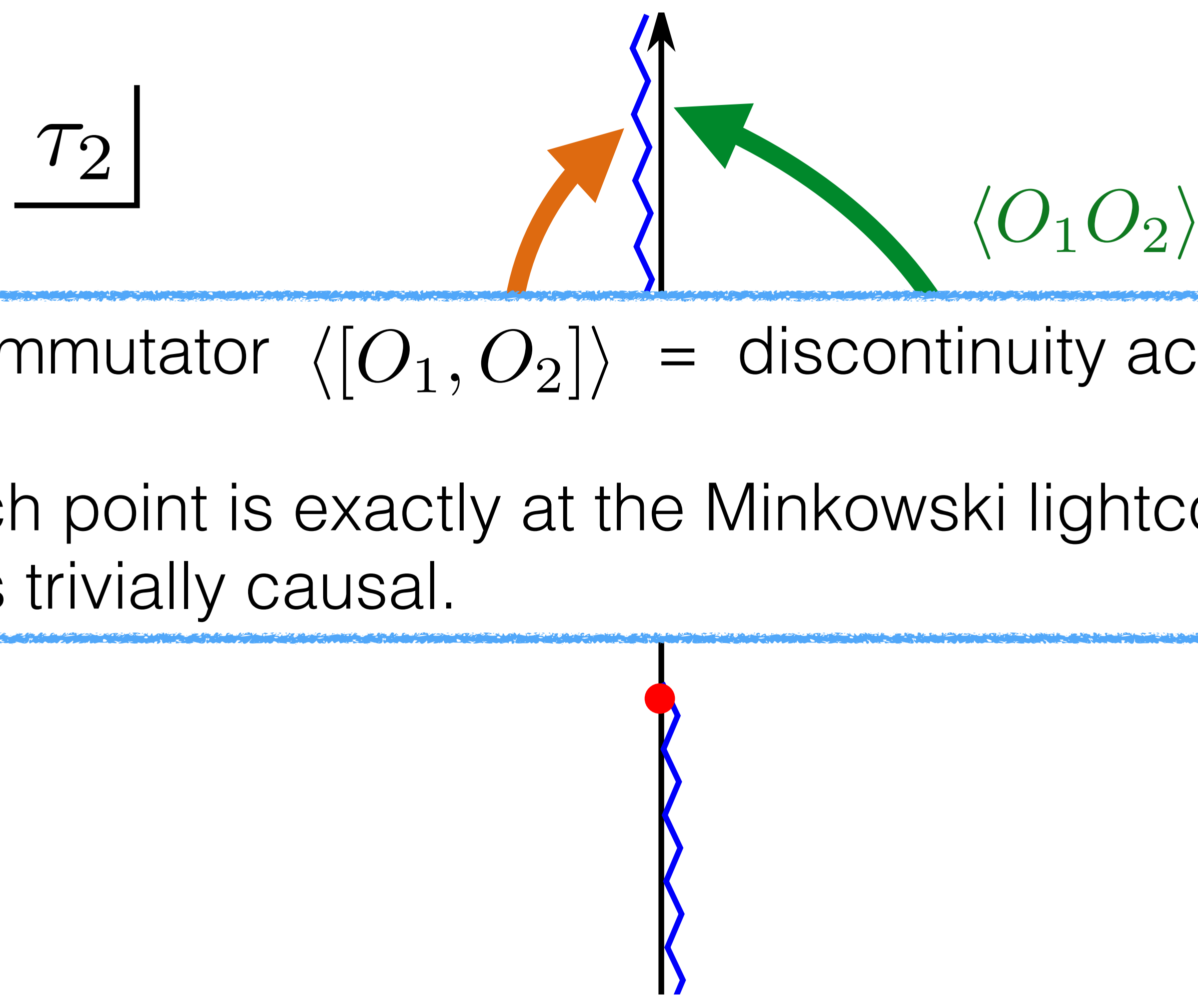
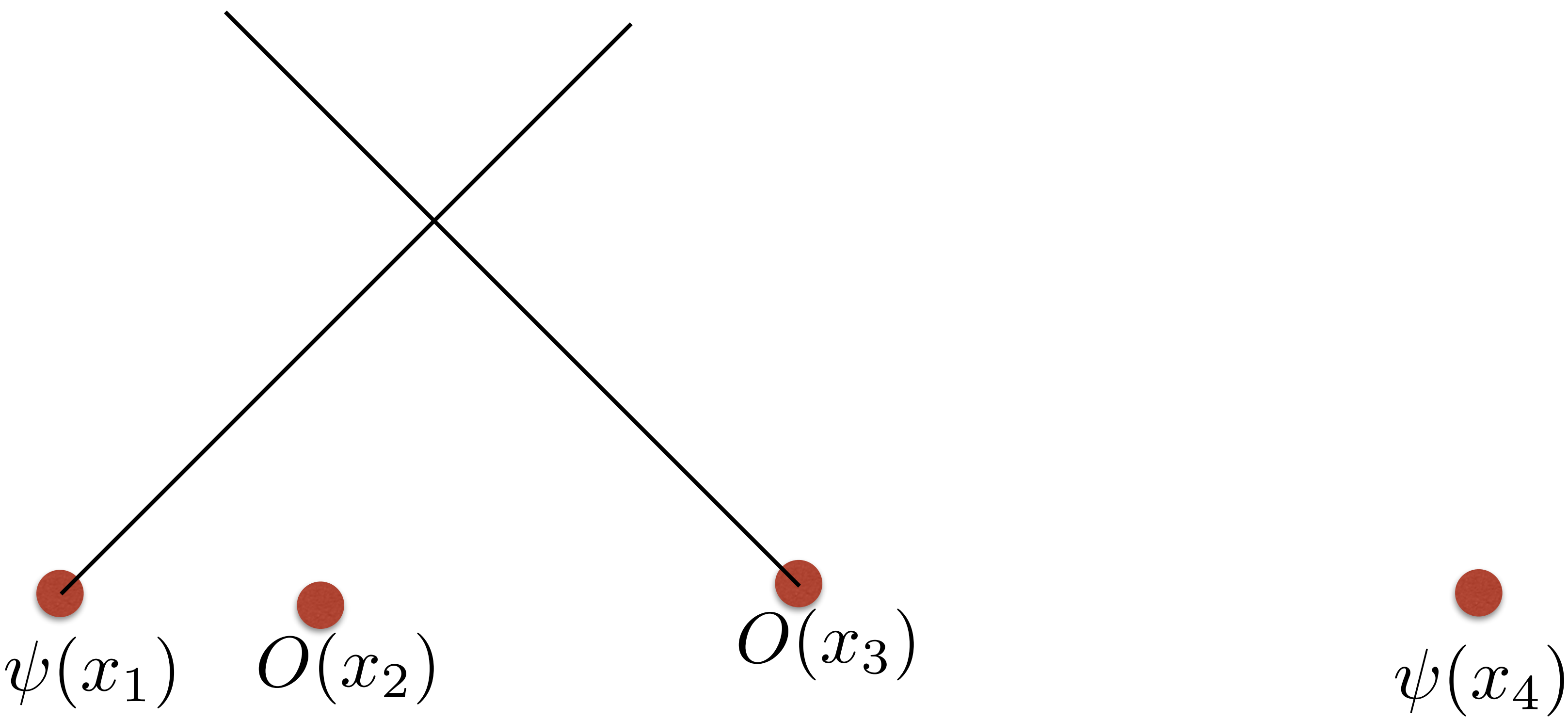


So: Commutator $\langle [O_1, O_2] \rangle$ = discontinuity across the cut.

The branch point is exactly at the Minkowski lightcone, so the 2pt function is trivially causal.

4-point functions

For now, we'll move just one operator *O(x_2)* to finite Lorentzian time:



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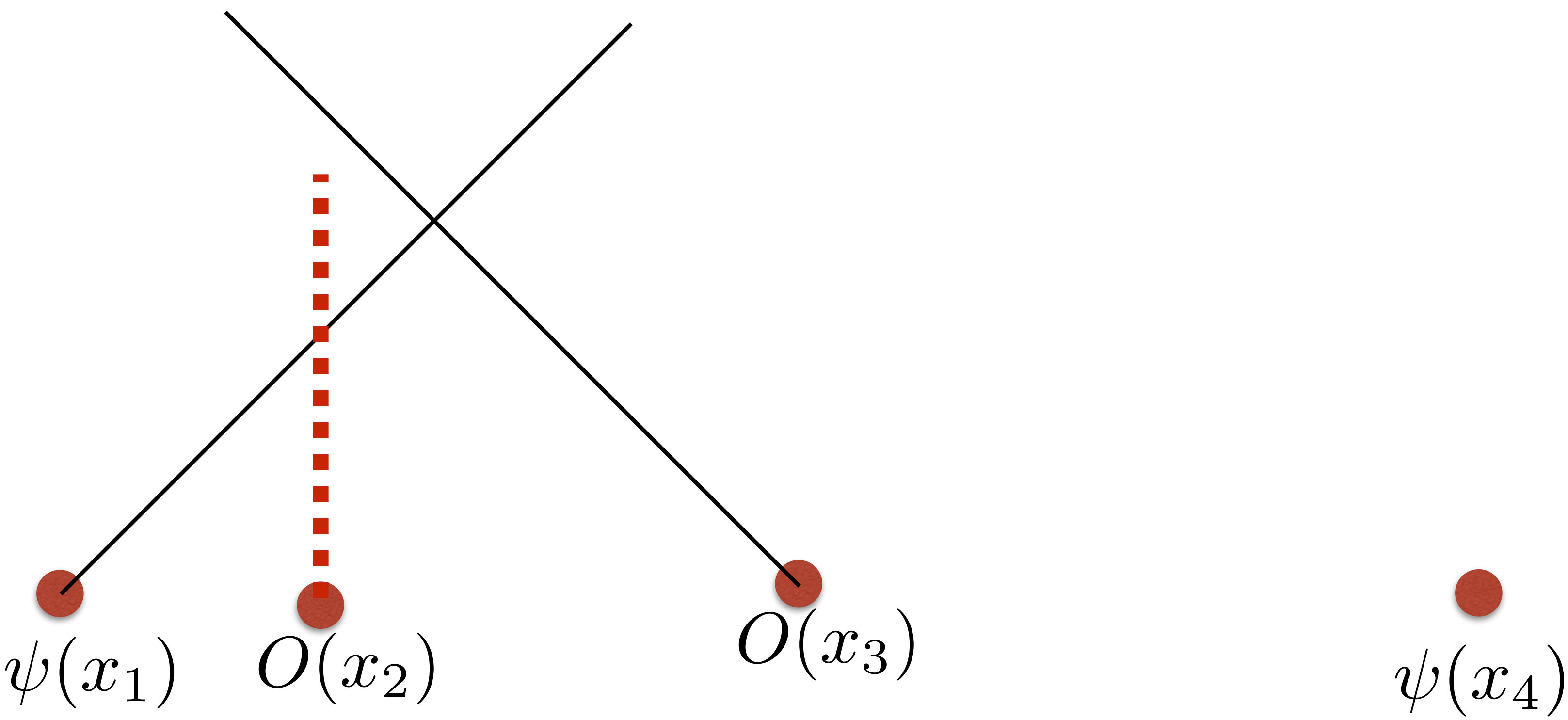

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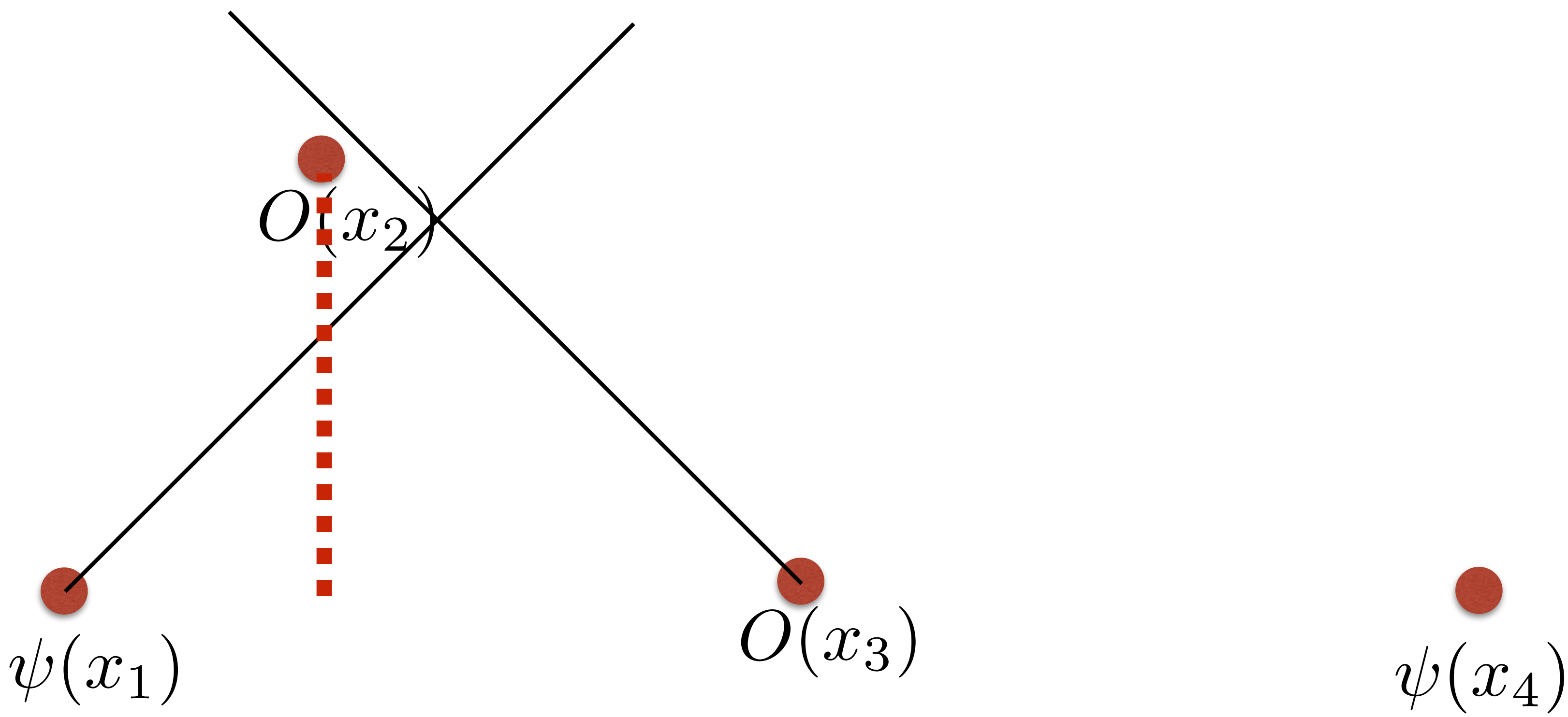
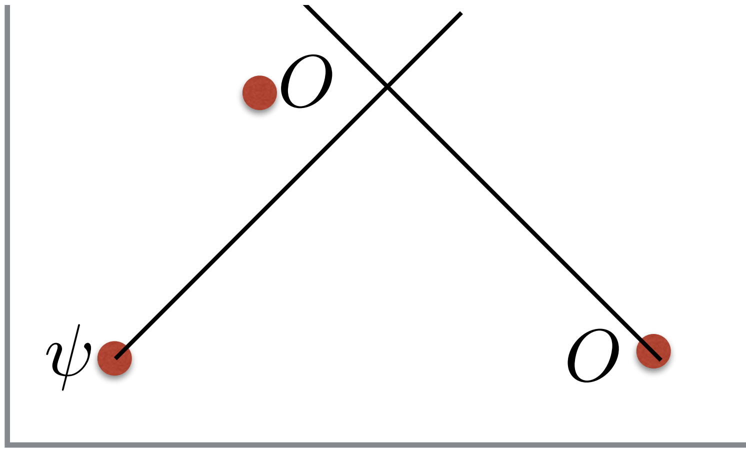




In this setup, causality requires

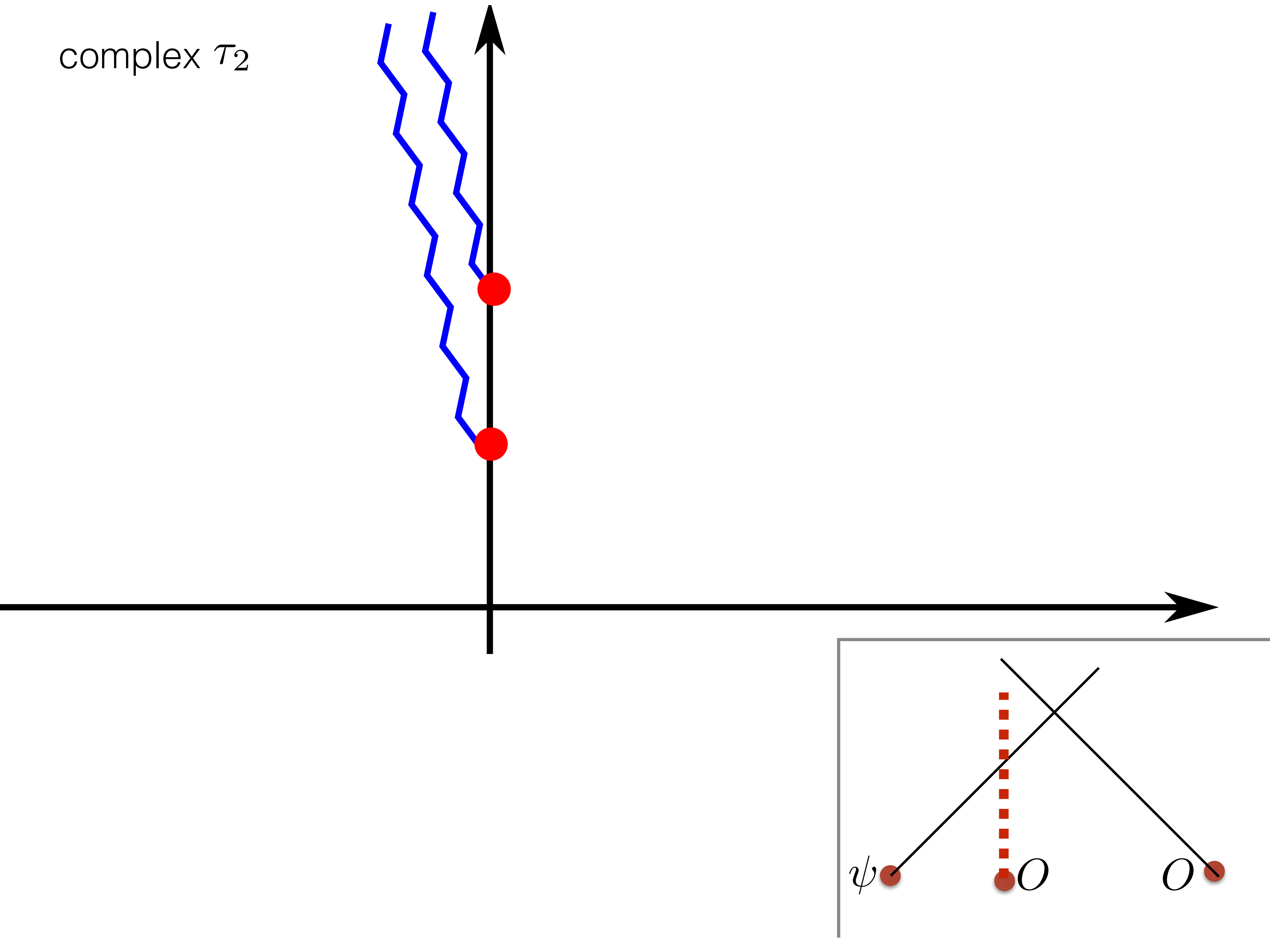
$$\langle [O, O]\psi\psi \rangle = 0$$

and

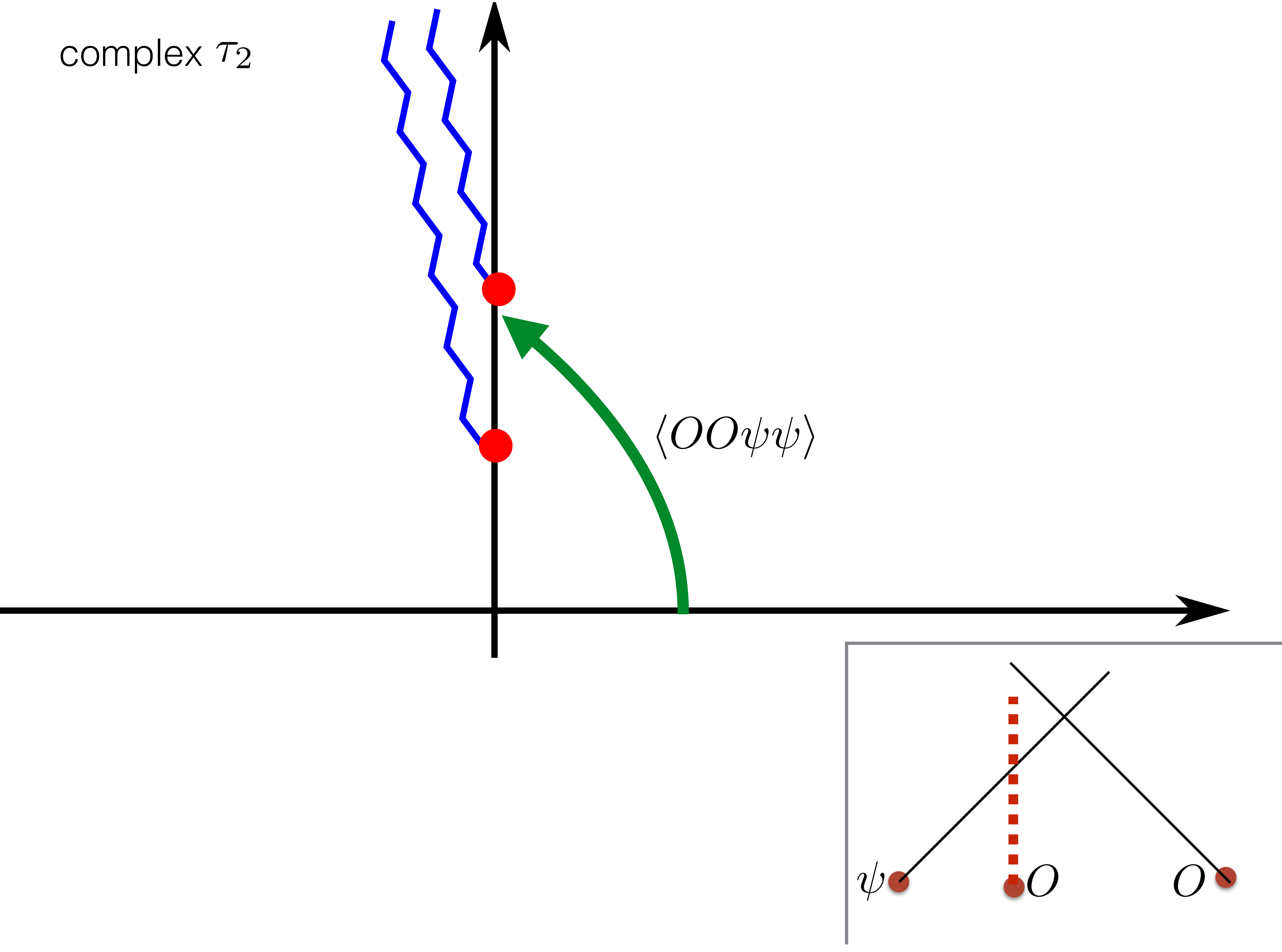
$$\langle \psi [O, O]\psi \rangle = 0$$

Claim:

- the first is obvious, like in the 2pt function.
- the second is highly nontrivial



complex τ_2
ψ
O
O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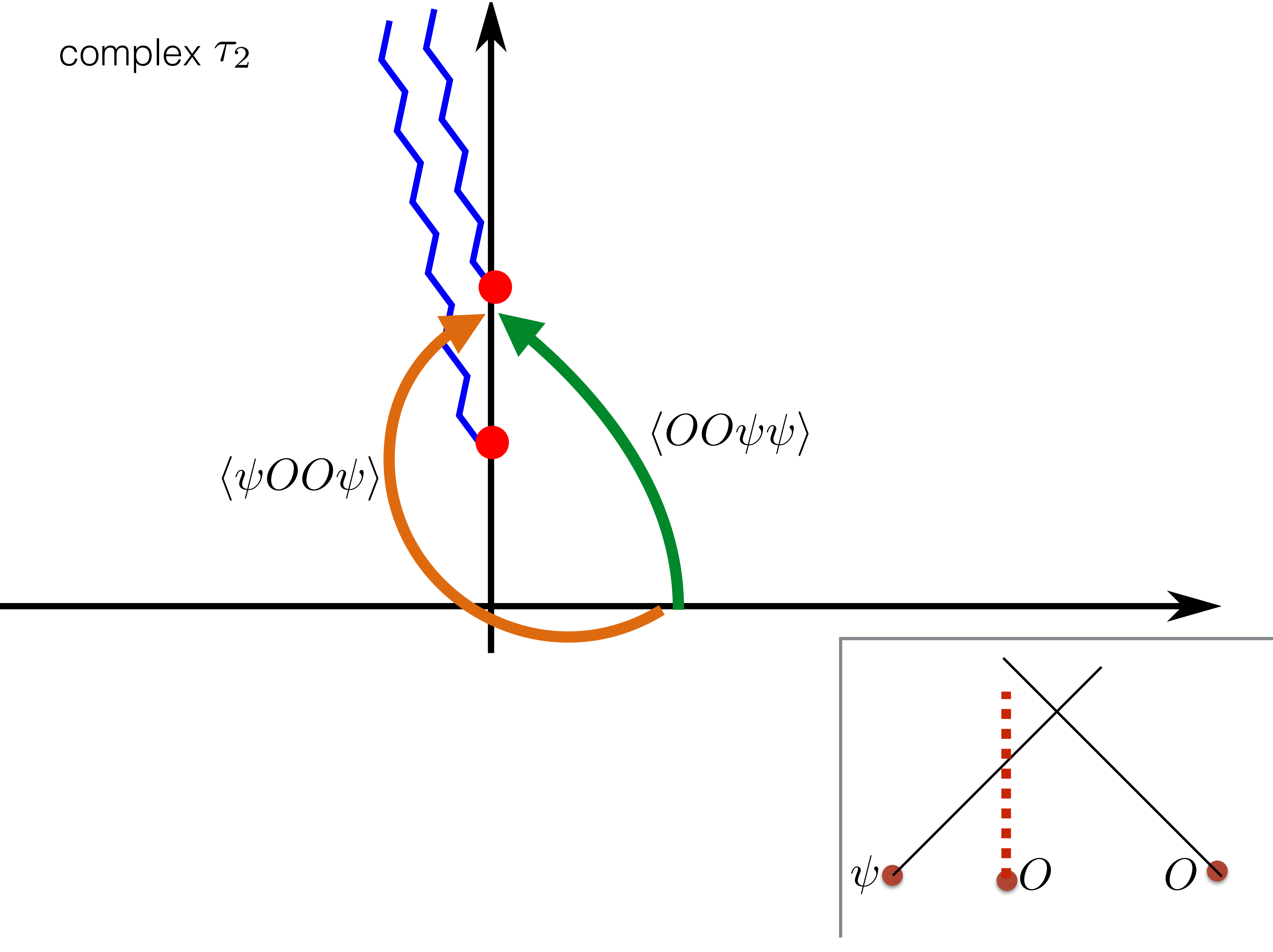
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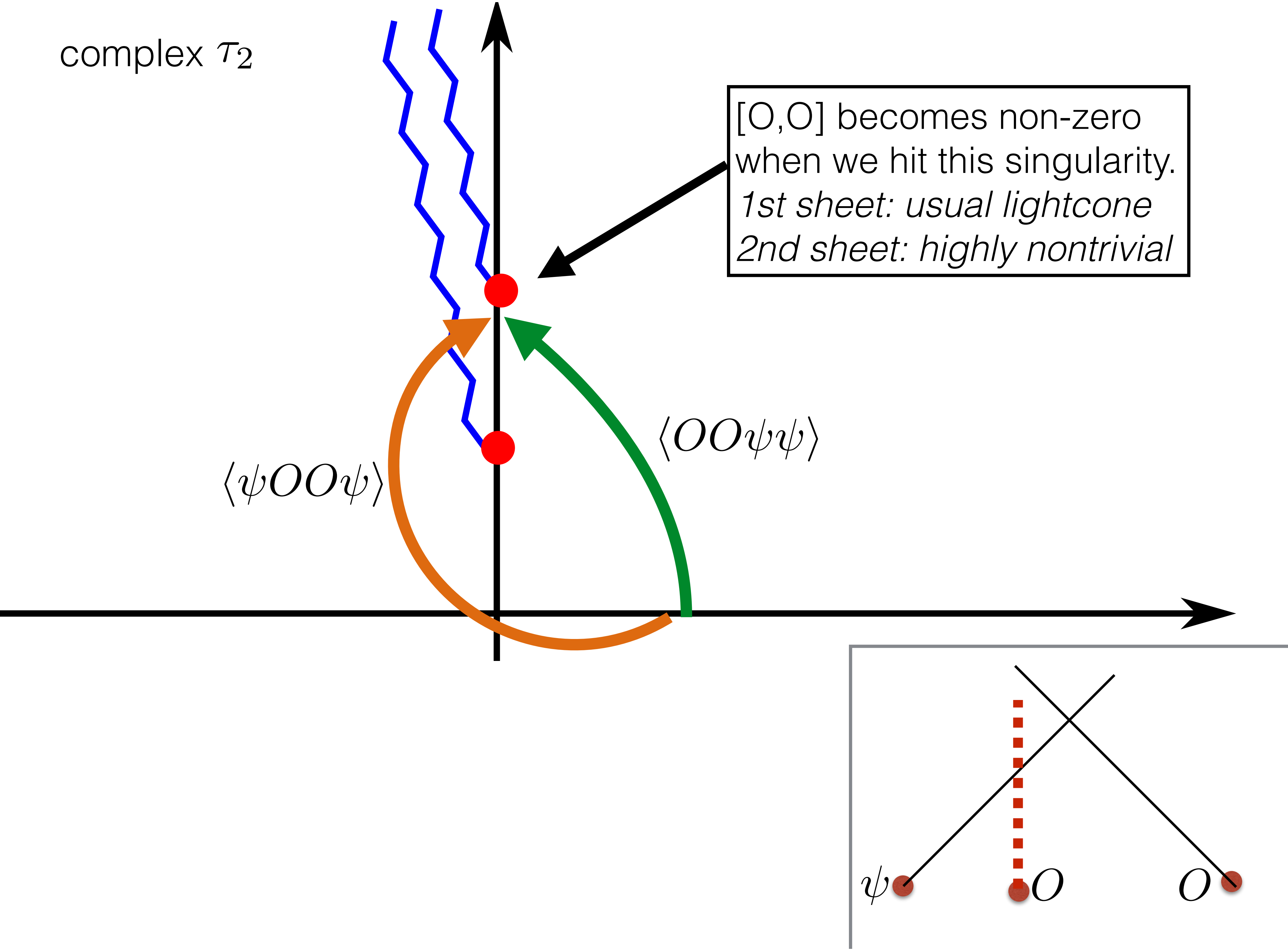

complex τ_2
[O,O] becomes non-zero when we hit this singularity.
1st sheet: usual lightcone
2nd sheet: highly nontrivial
$\langle OO\psi\psi\rangle$
$\langle \psi OO\psi\rangle$
ψ
O
O

CFT

[Luscher, Mack '74]

This was for a general QFT.

In CFT, we can rephrase all the same statements in terms of cross-ratios:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

With Euclidean insertions,

$$\bar{z} = z^*$$

Continuing to complex times means continuing to independent complex $z, \bar{z}$

On 1st sheet there are obvious branch points at

$$z = 0, 1, \infty, \quad \bar{z} = 0, 1, \infty$$

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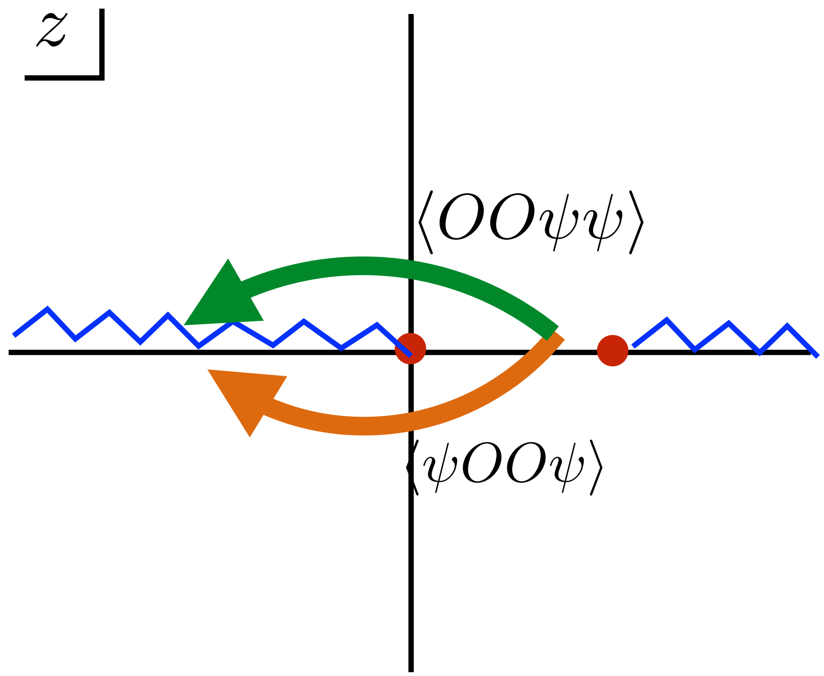

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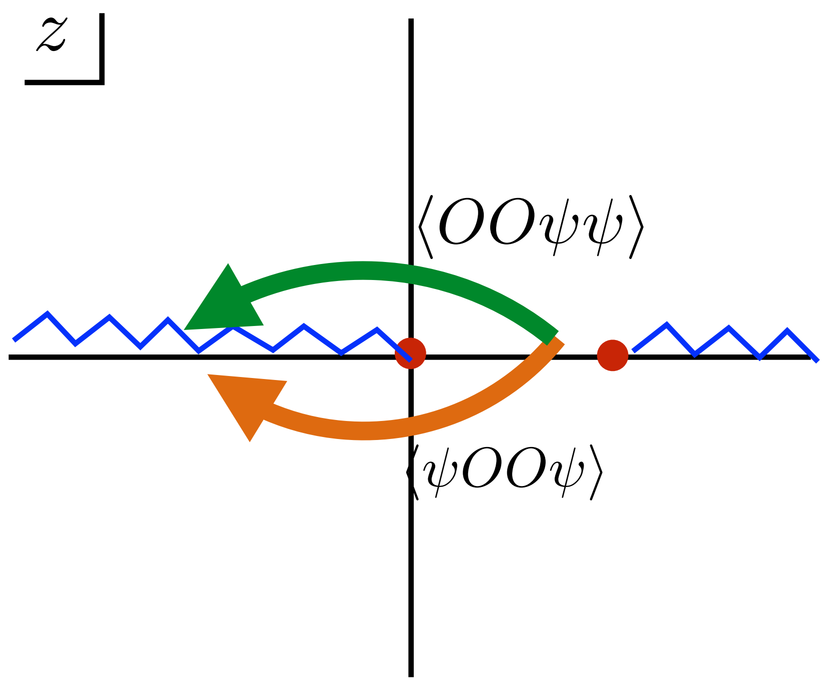

But on other sheets of this multivalued correlation function, location of singularities is not at all obvious.

Upshot so far:

Causality in a CFT 4-pt function is a question of how singularities on the complex $z, \bar{z}$ planes move around as we pass through branch cuts.

If a singularity moves to later t, then the theory has a time-delay.

If a singularity moves to earlier t, then the theory is acausal.

Aside: All of this can be rephrased in terms of a position-space $i\epsilon$ *prescription.*

(As in standard textbooks — Haag, Streater & Wightman)

The two approaches are completely equivalent.

Outline

Causality in quantum field theory

OPE at timelike separation

Shockwaves in CFT —> main theorem (causality sum rule)

Application: holographic derivation of $(\partial\phi)^4$ constraint

Consider

$$G = \langle \psi(x_1) O(x_2) O(x_3) \psi(x_4) \rangle$$

Throughout the talk, think of ψ as "background" and *O* as "probe" — we will always be interested in causality of [*O*,*O*].

Conformally map to

$$G(z, \bar{z}) = \langle \psi(0) O(z, \bar{z}) O(1) \psi(\infty) \rangle$$

Euclidean: $\bar{z} = z^*$.

Now expand using the $O\psi$ or OO OPE *s*-channel conformal block expansion:

$$G(z,\bar{z}) \sim \sum |c_{\psi O p}|^2 g_{\Delta_p,\ell_p}^{\Delta_{\psi O},-\Delta_{O\psi}}(z,\bar{z})$$

This converges for Euclidean $|z| < 1$

[Mack '77]
[Pappadopulo et al '12]

s-channel conformal block expansion:

$$G(z,\bar{z}) \sim \sum_{p} |c_{\psi O p}|^2 g_{\Delta_p,\ell_p}^{\Delta_{\psi O},-\Delta_{O\psi}}(z,\bar{z})$$

This converges for Euclidean $|z| < 1$

[Mack '77]
[Pappadopulo et al '12]

t channel:

$$G \sim \sum_p c_{\psi\psi p} c_{OOp} g_{\Delta_p,\ell_p}(1-z, 1-\bar{z})$$

Positive coefficients in the s-channel

Every coefficient in the *s*-channel expansion $(\psi O)(\psi O)$ is positive:

$$G \sim \sum_{\Delta,s} a_{\Delta,s} z^{\frac{1}{2}(\Delta-s)} \bar{z}^{\frac{1}{2}(\Delta+s)}$$

$$a_{\Delta,s} \geq 0$$

This can be derived from reflection positivity (or unitarity).

In *d=4*: easily checked via Dolan-Osborn conformal blocks.

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[Fitzpatrick, Kaplan, Poland, Simmons-Duffin]
[TH, Jain, Kundu]

Positive coefficients are very powerful.

Implies convergence in Lorentzian regime

$$-1 < z, \bar{z} < 1$$

And: will allow us to bound the magnitude of Lorentzian correlators — with independent complex cross-ratios — by Euclidean correlators:

$$\left| \sum a_{h,\bar{h}} z^h \bar{z}^{\bar{h}} \right| \leq \sum a_{h,\bar{h}} |z|^h |\bar{z}|^{\bar{h}}$$

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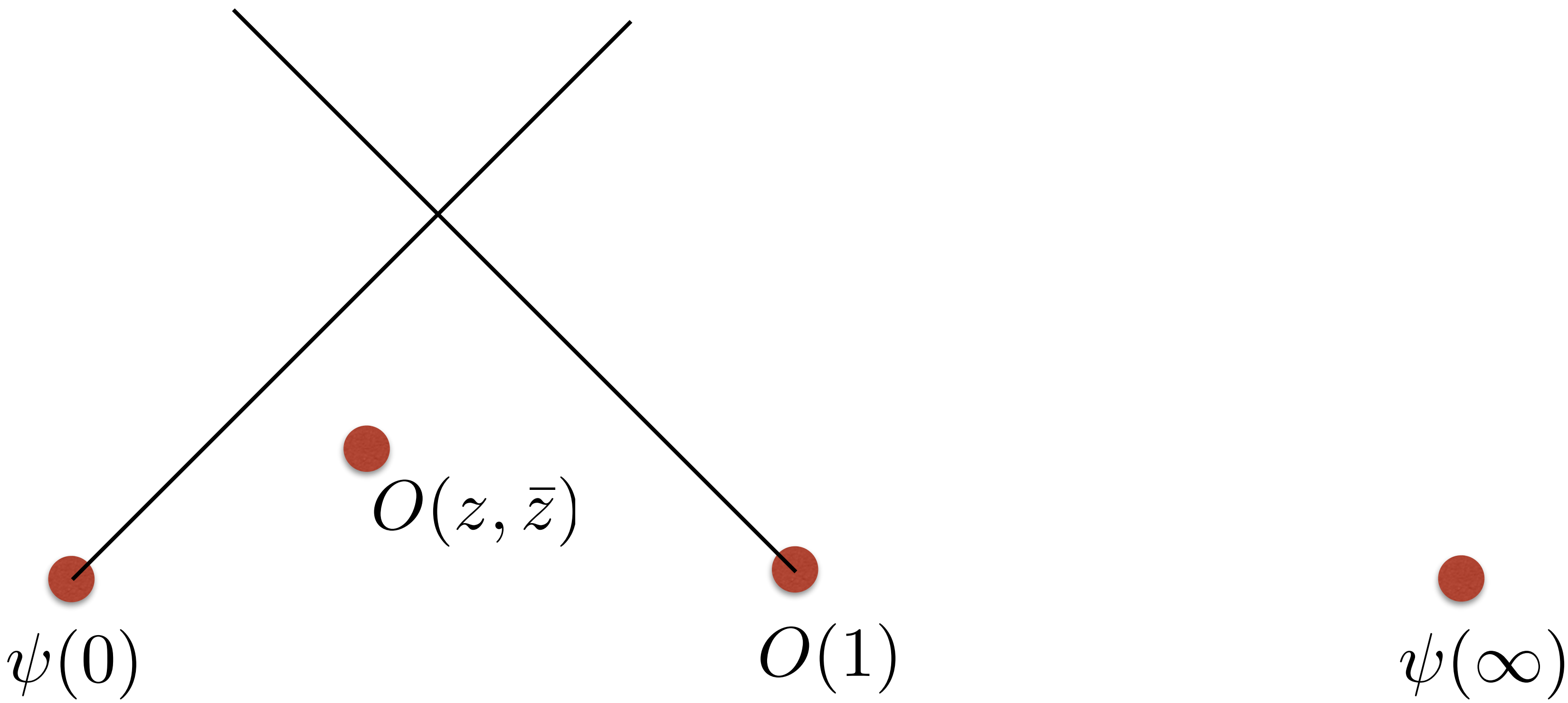
For the experts: There also positive coefficients in the rho-expansion [Hogervorst,Rychkov], after stripping off the correct prefactor. This is an *independent* statement.

So far we mostly considered Euclidean: $\bar{z} = z^*$

Now consider the simplest Minkowski configuration:



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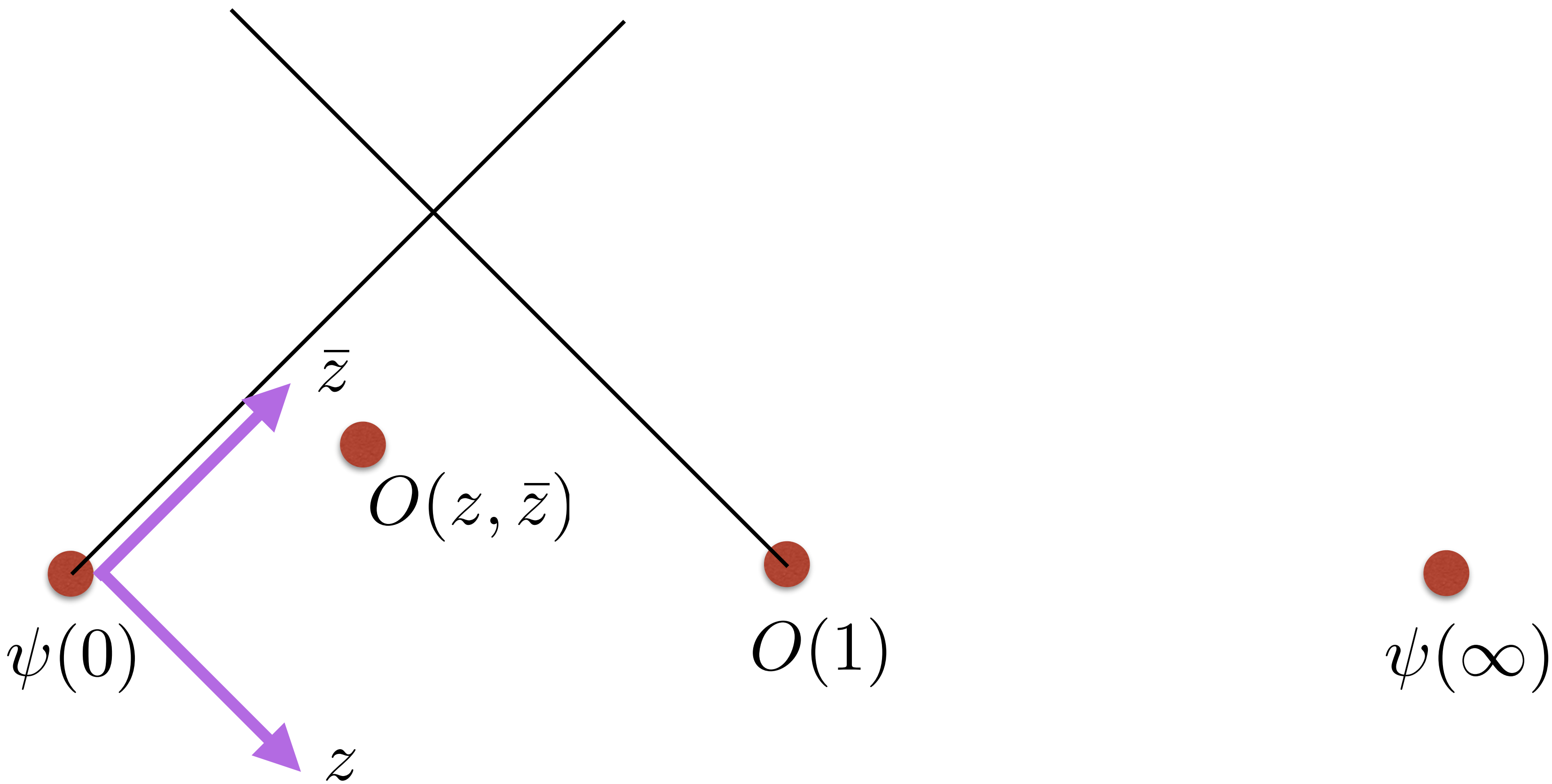

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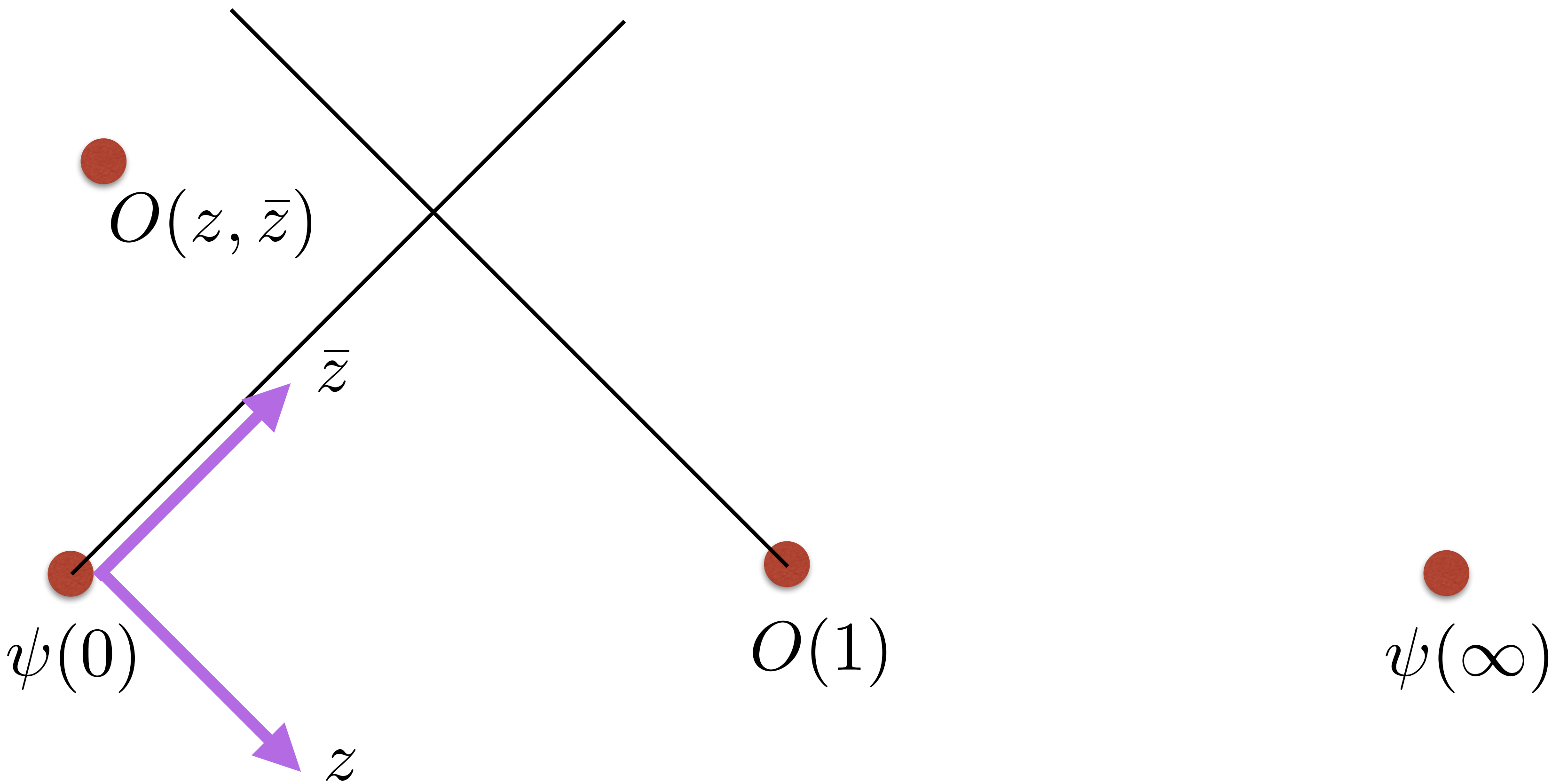

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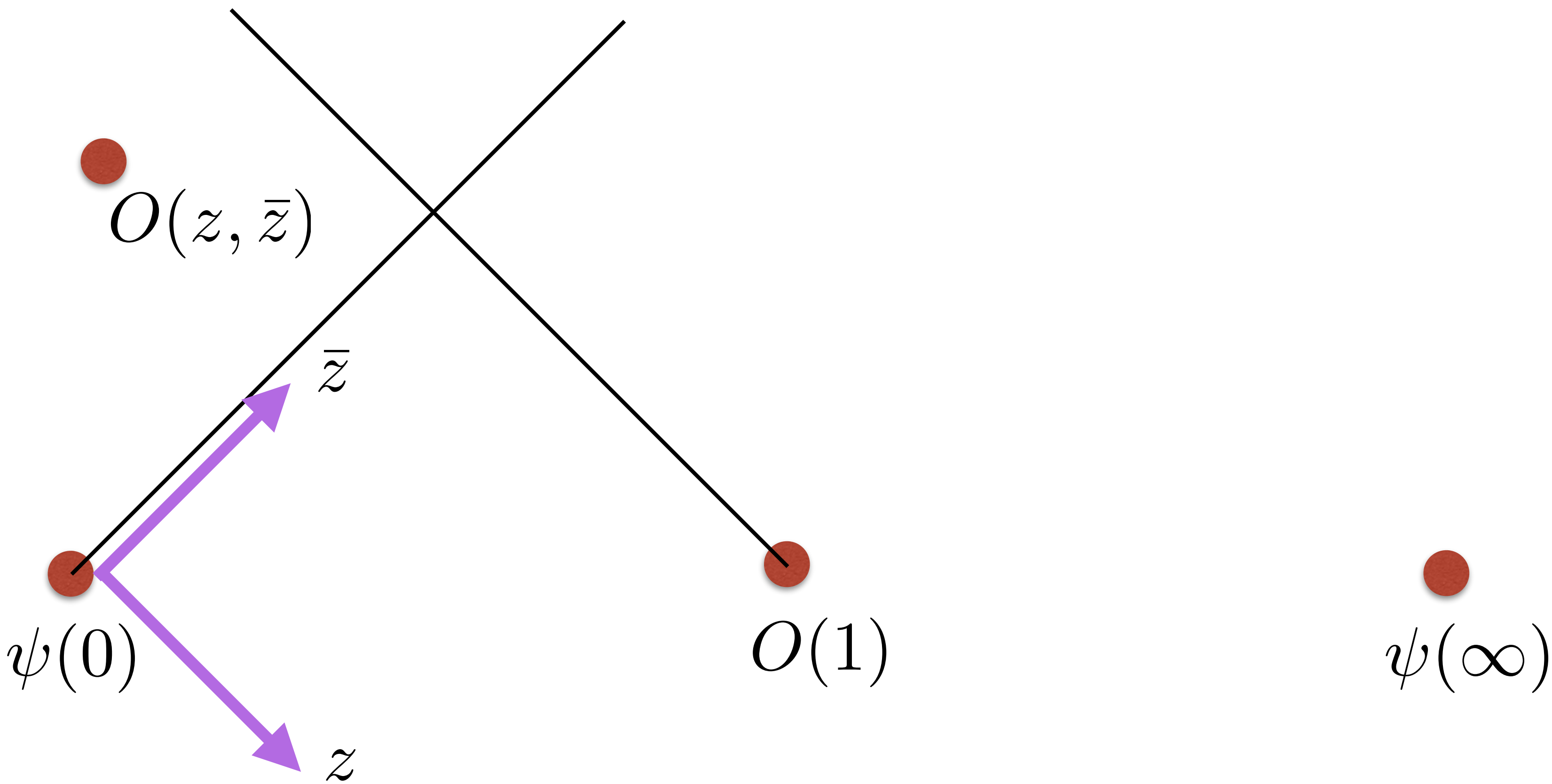


As a warm-up, we're going to prove [O,O] = 0 in this correlator.

We can reach timelike separation by sending

$$z \to -z$$

The *s* channel still converges:

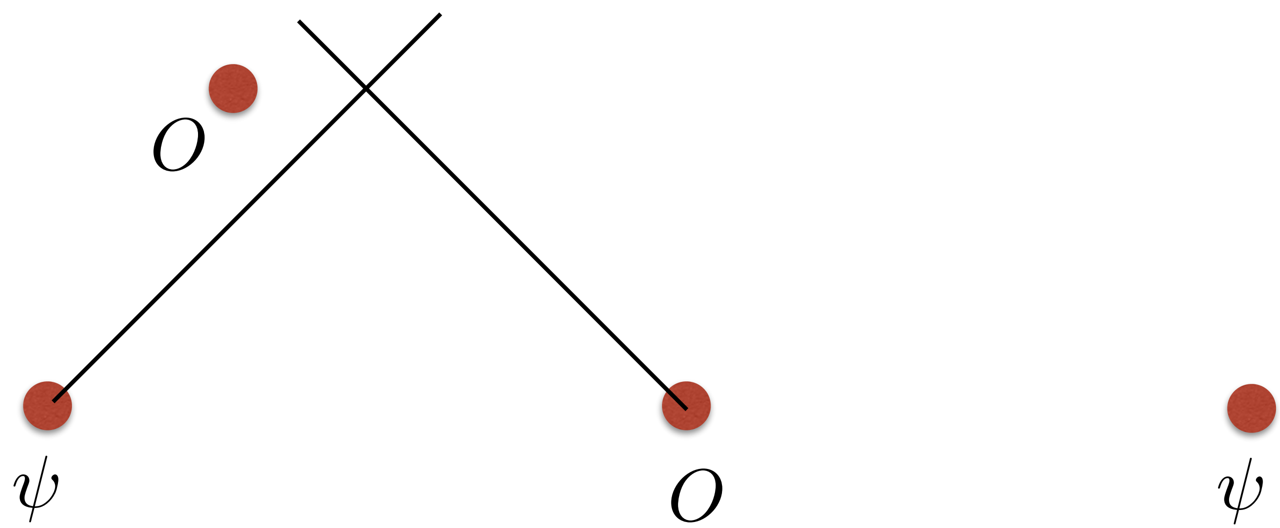
$$G \sim \sum_{\Delta,s} a_{\Delta,s} z^{\frac{1}{2}(\Delta-s)} \bar{z}^{\frac{1}{2}(\Delta+s)} (-1)^{\frac{1}{2}(\Delta-s)}$$

And, its magnitude *decreases*.

(*Careful:* The t-channel diverges! We'll come back to this.)

Since the magnitude decreases, there cannot be any new, unexpected singularities for negative *z.*

Therefore, in this configuration:



The correlator is causal: $\langle\psi[O,O]\psi\rangle = 0$

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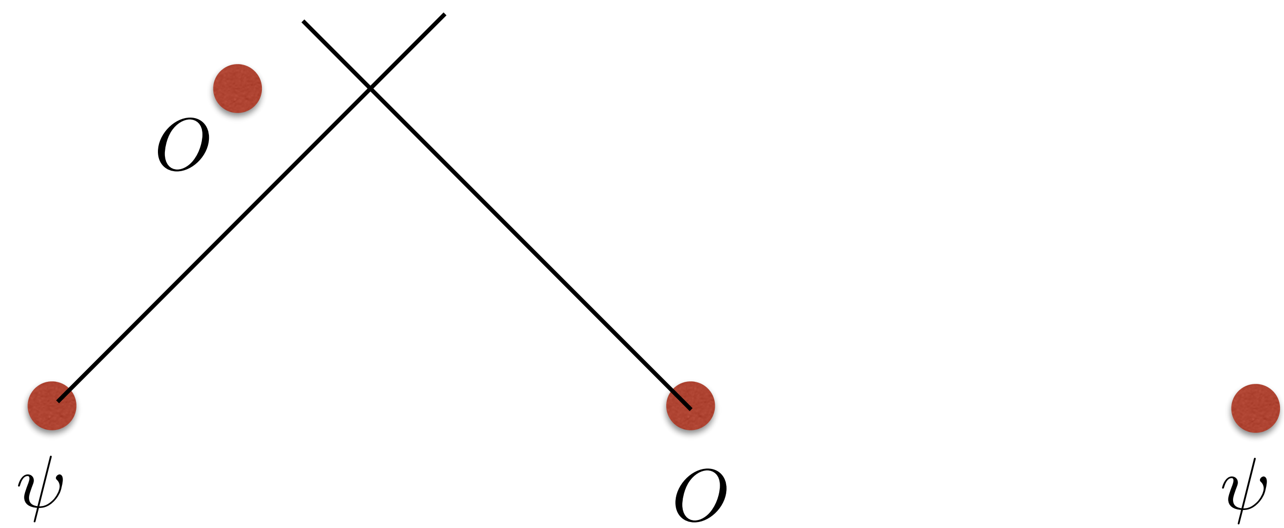

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This was just warm-up; it does not produce any interesting constraints.

We will find interesting constraints only be considering *two* timelike separations.

Outline

Causality in quantum field theory

OPE at timelike separation

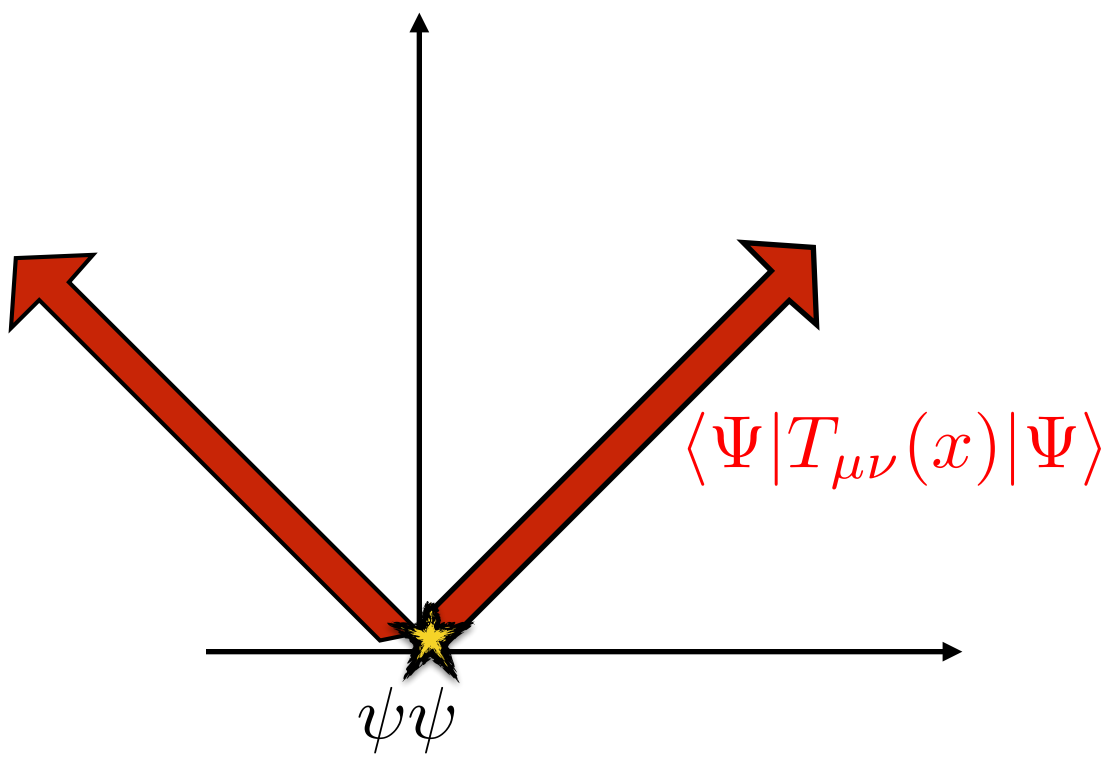
Shockwaves in CFT —> causality sum rule

Application: holographic derivation of $(\partial\phi)^4$ constraint

Define the “shockwave state”:

$$|\Psi\rangle \equiv \psi(t = i\delta, \vec{x} = 0)|0\rangle$$

For small δ this creates a stress tensor with support on an expanding null shell:



cf: previous work on shocks in AdS/CFT, esp. Cornalba et al

We will study the commutator

$$\langle\Psi|[O(x_2), O(x_3)]|\Psi\rangle$$

$$= \text{disc.}\ \langle\psi(-i\delta)O(x_2)O(x_3)\psi(i\delta)\rangle$$

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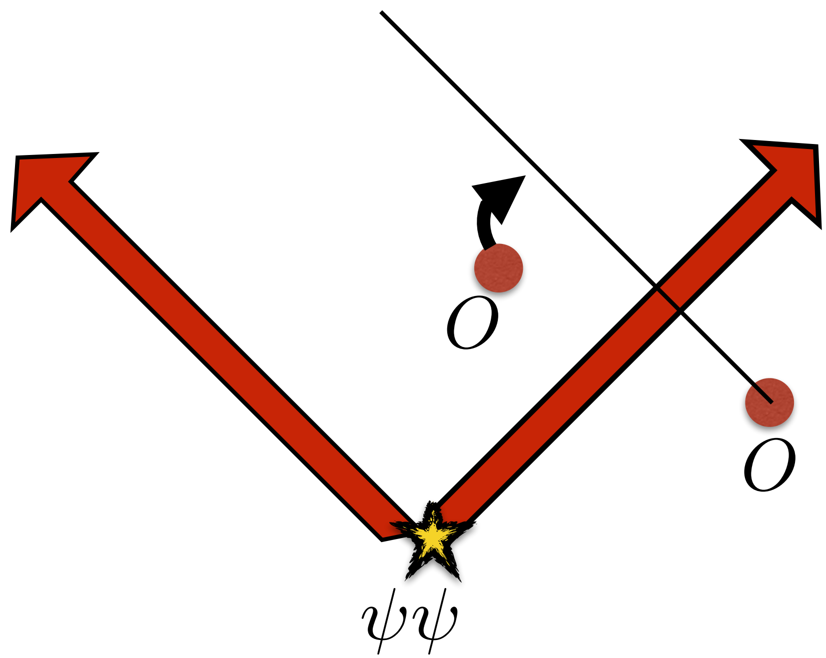
Follow the procedure described in the review:

- Draw contours on the complex-time planes
- "Go right" for time ordering, "Go left" for anti-time-ordering
- Then draw these same contours on z and $\bar{z}$ planes

Leads to:

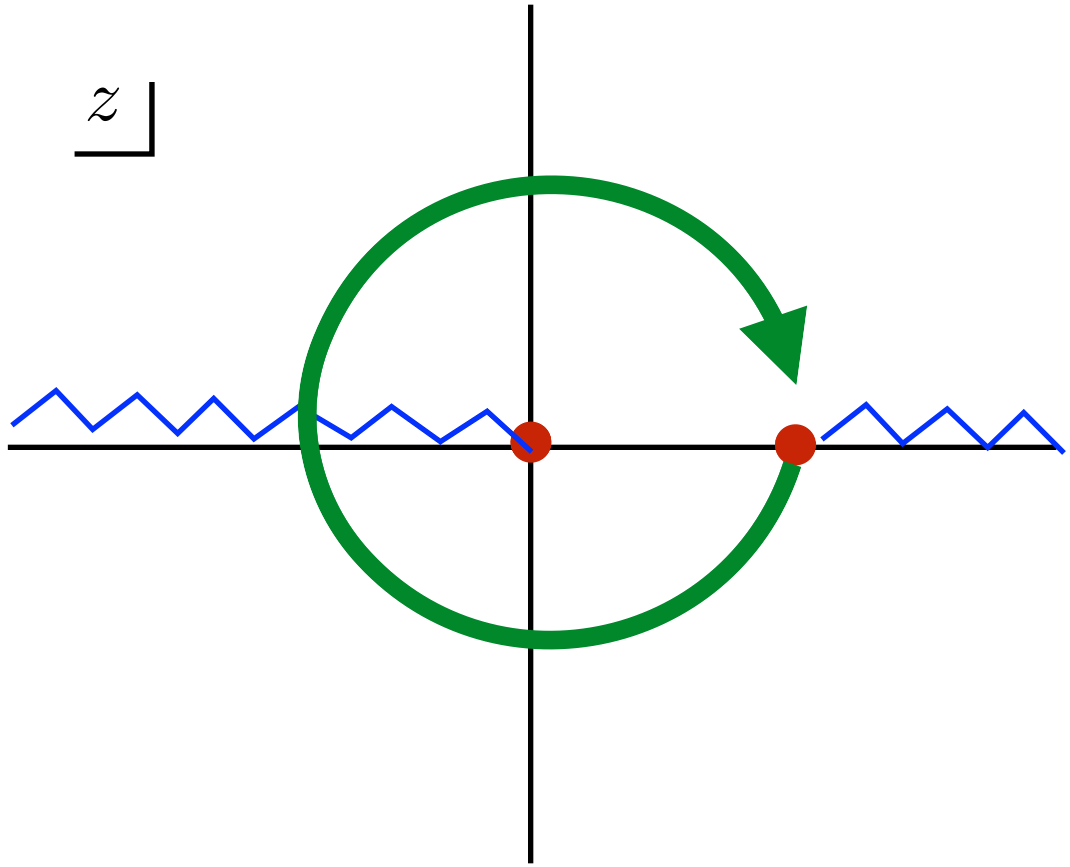
Causality in the situation



is translated into the following statement about the analytically-continued Euclidean correlator:

The Causality Requirement:

After taking *z* around zero,



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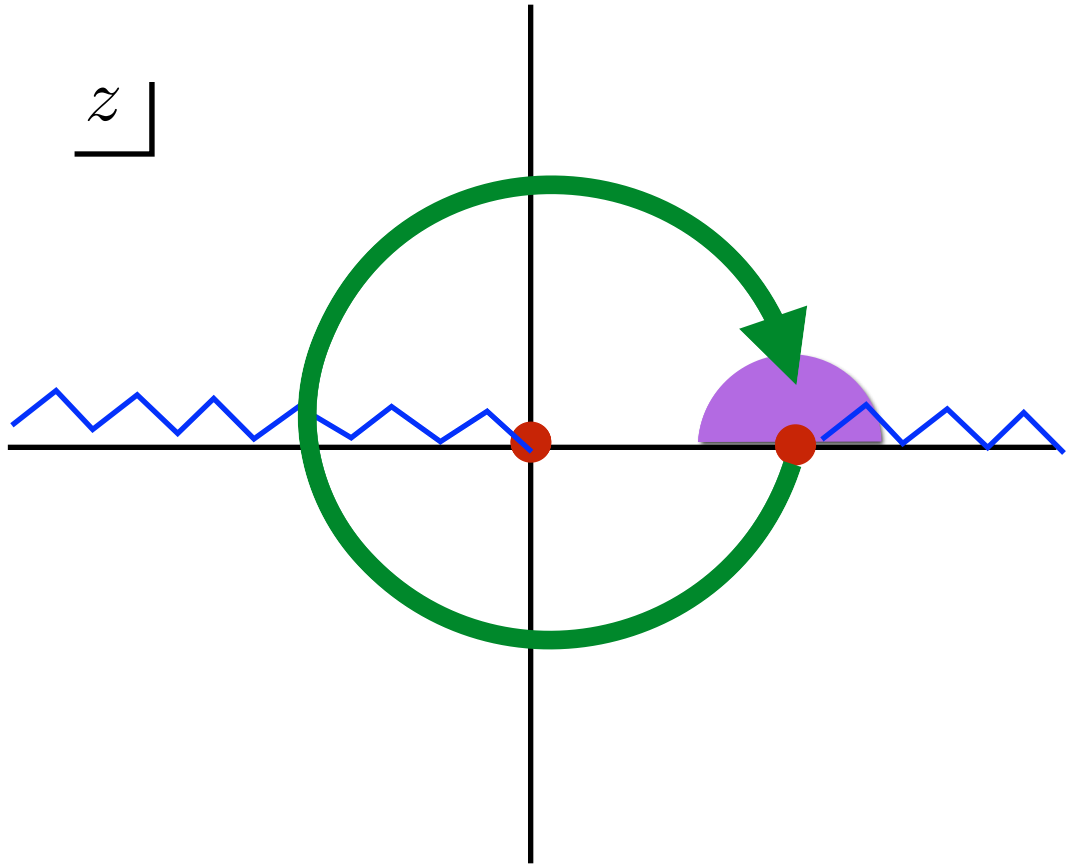

The lightcone singularity as *O* —> *O* must not appear in the purple region.

ie, it appears exactly at the red dot (=the Minkowski lightcone) or below it (=time delay), but not above it (=time advance)

Recap

Suppose we are given the Euclidean correlator

$$G(z,\bar{z}) = \langle \psi(0) O(z,\bar{z}) O(1) \psi(\infty) \rangle$$

Then

$$\langle \Psi | [O(x_2), O(x_3)] | \Psi \rangle$$

vanishes at spacelike separation if and only if

$$G(z e^{-2\pi i}, \bar{z})$$

is analytic for

$$z \sim 1 + i\epsilon, \quad \bar{z} \sim 1 + i\bar{\epsilon}$$

Claim:

Reflection positivity ==> Causality criterion is satisfied.

Proof is just like the “warm-up” argument a few slides back, but in “ρ-variable” conformal frame of [Pappadopulo et al] and [Hoogervorst, Rychkov].

TH, Jain & Kundu

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ie: reflection positivity

=> positive coefficients in the rho-expansion

=> bound magnitude of the correlator on the 2nd sheet by correlator on the 1st sheet

=> No new (causality-violating) singularities on 2nd sheet.

So far, most of what I’ve said was probably known in the 70’s in some form or another (via reconstruction theorems).

We invoked only the *s*-channel.

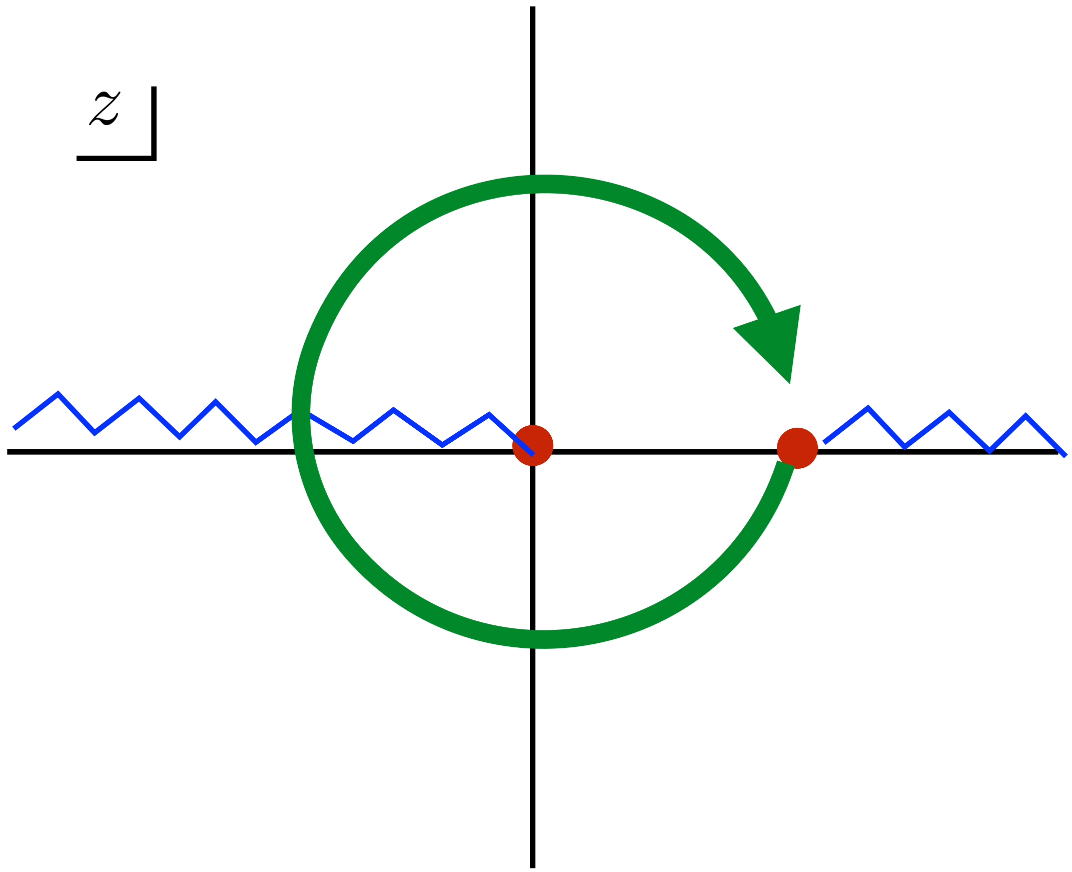
Next, we’ll show that this constrains the couplings of light operators in the *t*-channel.

This is closely related to the recent bound on chaos of [Maldacena, Shenker, Stanford].

The regime of the correlator is different — and the constraints do not (I believe) follow from the chaos bound, except at large N — but derivation is similar.

Goal of the rest of the talk is to show that

Crossing + Analyticity on the purple region:



restrict the allowed couplings among light operators, if one of the operators has spin > 1.

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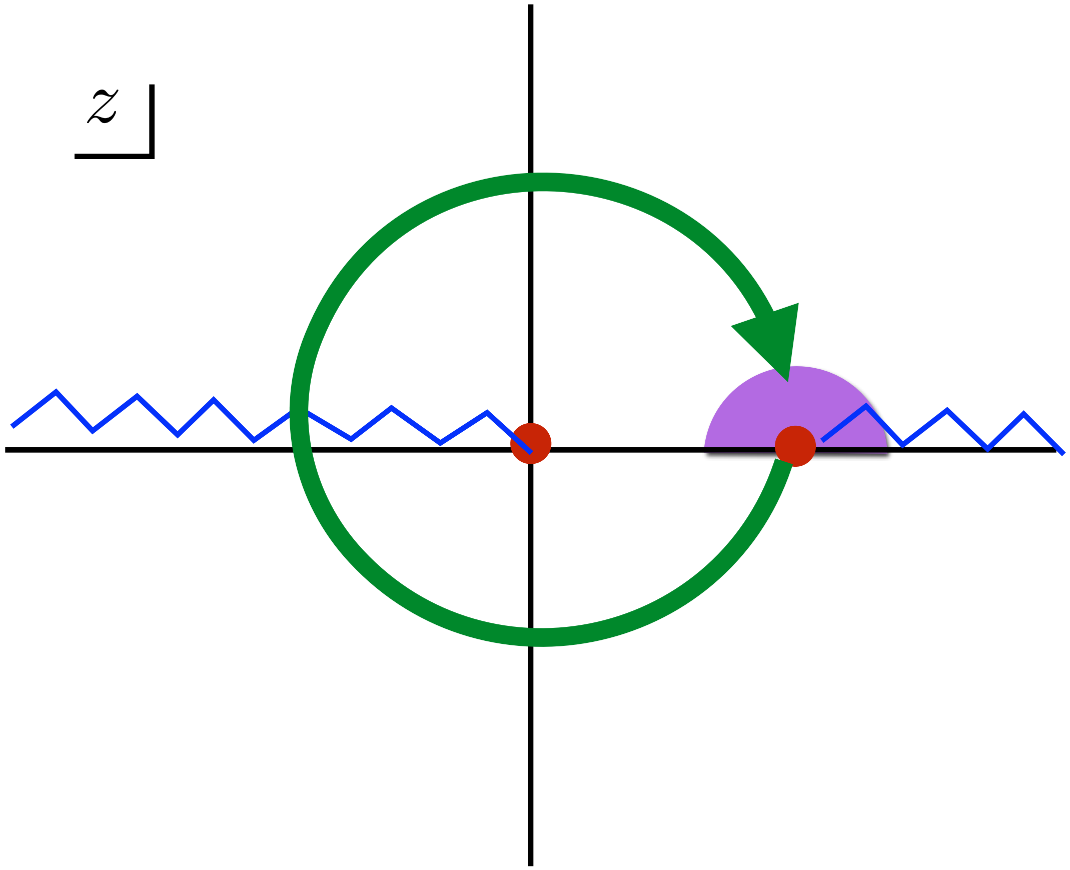
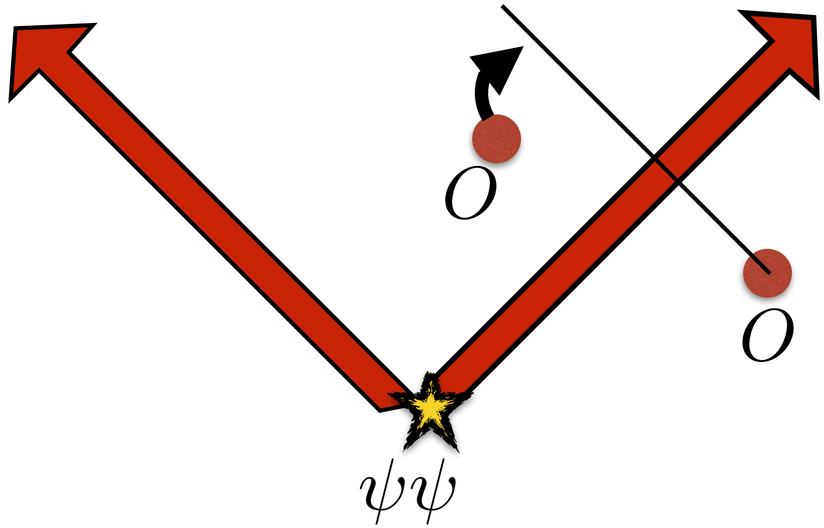


restrict the allowed couplings among light operators, if one of the operators has spin > 1.

Expand in the t channel *O* -> *O*



- The conformal block expansion diverges!

- However, it is a reliable asymptotic series in the lightcone limit $\bar{z} \to 1 \quad (z \text{ fixed})$

[TH, Jain, Kundu]

In the lightcone limit, the *t*-channel double-sum

$$\sum_{\Delta,\ell} c_{\Delta,\ell}\; g_{\Delta,\ell}(1-z, 1-\bar{z})$$

is re-organized as an expansion in twist:

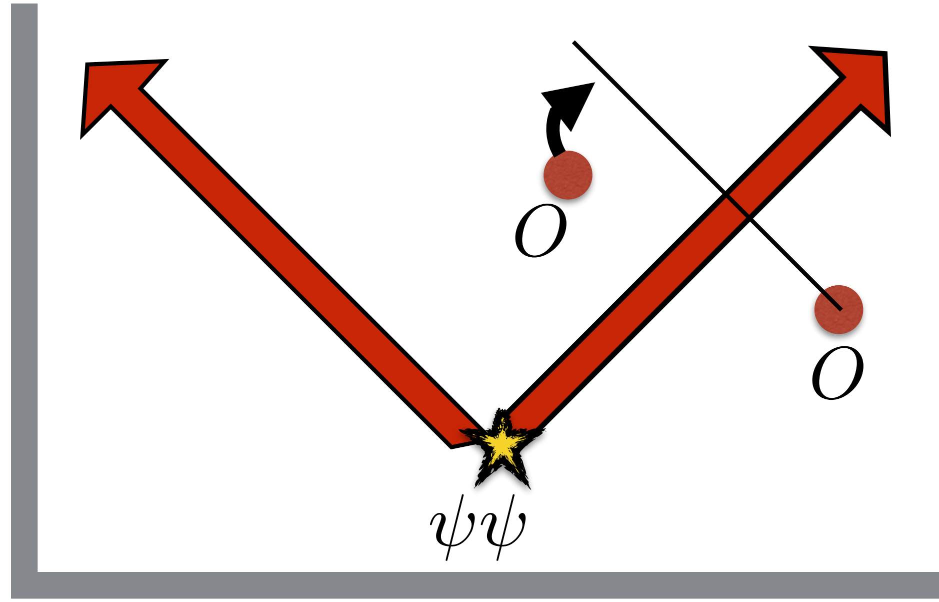
$$\sum_h a_h (1-\bar{z})^h f_h(z) \qquad\qquad h = \frac{1}{2}(\Delta - \ell)$$

where *f* is the lightcone (or colinear) conformal block.

[Komargodski, Zhiboedov;
Fitzpatrick et al; Alday et al; etc]

Let's evaluate the leading term in the shockwave kinematics:

Set *d=4* and assume *T* is the minimal-twist operator.



The stress-tensor lightcone block is

$$\tilde{g}_T(1-z) = -\frac{15(3-3z^2+(1+4z+z^2)\log z)}{2(1-z)^2}$$

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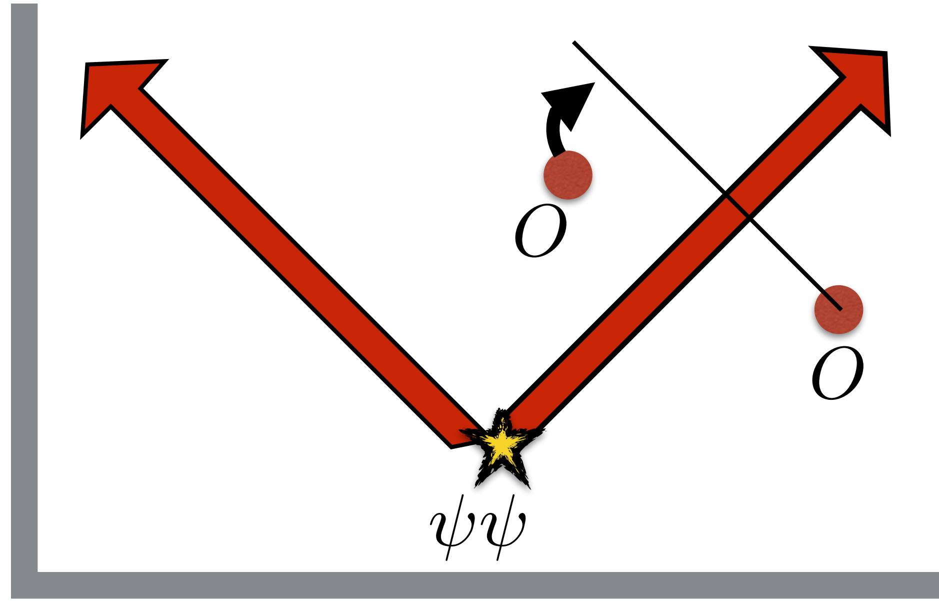
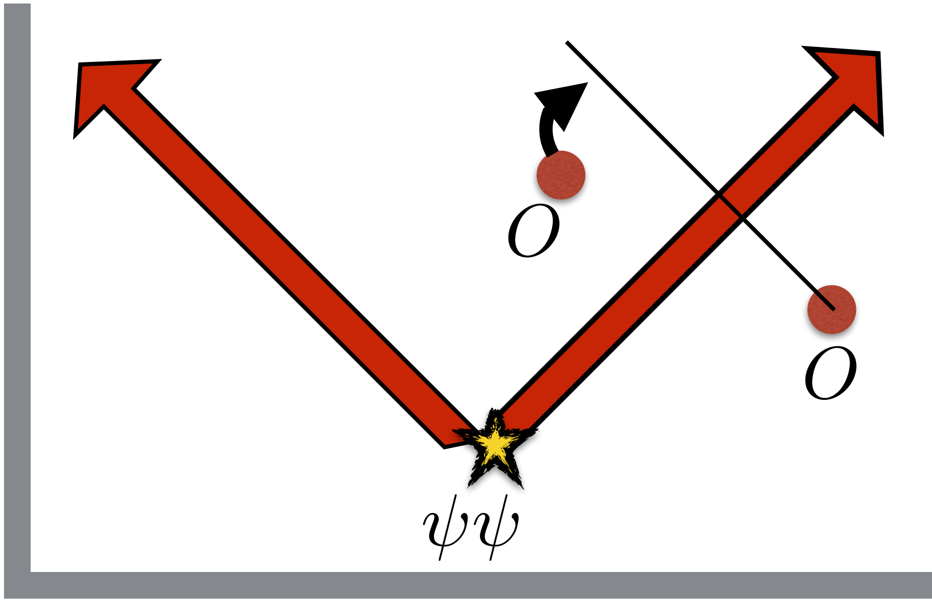


$$\tilde{g}_T(1-z) = -\frac{15(3-3z^2+(1+4z+z^2)\log z)}{2(1-z)^2}$$

Expand on 1st sheet: $1-\bar{z} \ll 1-z \ll 1$

$$\langle O\psi O\psi\rangle \sim 1 + a_T(1-\bar{z})(1-z)^3 + \cdots$$

Set d=4 and assume T is the minimal-twist operator.



The stress-tensor lightcone block is

$$\tilde{g}_T(1-z) = -\frac{15(3-3z^2+(1+4z+z^2)\log z)}{2(1-z)^2}$$

Expand on 1st sheet: $1-\bar{z} \ll 1-z \ll 1$

$$\langle O\psi O\psi \rangle \sim 1 + a_T(1-\bar{z})(1-z)^3 + \cdots$$

Expand on 2nd sheet: $1-\bar{z} \ll 1-z \ll 1$

$$\langle \psi O O\psi \rangle \sim 1 - 2\pi i a_T \frac{1-\bar{z}}{(1-z)^2} + \cdots$$

Logs —> Potentially large corrections on 2nd sheet!

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The limit of a high-energy shockwave is

$$\delta \to 0, \quad z, \bar{z} \to 1$$

("Regge limit").

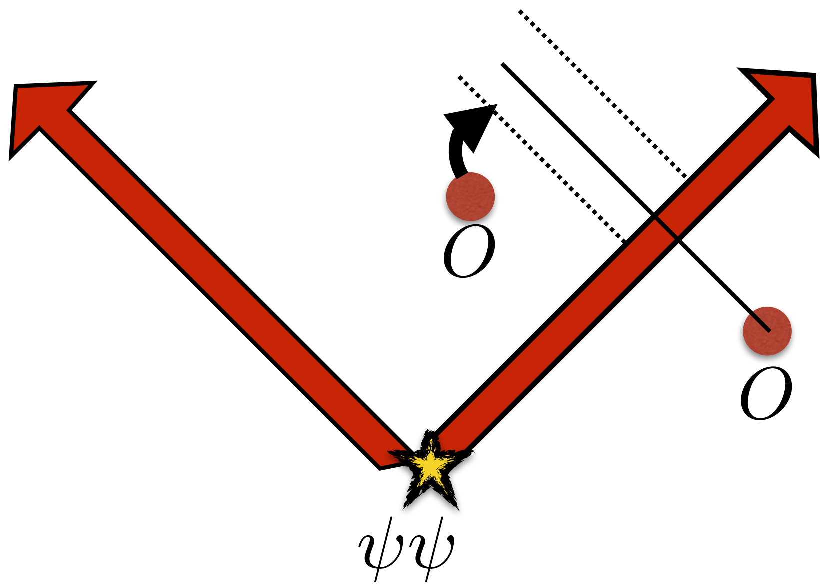
The correction looks singular in this limit $\sim \delta^{-(s-1)}$

However, it can't be trusted, since we assumed the lightcone limit

$$\bar{z} \to 1 \quad (z \text{ fixed})$$

Intuition:

Negative powers are a "perturbative hint" of causality violation if the coefficient has the wrong sign. For example if a lightcone shifts:



we might expect perturbatively something like

$$(1+\delta t-\bar{z})^{-\Delta_O} \approx (1-\bar{z})^{-\Delta_O}\left[1-\delta t\frac{\Delta_O}{1-\bar{z}}+O(\delta t)^2\right]$$

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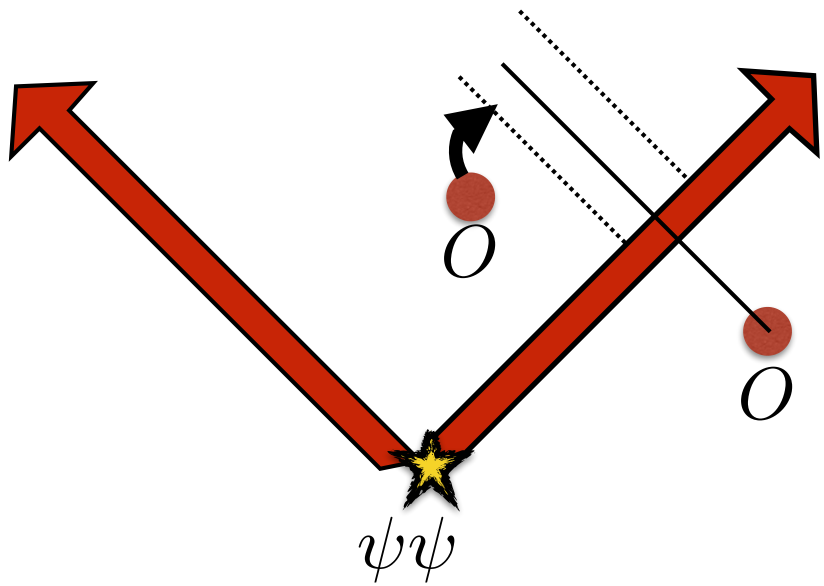

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negative power!

This is just for intuition.

Causality is a statement about singularities, and cannot be diagnosed from small perturbative terms alone.

So now, we need to argue that a small term like this with the wrong sign *always* re-sums into a full-blown, causality-violating singularity.

Aside:

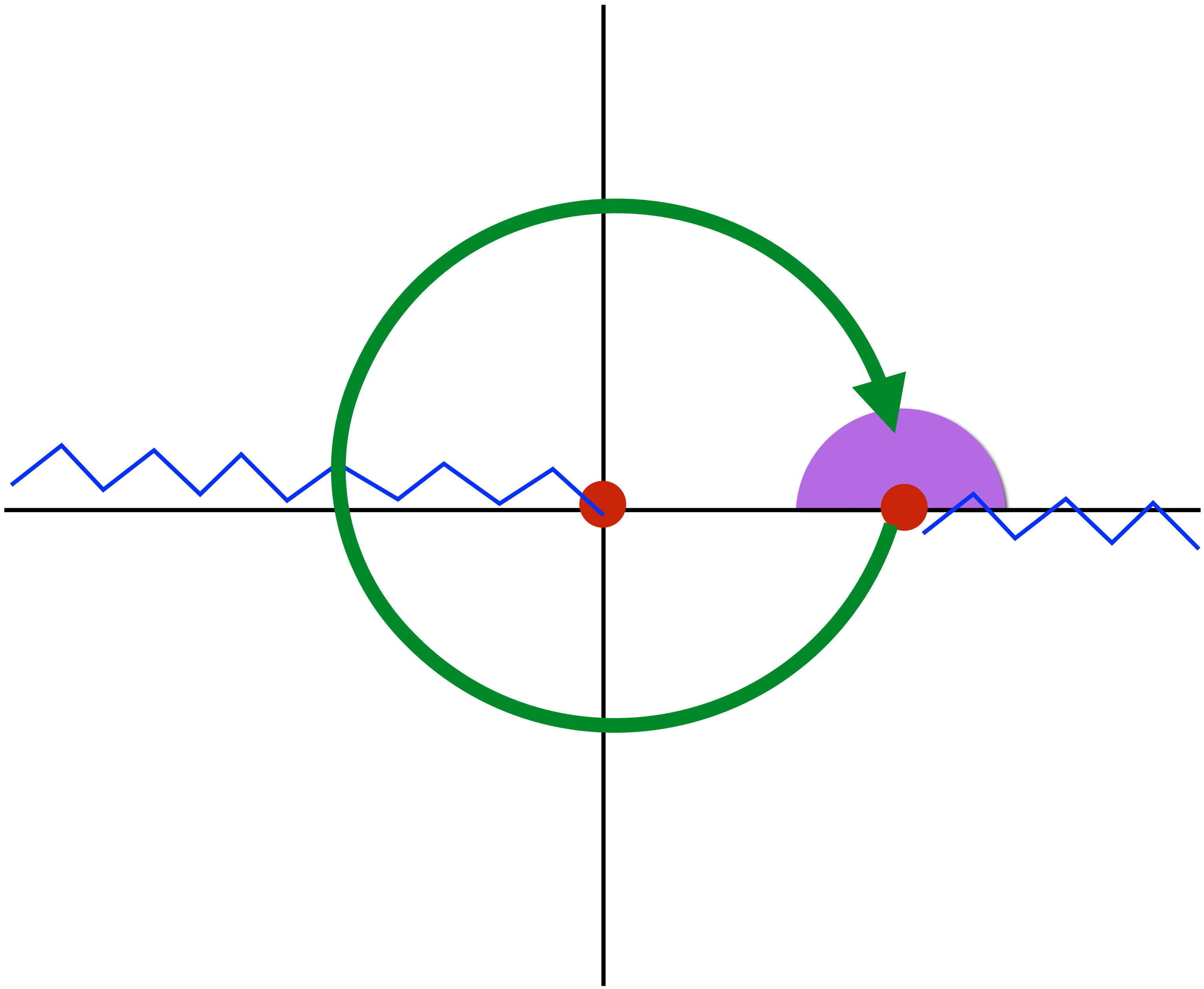
These same logs played a starring role in the lightcone bootstrap

[Komargodski and Zhiboedov]
[Fitzpatrick, Kaplan, Poland, Simmons-Duffin]
etc

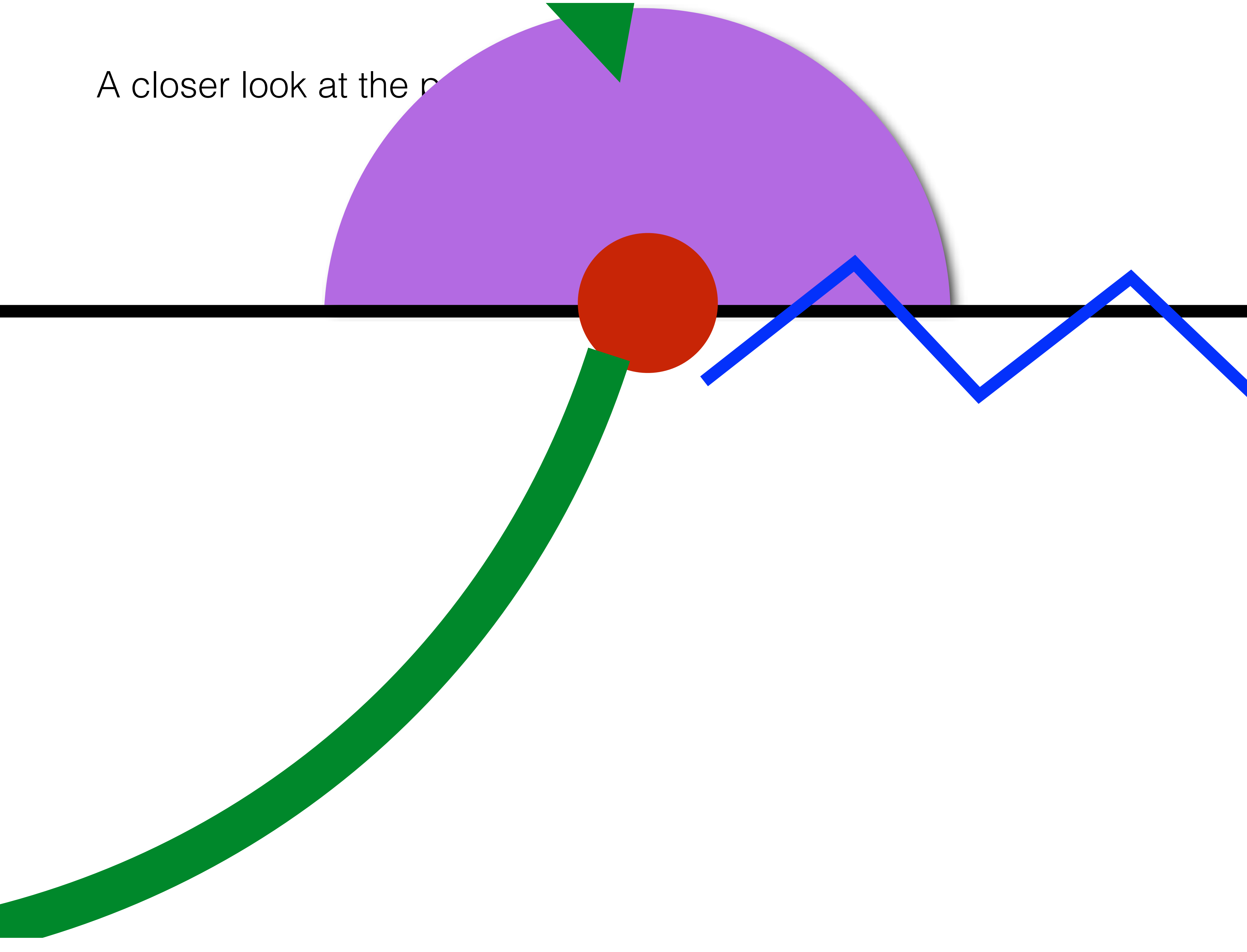
Log coefficients fix the anomalous dimensions of certain high spin double-trace operators in the dual channel.

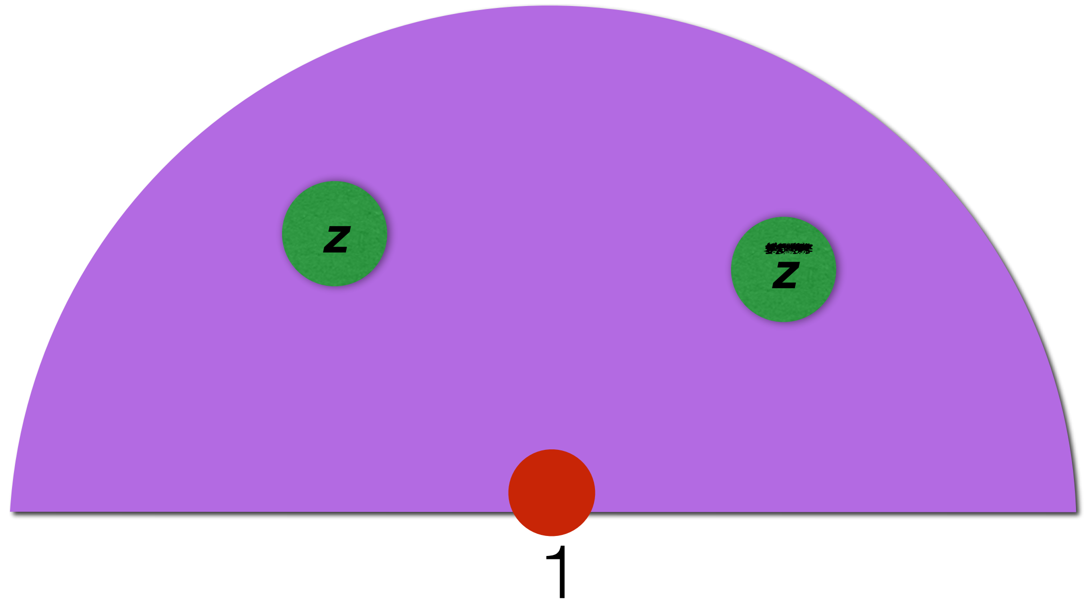
We will prove the log coefficients are positive; so this also proves those anomalous dimensions are negative.

A closer look at the purple region:

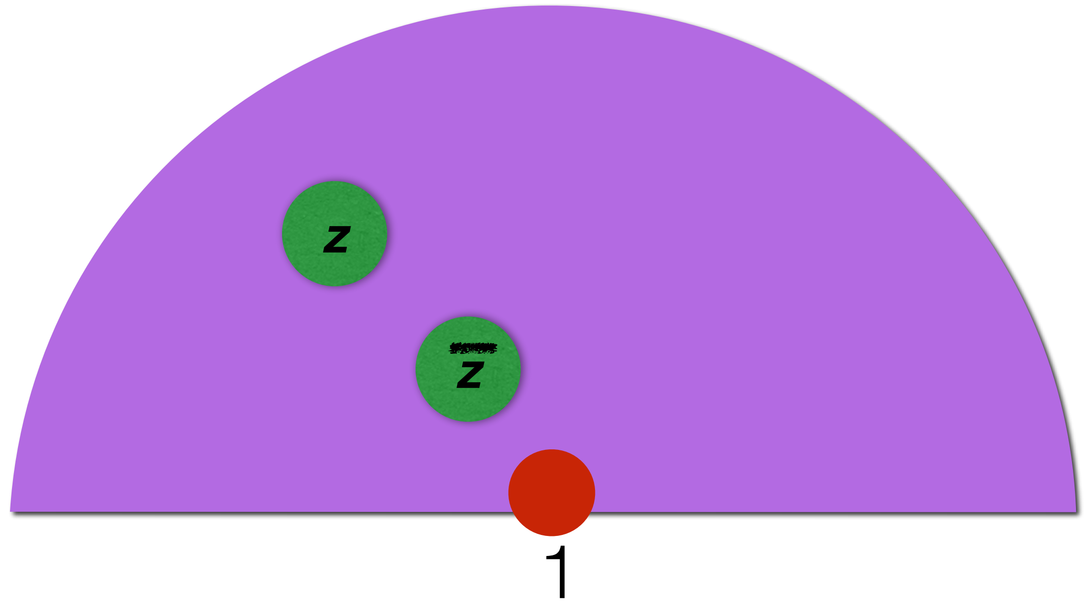


A closer look at the p





Regimes:



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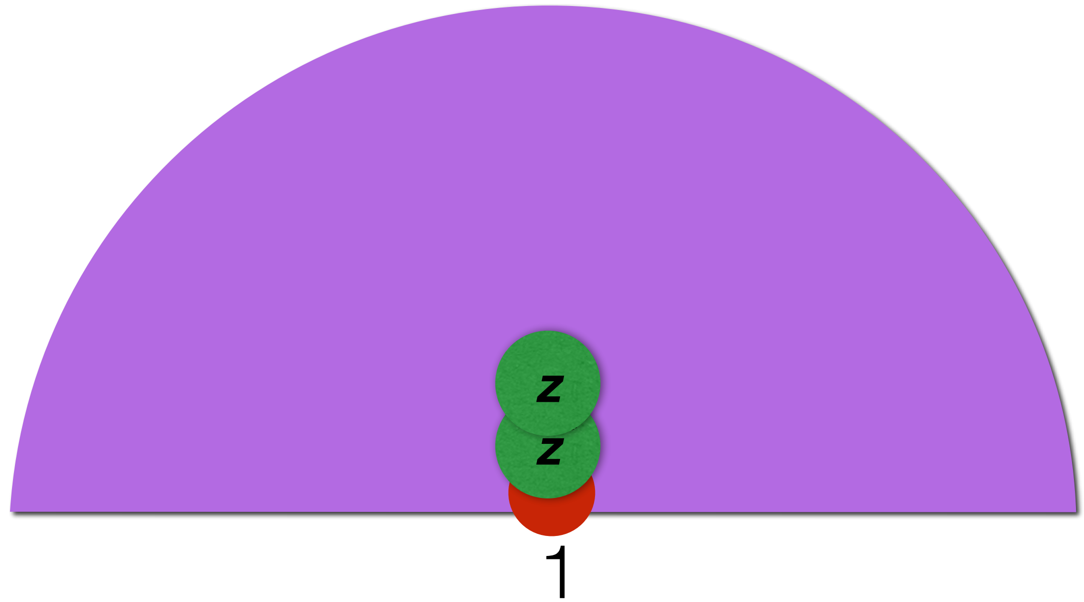
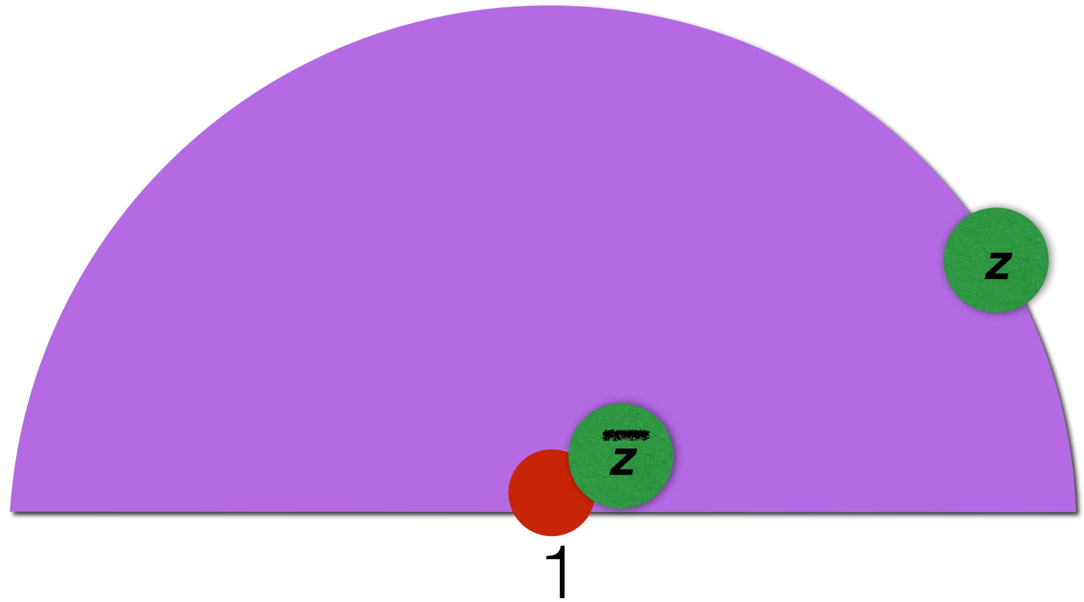


Regimes:

- Regge (strong shockwave) $z, \bar{z} \sim 1$

-we know almost nothing about the correlator — no OPE!
-exceptions: Chaos bound [Maldacena, Stanford, Shenker]
-and large N [esp. Cornalba, Costa, Penedones, Schiappa]



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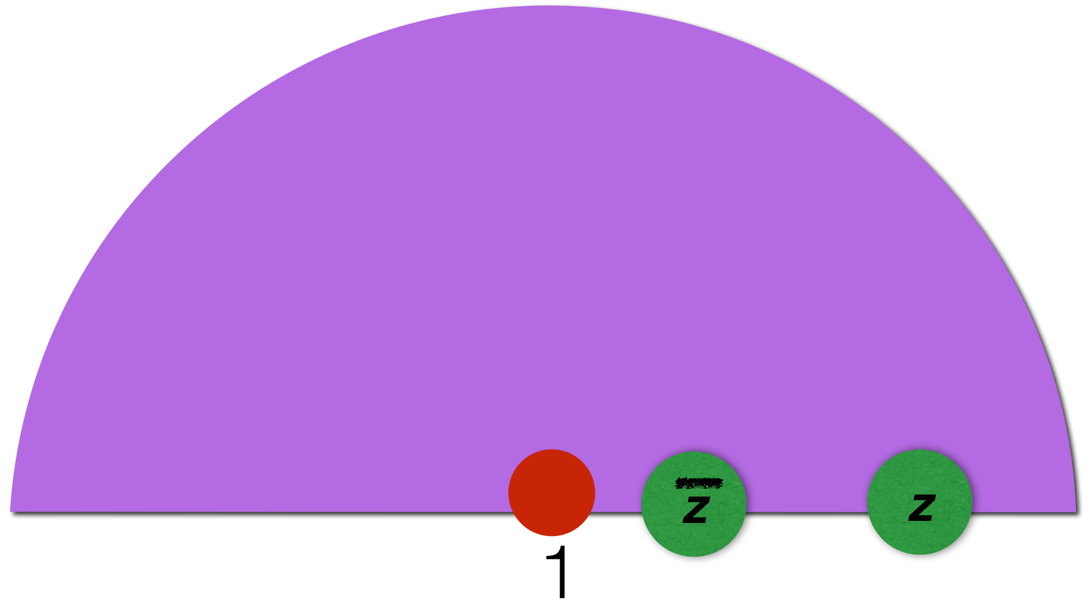
  calculate correlator by asymptotic expansion in *t*-channel



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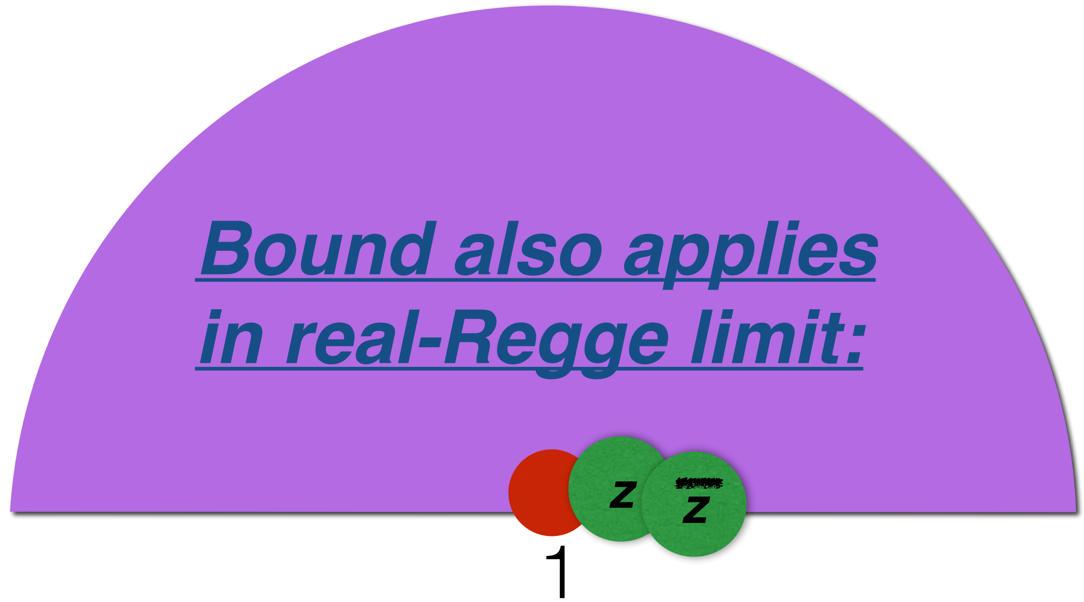

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Theorem

A function with these properties, and the lightcone expansion

$$\langle \psi O O \psi \rangle \sim 1 - 2\pi i a_T \frac{1-\bar{z}}{(1-z)^2} + \cdots$$

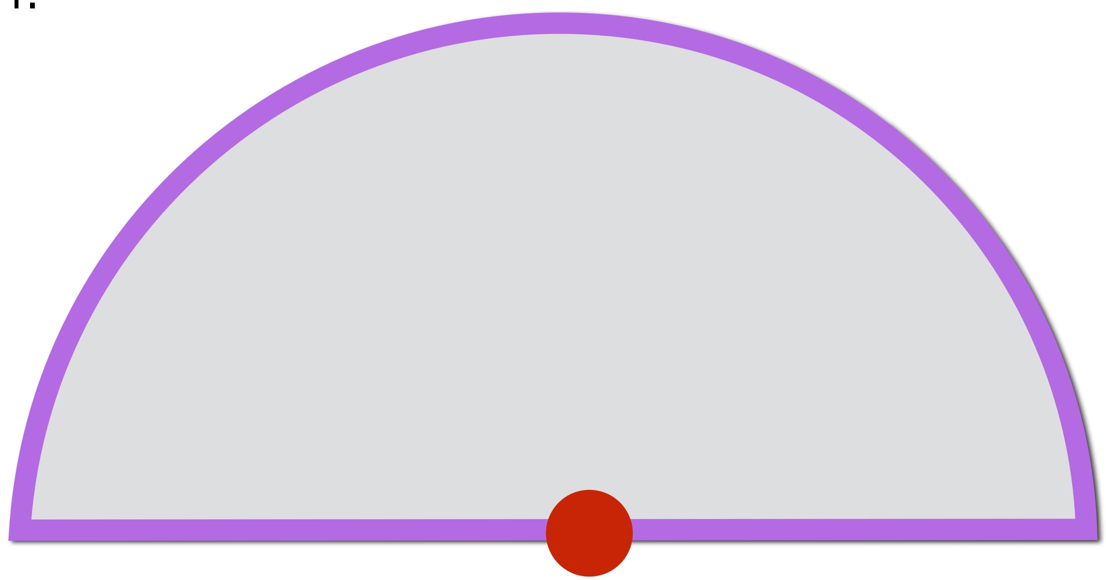
either has

$$a_T \geq 0$$

or has a causality-violating singularity in the purple region.

Proof

Integrate $\mathrm{Re} \oint G(z, \bar{z})$ over the boundary of the purple region:



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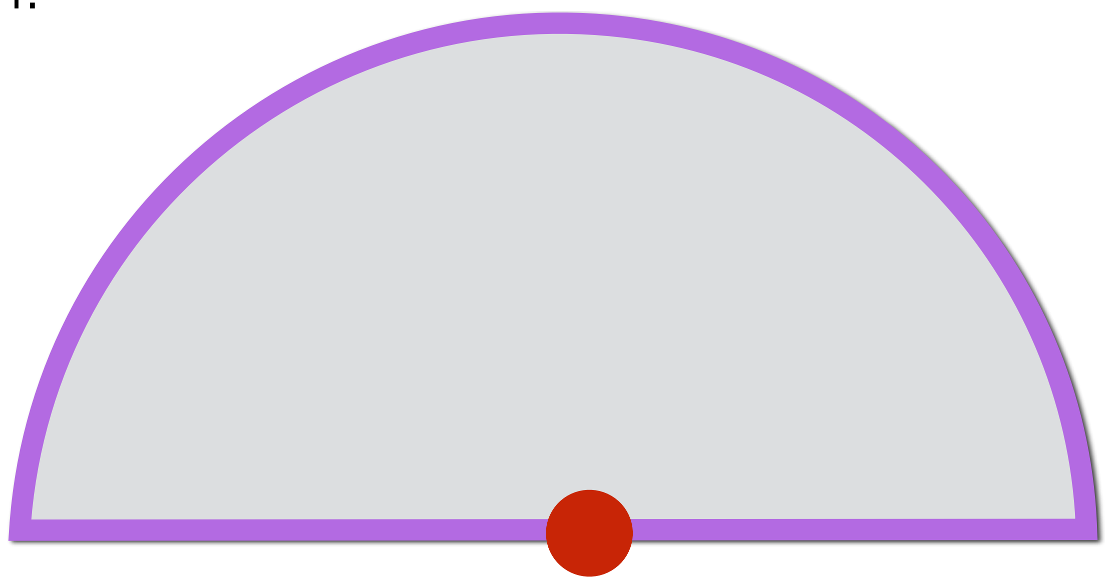

(This is the *z*-contour, with the z-bar contour defined by

$$1 - \bar{z} = \eta(1 - z) \qquad \eta \ll 1 \quad \text{fixed}$$

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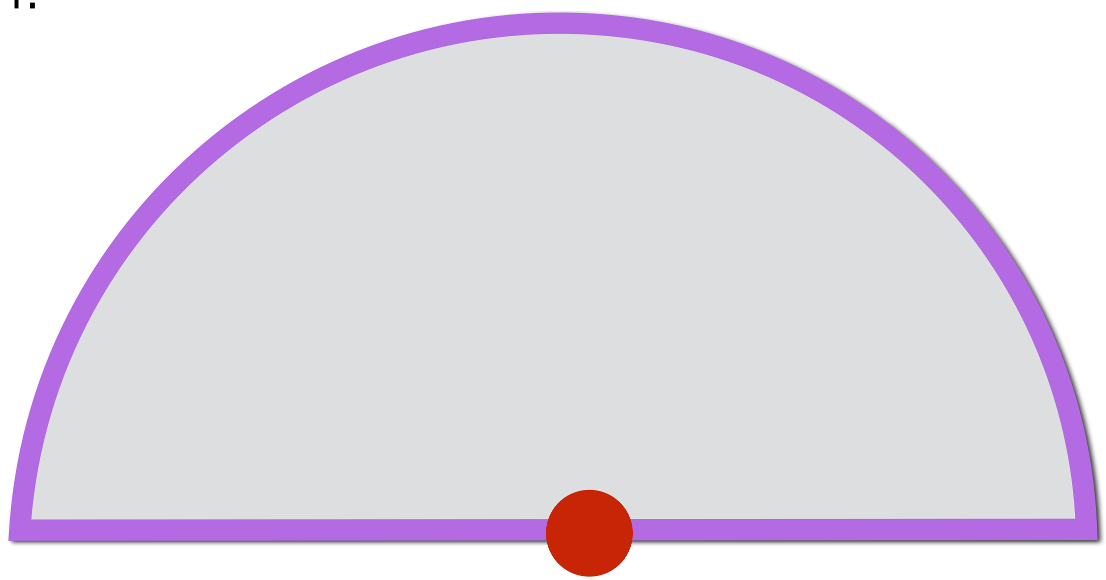

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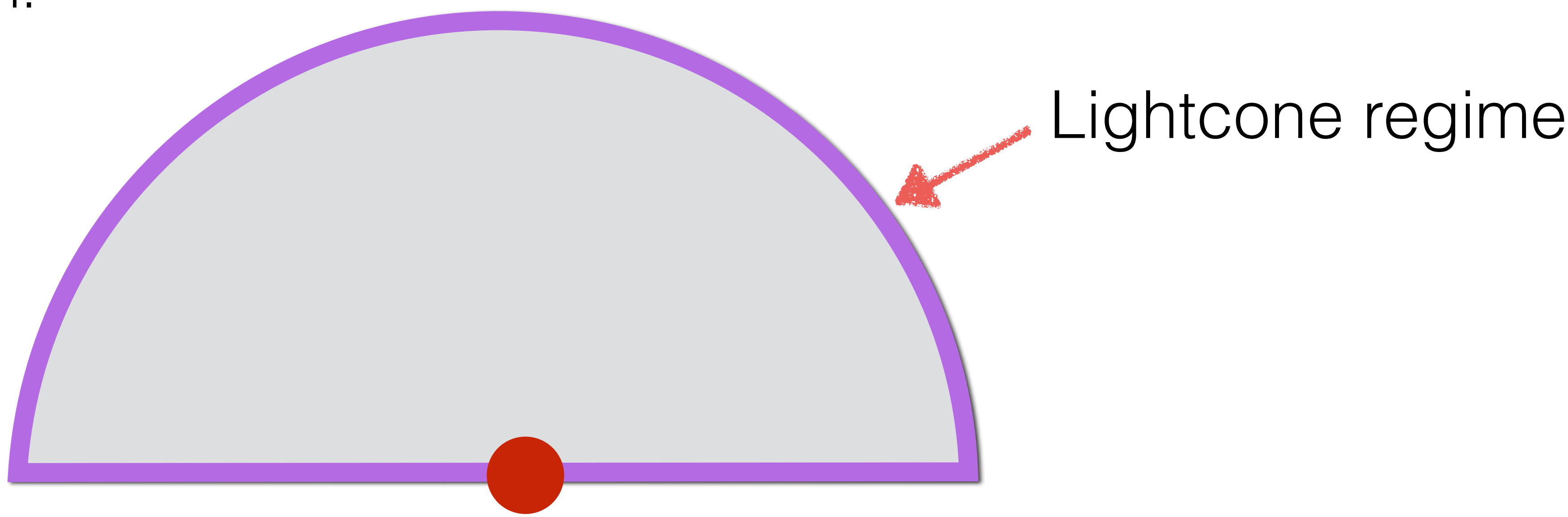

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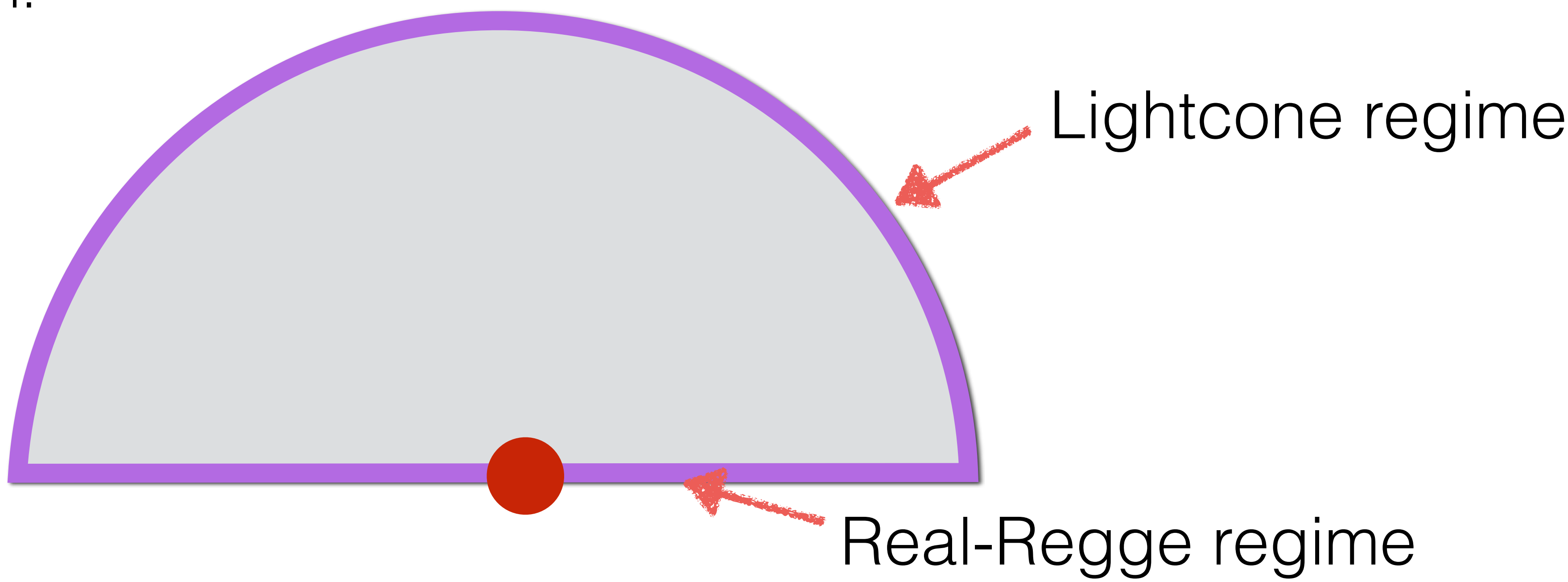

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The sum rule is

$$
\begin{aligned}
0 &= \mathrm{Re} \oint G \\
&= \mathrm{Re} \int_{lightcone} G + \mathrm{Re} \int_{Real-Regge} G
\end{aligned}
$$

Plugging in known lightcone *G* and massaging gives:

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a_T &= \int_{Real-Regge} \mathrm{Re} \quad (\langle O\psi O\psi \rangle - \langle \psi O O \psi \rangle) \\
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$$a_T = \int_{Real-Regge} \text{Re} \quad (\langle O\psi O\psi \rangle - \langle \psi O O \psi \rangle)$$
$$> 0$$

“position space optical theorem”
analogous theorem holds in non-conformal QFT

Conversely,

$$a_T < 0$$

implies the correlation function has a causality-violating singularity (somewhere in the purple region), or violates crossing symmetry.

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What actually went into this argument?

- Lightcone blocks in the t-channel
- Positive coefficients in the *s* and *u* channels

By now somebody has probably objected that the conformal Ward identity fixes the coupling of scalars to the stress tensor:

$$a_T = (positive) \times \frac{\Delta_O}{c}$$

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However, an identical argument produces nontrivial bounds. Three examples:

- Spin > 2 currents: *both* signs ruled out! [Maldacena, Zhiboedov]
- Leading term is some *other* spin-2 operator
- External operators with spin

Outline

Causality in quantum field theory

Conformal bootstrap at timelike separation

Shockwaves in CFT —> main theorem (causality sum rule)

Application: holographic derivation of $(\partial\phi)^4$ constraint

A scalar EFT in flat space with Lagrangian

$$(\partial\phi)^2 + \lambda(\partial\phi)^4 + \cdots$$

is causal if and only if

$$\lambda > 0$$

Now we'll derive this in AdS, from the dual CFT.

Gravity is decoupled in this EFT; thus the stress tensor in the dual CFT is decoupled.

Following [Heemskerk, Penedones, Polchinski, Sully],

this bulk interaction translates into an anomalous dimension for the double-trace operators in the CFT:

$$O\Box^n \partial_\mu^\ell O$$

$$\Delta = 2\Delta_O + 2n + \ell + \gamma_{n,\ell}$$

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This anomalous dimension appears in front of a log in the conformal block expansion.

And its sign is fixed by our argument:

$$\gamma_{n,\ell} < 0 \Rightarrow \lambda_{bulk} > 0$$

We also find nontrivial bounds on spinning correlators, *eg*

$$\langle\psi\ T_{\mu\nu}T_{\sigma\rho}\ \psi\rangle$$

cf:

Hofman & Maldacena;

Camanho, Edelstein, Maldacena, Zhiboedov;

etc.

[TH, Jain, and Kundu, *in progress*]

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Thank you.