Inhomogeneous chiral condensates in the QCD phase diagram

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Motivation

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- Standard argument for the possible existence of the chiral critical point:
  - Lattice QCD: crossover at $\mu = 0$
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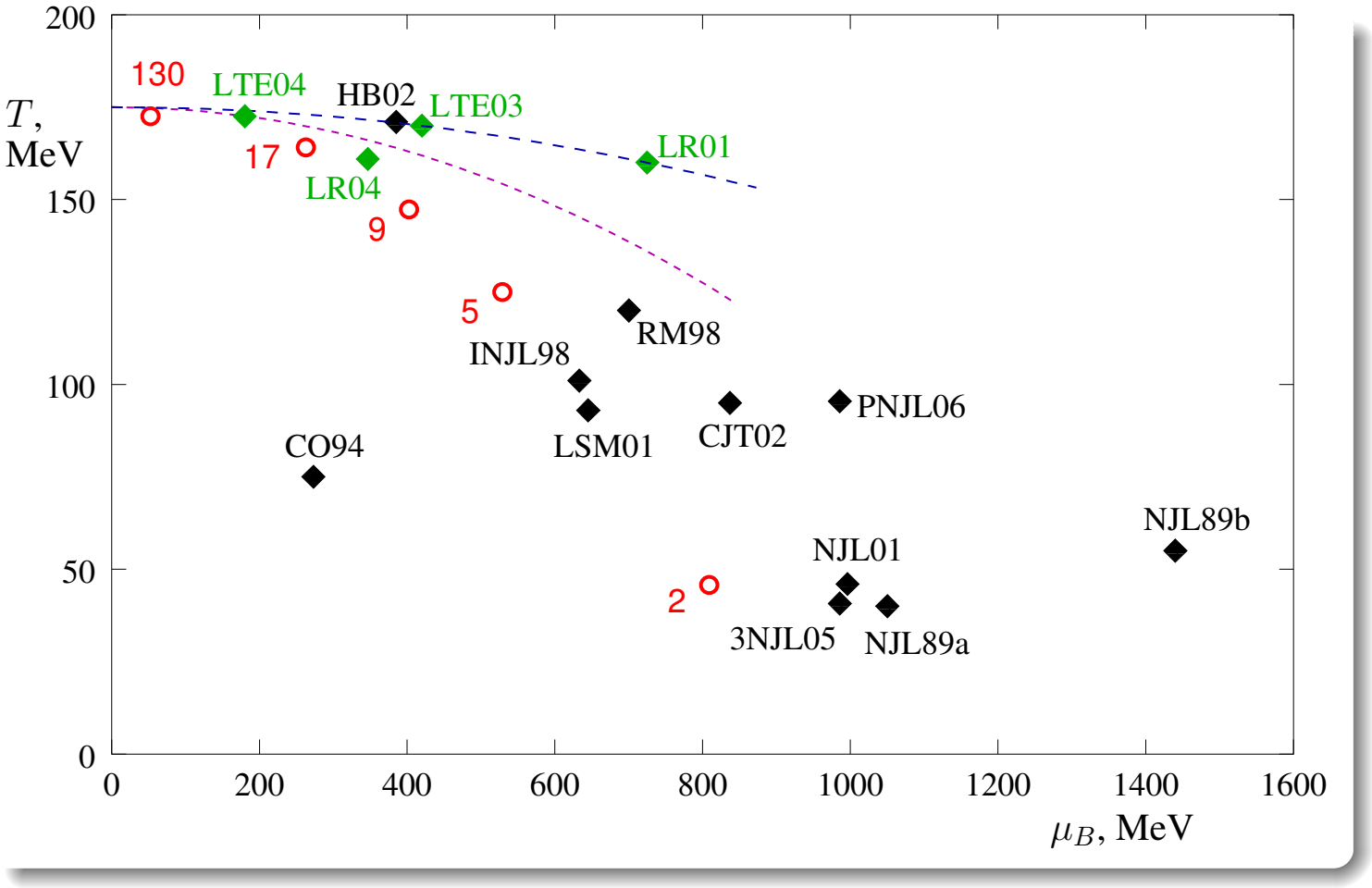
Compilation of critical points

[M. Stephanov, PoSLAT (2006)]



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NJL model [D. Nickel, PRD (2009)]

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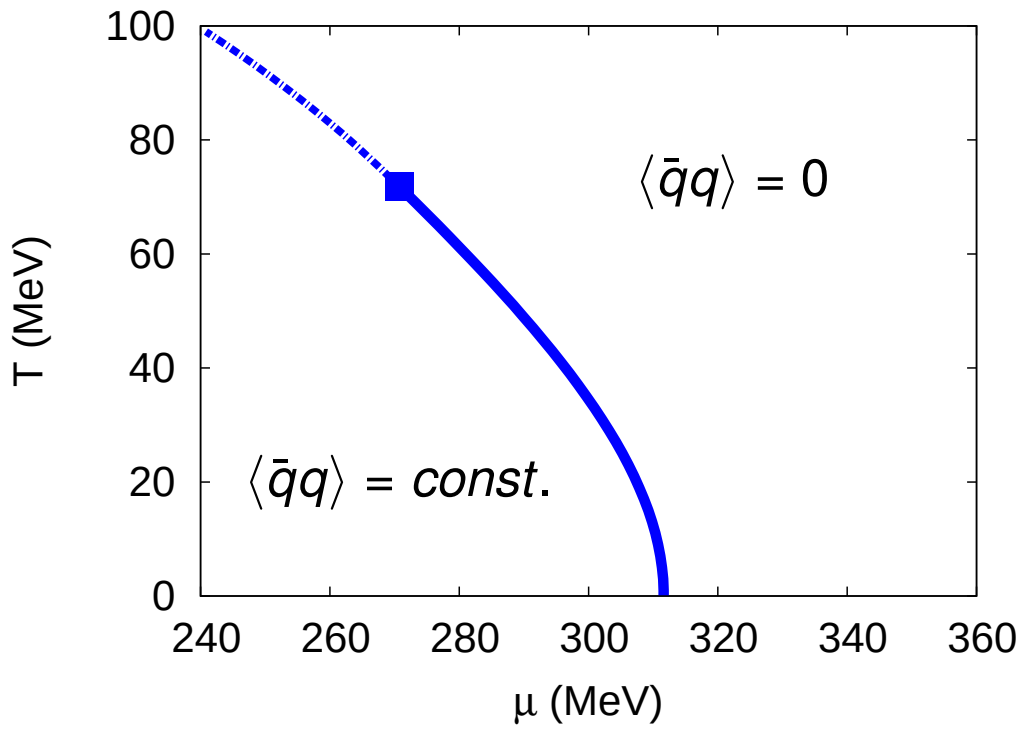

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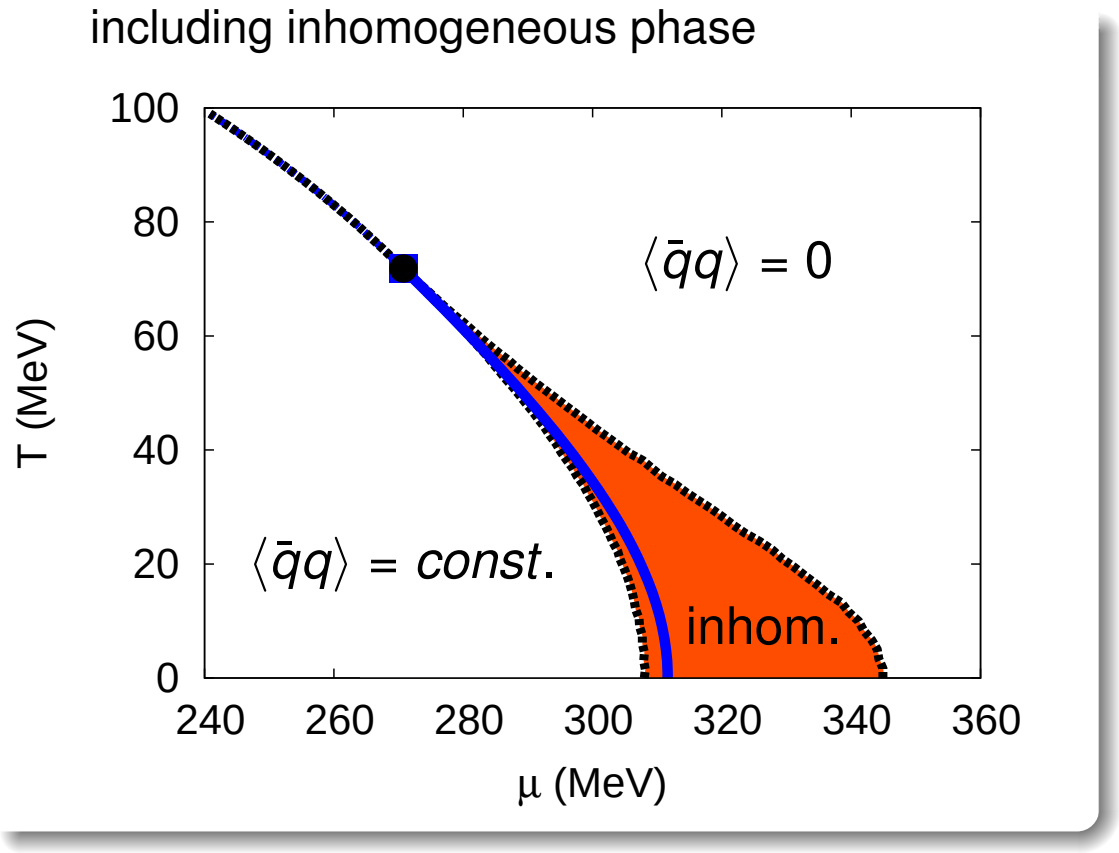
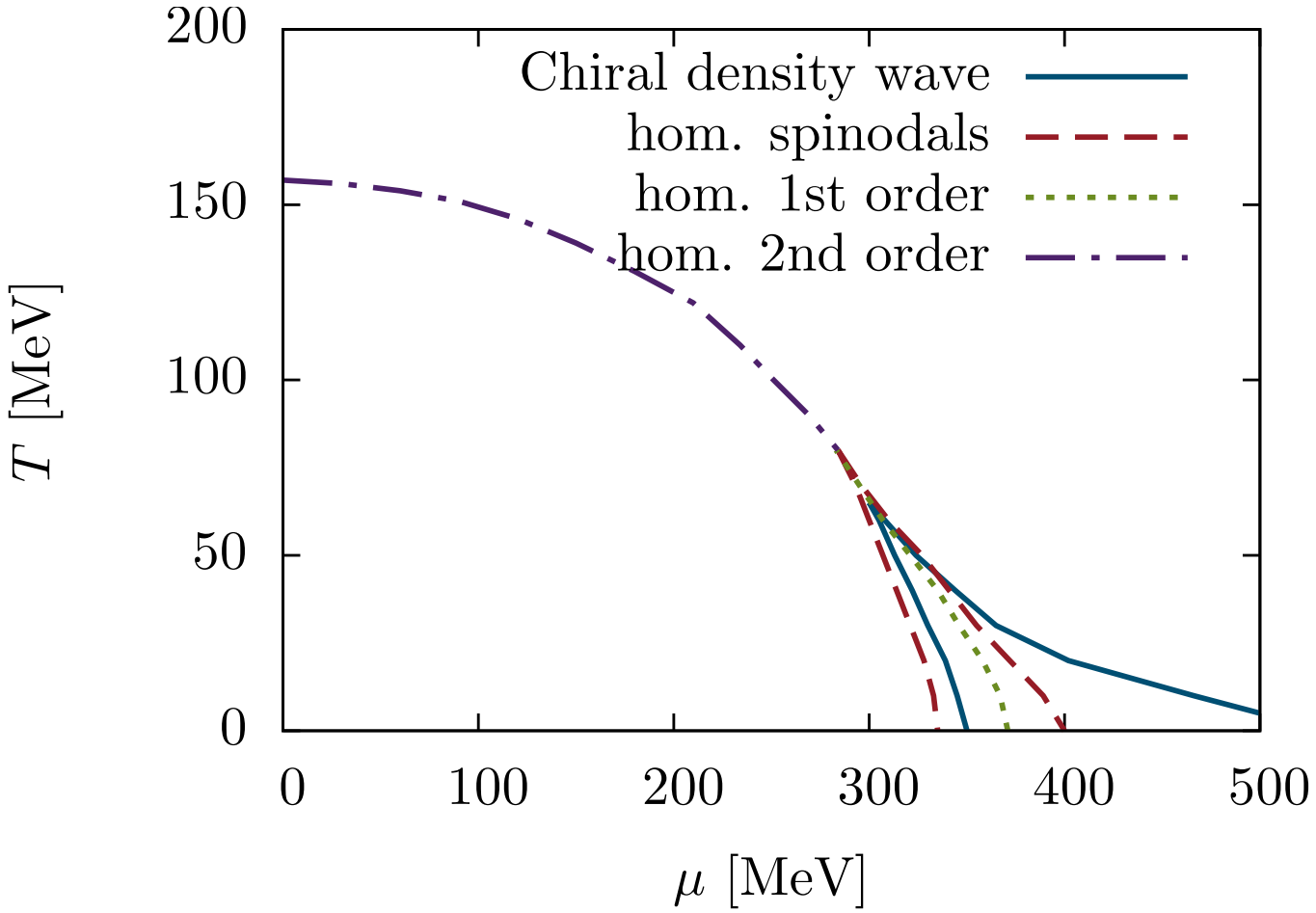


- Inhomogeneous phase covers 1st-order phase boundary completely
- Critical point $\rightarrow$ Lifshitz point

Dyson-Schwinger QCD [D. Müller et al. (2013)]



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$\rightarrow$ To what extent is this a generic feature?

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- Lagrangian: $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G_S\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$
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$$\Omega(T,\mu) = -\frac{T}{V}\ln\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L} + \mu\bar{\psi}\gamma^0\psi)\right)$$

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$$\Rightarrow \quad \mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ + 2G_S(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)\psi - G_S\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

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- Mean-field approximation:

$$\sigma(x) \to \langle\sigma(x)\rangle \equiv S(\vec{x}), \quad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv P(\vec{x})\,\delta_{a3}$$

  - $S(\vec{x})$, $P(\vec{x})$ time independent classical fields
  - Retain space dependence !

Mean-field model

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- Mean-field Lagrangian:

$$\mathcal{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G_S\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

  - inverse dressed propagator: $\mathcal{S}^{-1}(x) = i\partial\!\!\!/ + 2G_S\left(S(\vec{x}) + i\gamma_5\tau_3 P(\vec{x})\right)$
  - bilinear in ψ and $\bar{\psi}$ $\Rightarrow$ quark fields can be integrated out!

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- Thermodynamic potential: $\Omega_{MFA} = \Omega_q + \Omega_{cond}$

  - $\Omega_{cond} = \frac{G_S}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$ straightforward
  - $\Omega_q = -\frac{T}{V}\mathbf{Tr}\,\mathrm{Log}\left[\frac{1}{T}\left(\mathcal{S}^{-1}[S,P] + \mu\gamma^0\right)\right]$ difficult
    - no general solution for arbitrary $S(\vec{x})$, $P(\vec{x})$
    - tractable for certain ansätze

Condensate modulations

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- Constituent mass functions: $M(\vec{x}) = -2G_S\left(S(\vec{x}) + iP(\vec{x})\right)$

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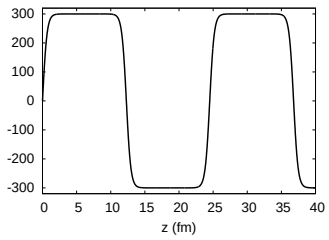
- Modulations with analytical solutions:
  - homogeneous matter: $M = const.$
  - chiral density wave (CDW): $M(z) = \Delta e^{iqz}$
  - real kink crystal (RKC): $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu)$



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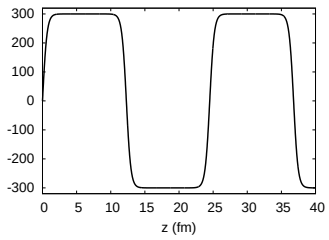
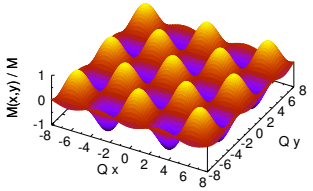
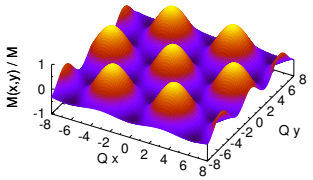


- Ansätze which have been treated numerically:
  - 1d cosine: $M = \Delta\cos Qz$
  - 2d square lattice: $M(x,y) = M\cos(Qx)\cos(Qy)$
  - hexagon:
    $M(x,y) = \frac{M}{3}\left[2\cos(Qx)\cos\left(\frac{1}{\sqrt{3}}Qy\right) + \cos(\frac{2}{\sqrt{3}}Qy)\right]$





Results

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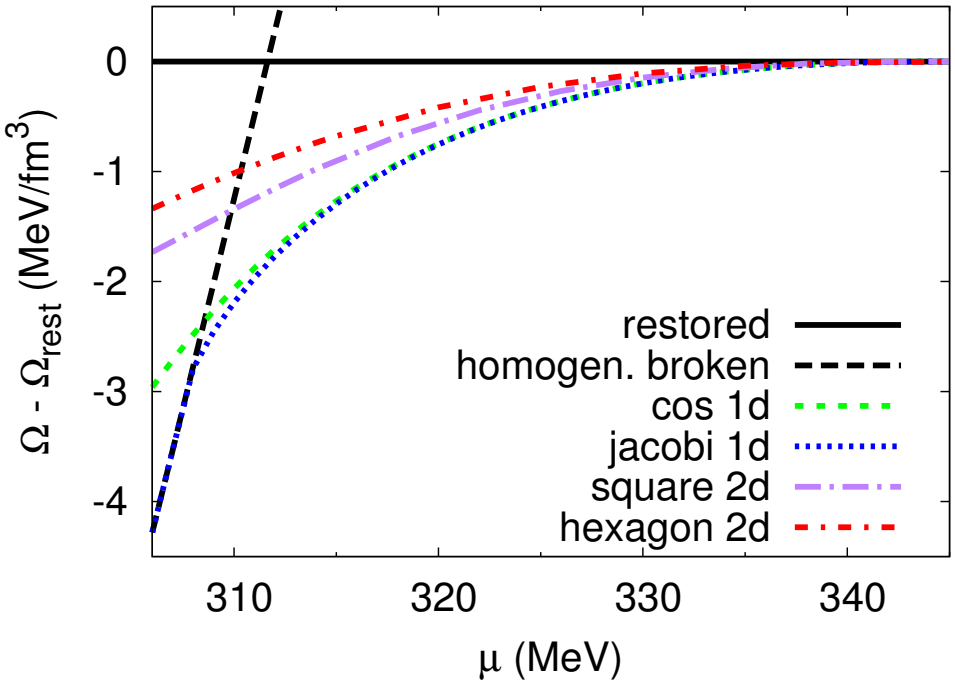
- Comparison of the free energies ($T = 0$):

  [S. Carignano, MB, PRD (2012)]



- similar windows for all non-uniform ansätze (identical restoration points)
- 2d solutions not favored (but still more than CDW)
- RKC most favored (true absolute minimum?)

Ginzburg-Landau analysis

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- Expansion of the thermodynamic potential:

$$\Omega(M) = \Omega(0) + \frac{1}{V}\int d^3x \, \left\{ \frac{1}{2}\gamma_2 |M(\vec{x})|^2 + \frac{1}{4}\gamma_{4,a}|M(\vec{x})|^4 + \frac{1}{4}\gamma_{4,b}|\nabla M(\vec{x})|^2 + \dots \right\}$$

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- Striking result: $\beta_{4,a} = \beta_{4,b} \Rightarrow \gamma_{4,a} = \gamma_{4,b} \Rightarrow$ CP = LP ! [D. Nickel, PRL (2009)]

NJL model [D. Nickel, PRD (2009)]

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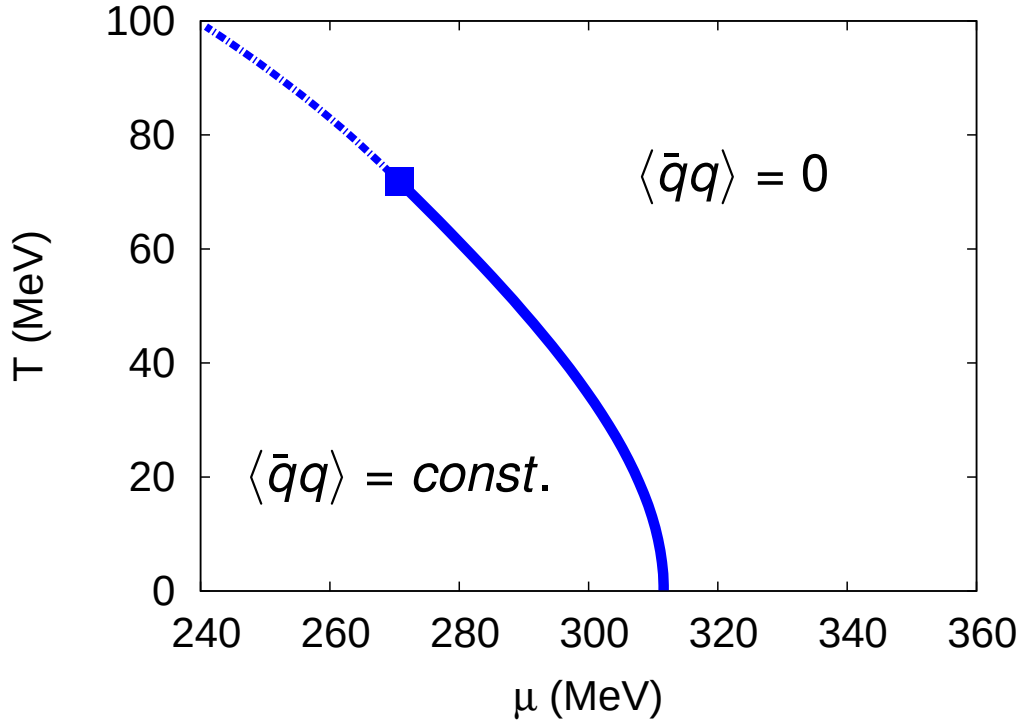

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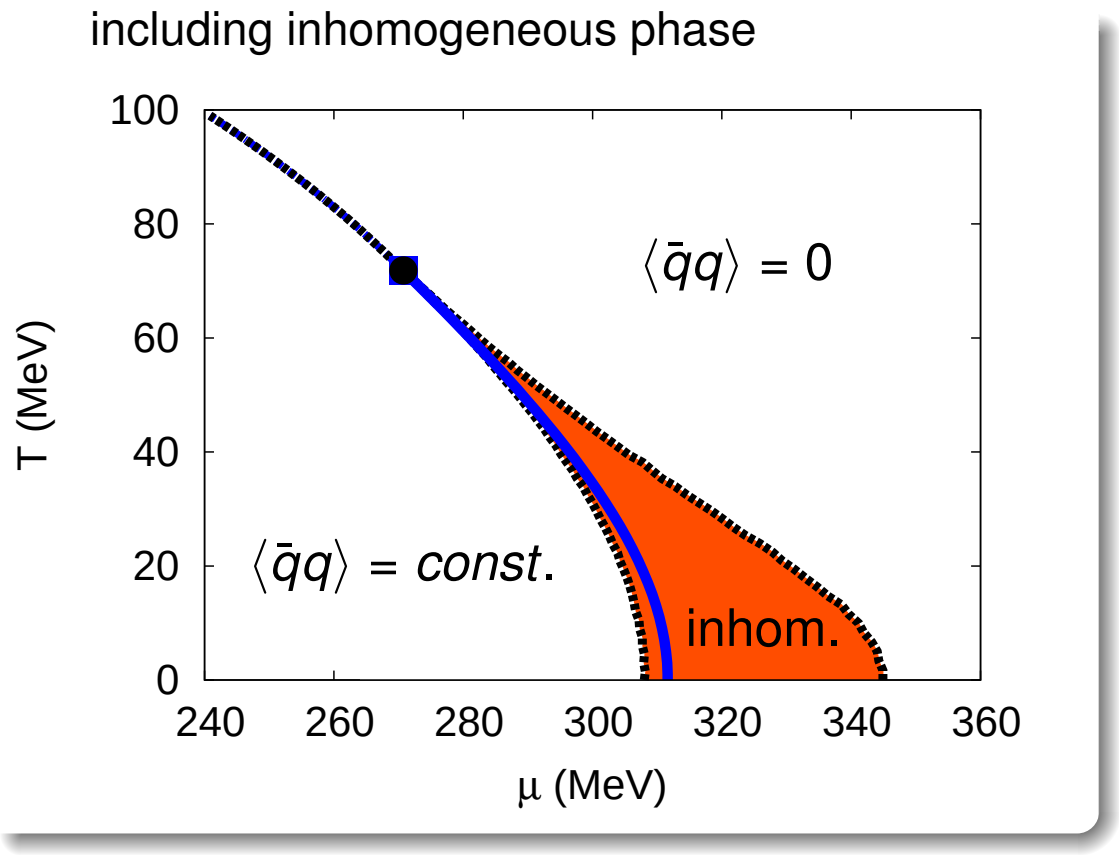

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  - the form of the modulations
  - model parameters (G_S, Λ)
- Can be different
  - for model extensions
  - beyond mean field
  - in other models or QCD

Including vector interactions

[S. Carignano, D. Nickel, MB, PRD (2010)]

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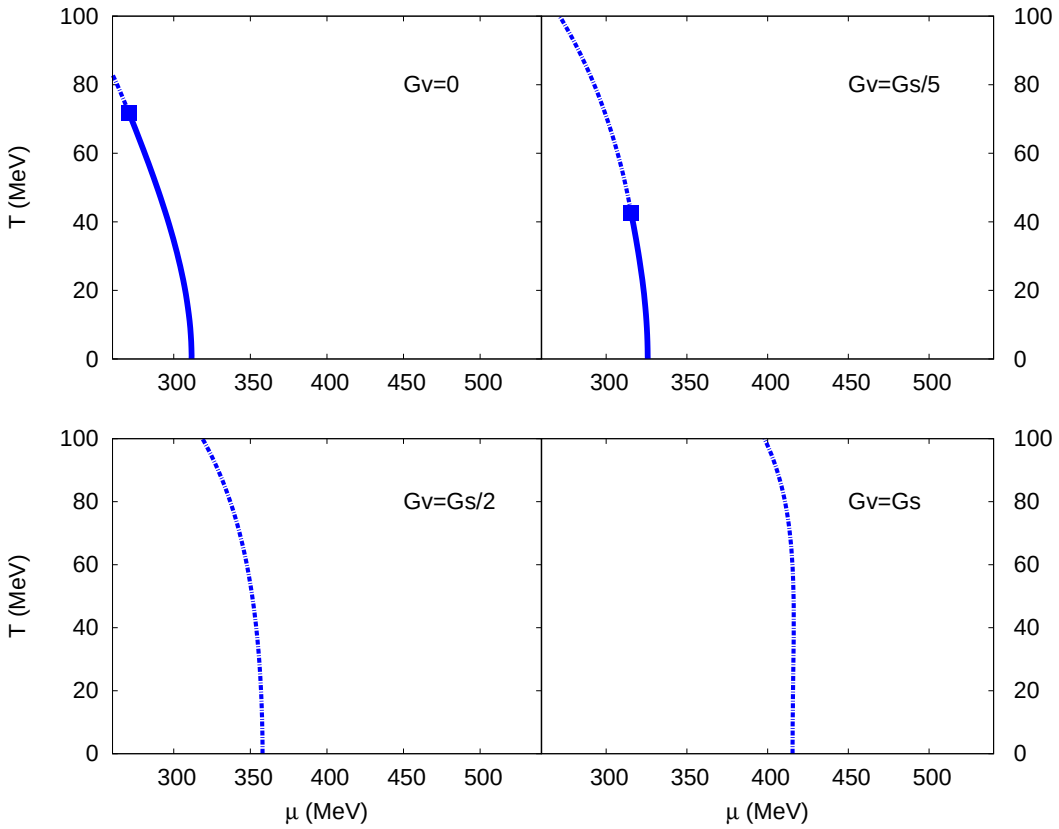

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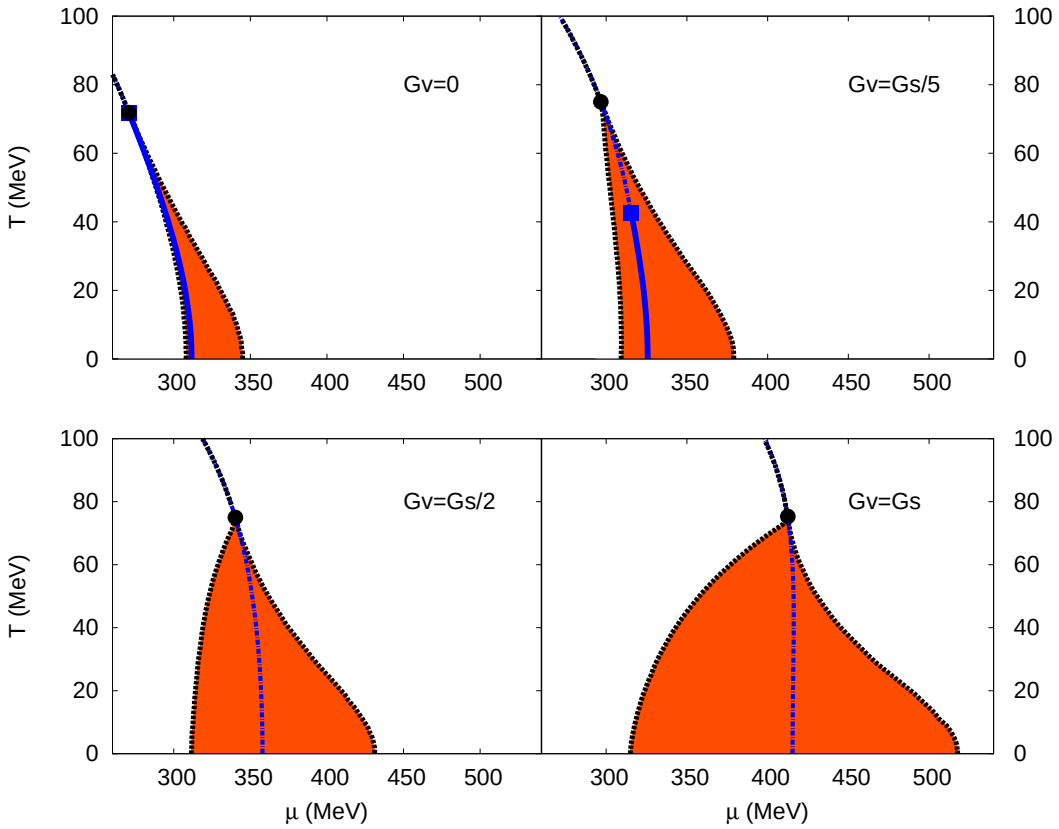

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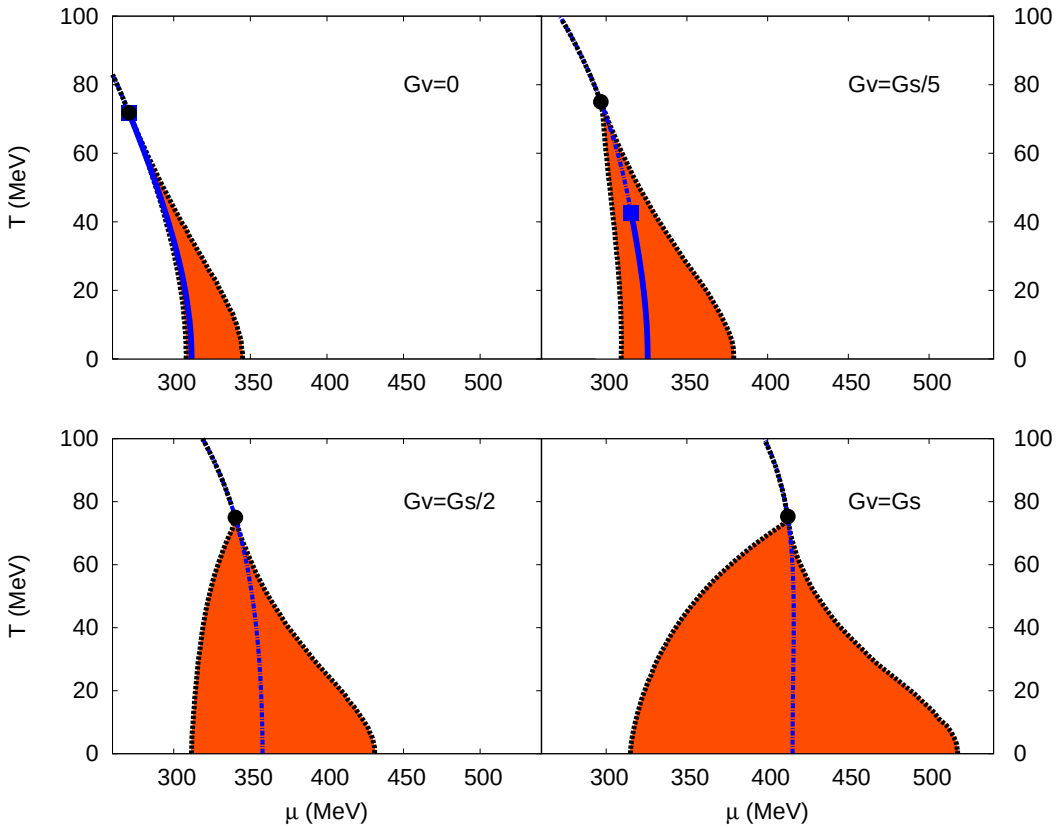
$\Rightarrow$ CP covered by inhom. region



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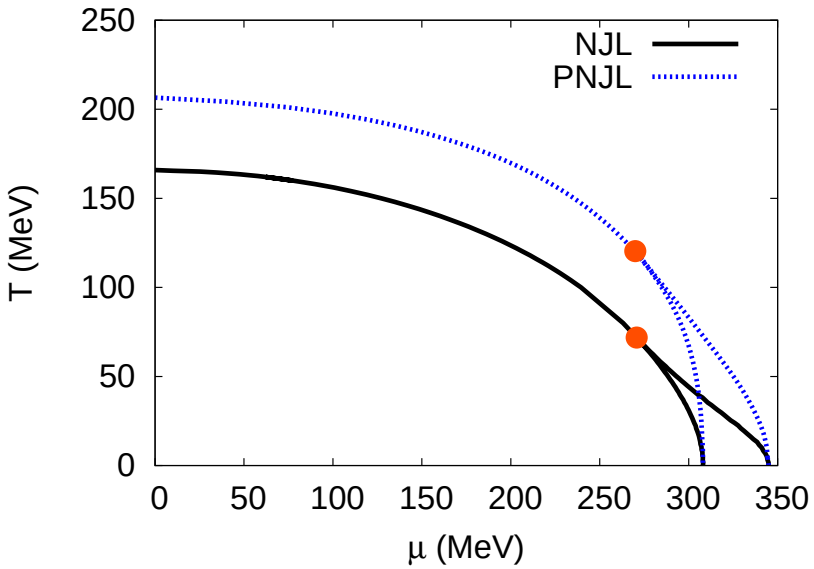

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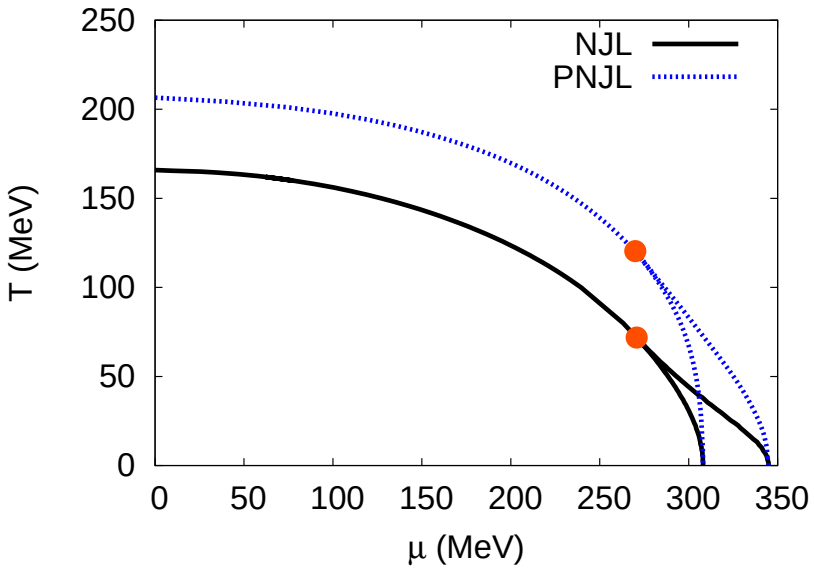

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- Ginzburg-Landau: CP outside of the inhomogeneous region!

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[S. Carignano, MB, B.-J. Schaefer, PRD (2014)]

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- Thermodynamic potential: $\Omega_{MFA} = \Omega_{mes} + \Omega_q$
  - $\Omega_{mes} = \frac{1}{V}\int_V d^3x \left(\frac{1}{2}\left((\vec{\nabla}\sigma)^2 + (\vec{\nabla}\vec{\pi})^2\right) + U(\sigma,\vec{\pi})\right)$ straightforward
  - $\Omega_q = -\frac{T}{V}\mathbf{Tr}\,\mathrm{Log}\left[\frac{1}{T}\left(i\partial\!\!\!/ - g\left(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi}\right) + \mu\gamma^0\right)\right]$ difficult

  equivalent to NJL $\rightarrow$ use same techniques

Standard and extended MFA

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- Result:

$$\Omega_q = -\int_0^\infty dE\, \rho(E;\sigma,\vec{\pi})\left\{E + T\log\left[1 + e^{-\frac{E-\mu}{T}}\right] + T\log\left[1 + e^{-\frac{E+\mu}{T}}\right]\right\}$$

  - $\rho(E;\sigma,\vec{\pi})$: density of states,
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- extended mean-field approximation (eMFA): include the Dirac sea

Fixing the parameters

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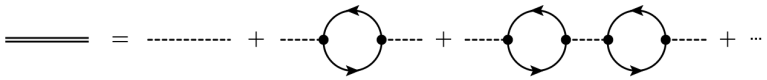
- Fit g, λ, and v to three vacuum "observables" : $M_{vac} = g\langle\sigma\rangle, \quad m_\sigma, \quad f_\pi$
- sMFA: $m_\sigma^2 = \frac{\partial^2 U}{\partial\sigma^2}\Big|_{\sigma=\langle\sigma\rangle,\vec{\pi}=\vec{0}} \equiv m^2_{\sigma,\text{tree}}, \qquad f_\pi = \langle\sigma\rangle$

  (Goldberger-Treiman: $M_{vac} = g_\pi\, f_\pi$)
- Including the Dirac sea (usual identification):

$$m_\sigma^2 = \frac{\partial^2\Omega}{\partial\sigma^2} r\Big|_{\sigma=\langle\sigma\rangle,\vec{\pi}=\vec{0}} \equiv m^2_{\sigma,curv}, \qquad f_\pi = \langle\sigma\rangle$$

- Correct procedure

  meson propagator with loop corrections:



$$D_j(q^2) = \frac{1}{q^2 - m^2_{j,tree} + g^2\Pi_j(q^2) + i\epsilon} = \frac{Z_j}{q^2 - m^2_{j,pole} + i\epsilon} + \text{reg. terms}, \qquad j = \sigma, \pi$$

$$\rightarrow \qquad m_\sigma \equiv m_{\sigma,pole}, \qquad f_\pi = \frac{M_{vac}}{g_{\pi,\text{ren}}} = \frac{1}{\sqrt{Z_\pi}}\langle\sigma\rangle$$

Fixing the parameters

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- "Hands-on renormalization":
  - Introduce regularization for the loop integrals (here: Pauli-Villars).
  - Increase cutoff Λ , keeping M_{vac}, m_σ, and f_π fixed.

Fixing the parameters

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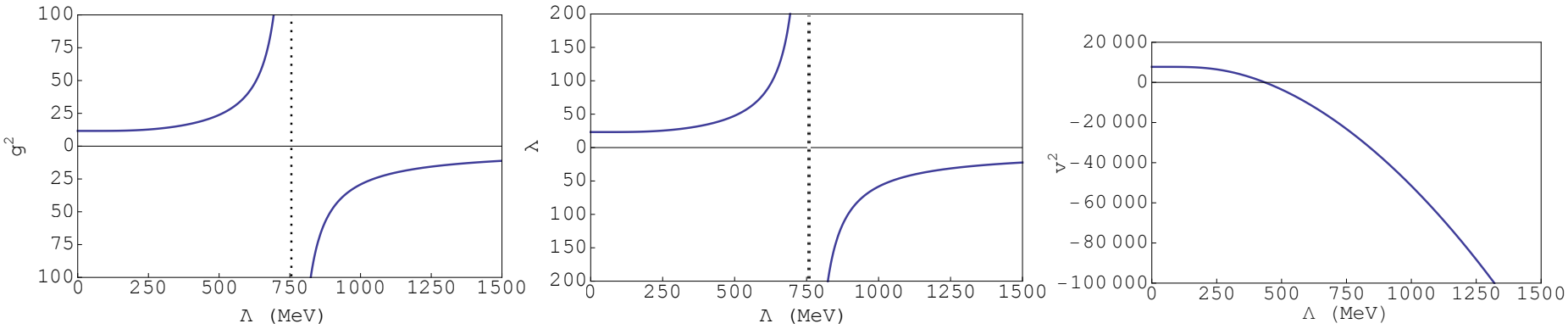
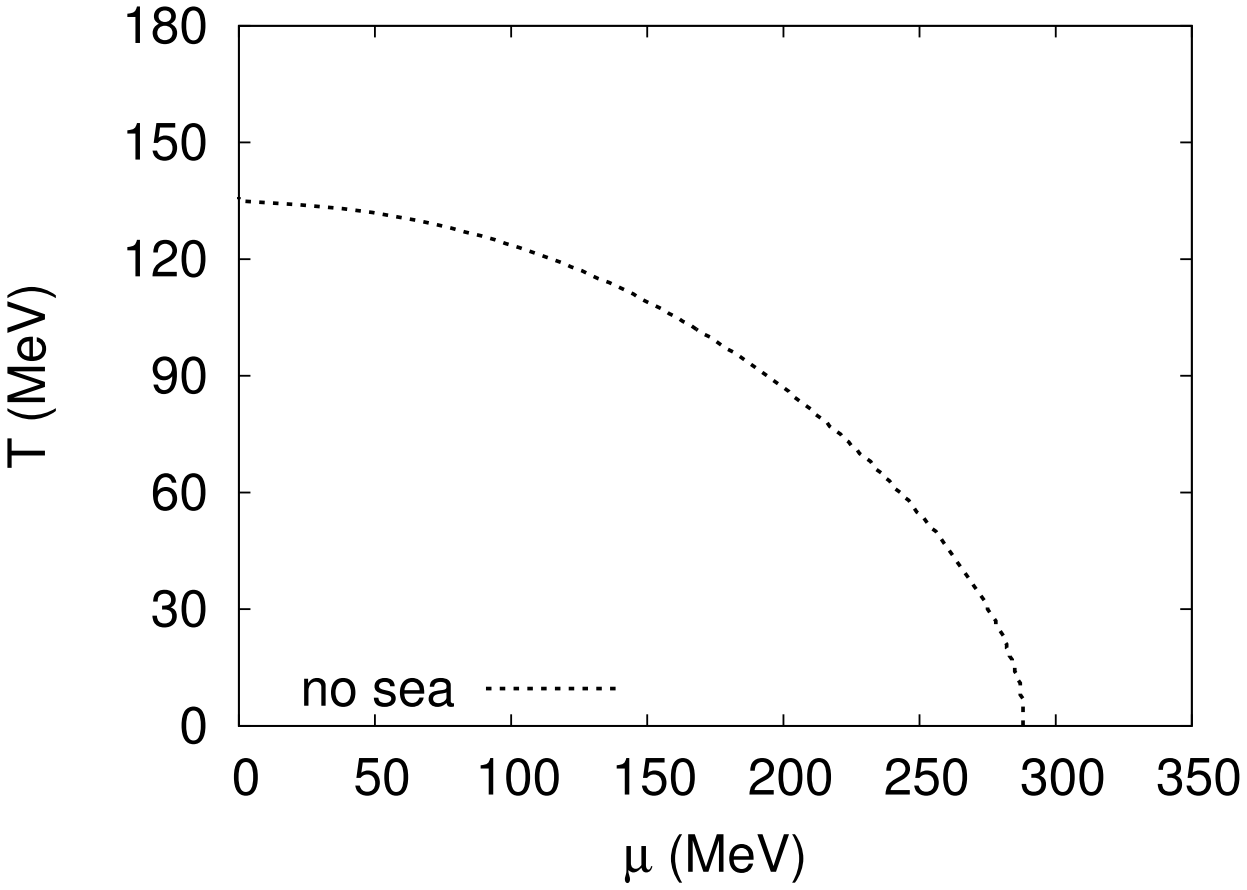
- "Hands-on renormalization":
  - Introduce regularization for the loop integrals (here: Pauli-Villars).
  - Increase cutoff Λ , keeping M_{vac}, m_σ, and f_π fixed.
- parameters for M_{vac} = 300 MeV, f_π = 88 MeV, m_σ = 600 MeV:



Phase diagram for homogeneous matter

Phase diagram for homogeneous matter



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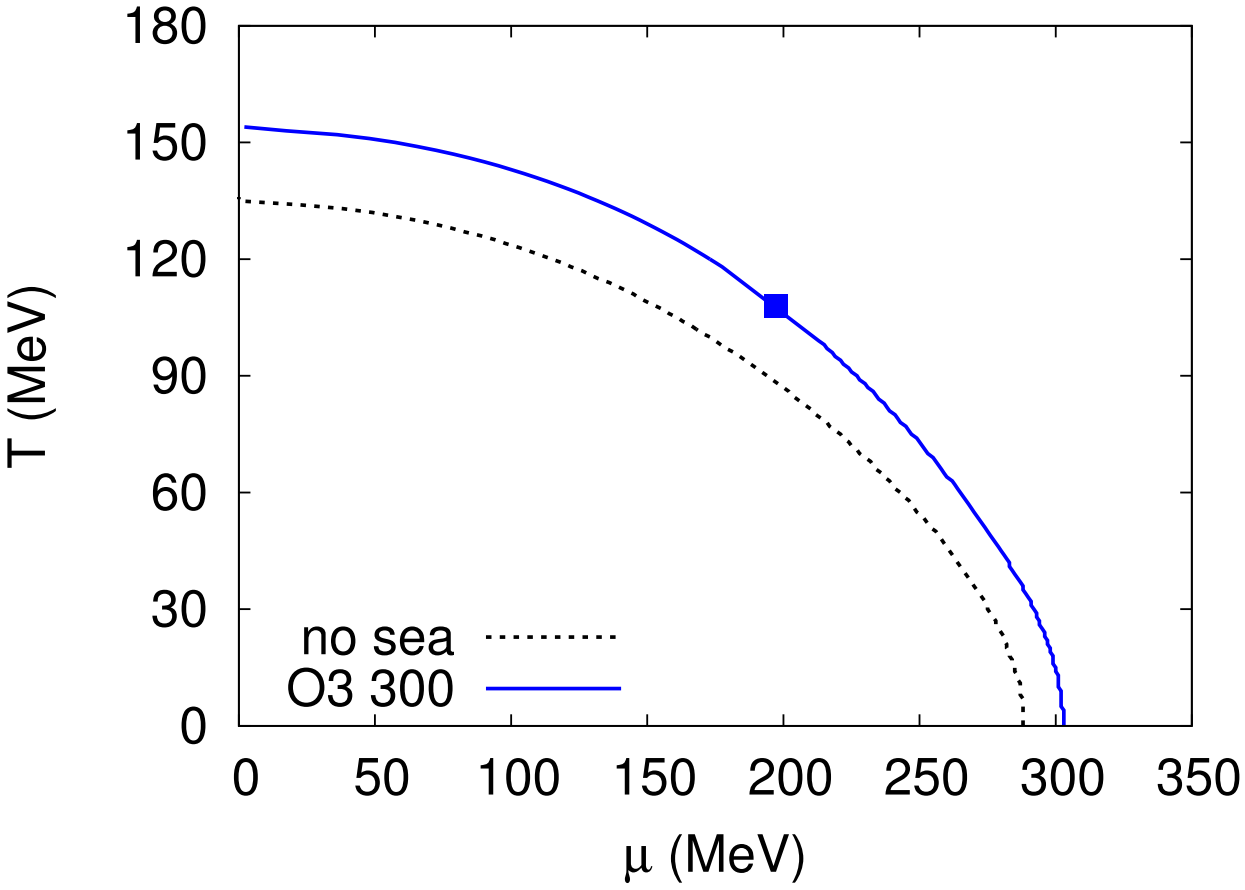

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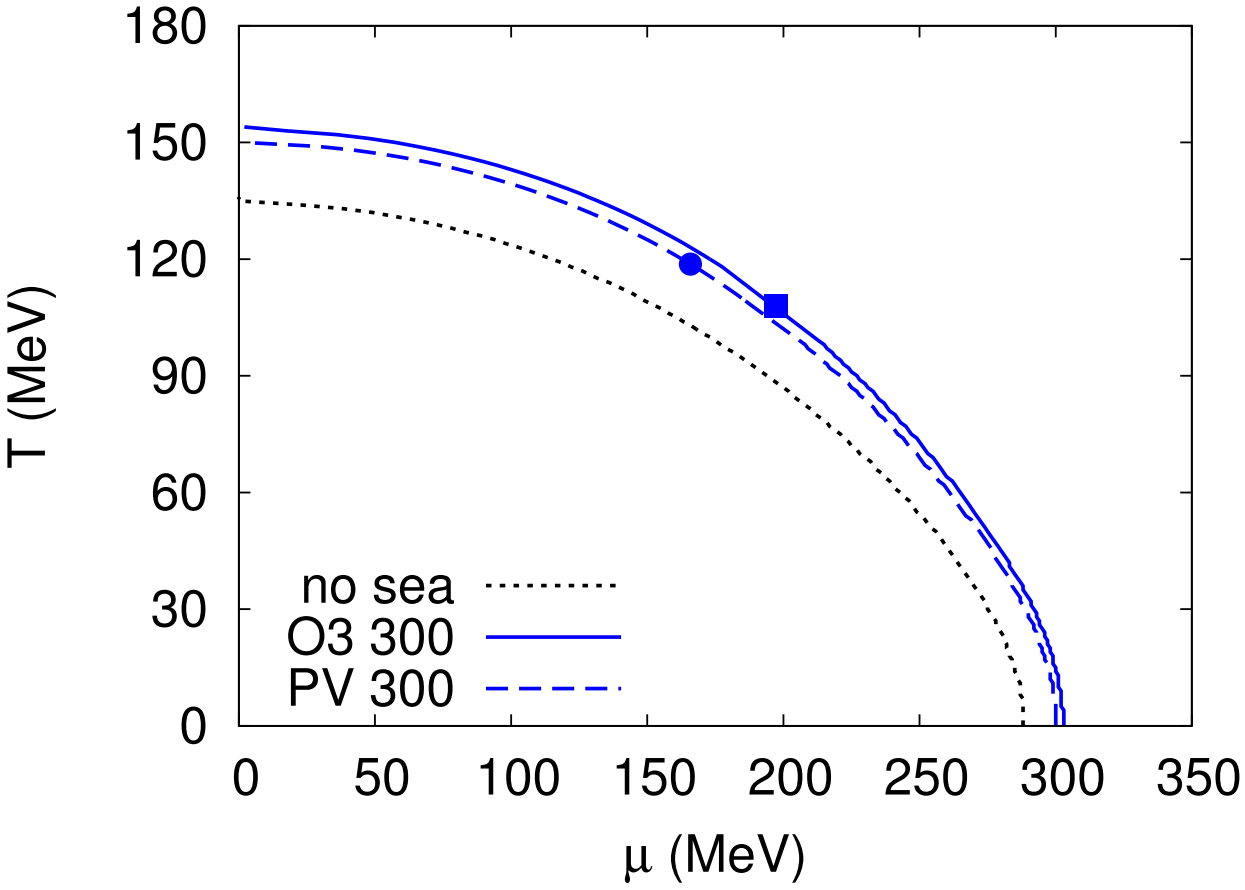

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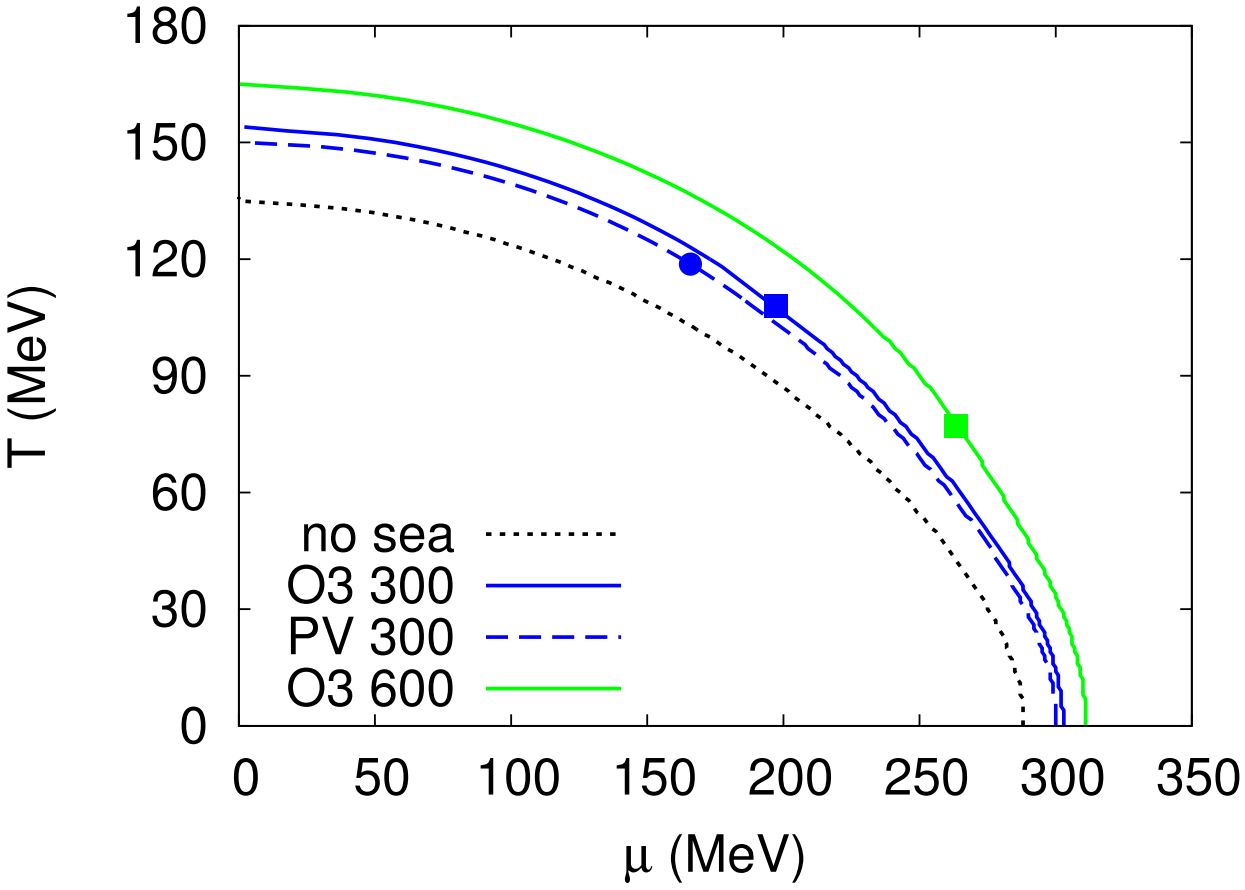

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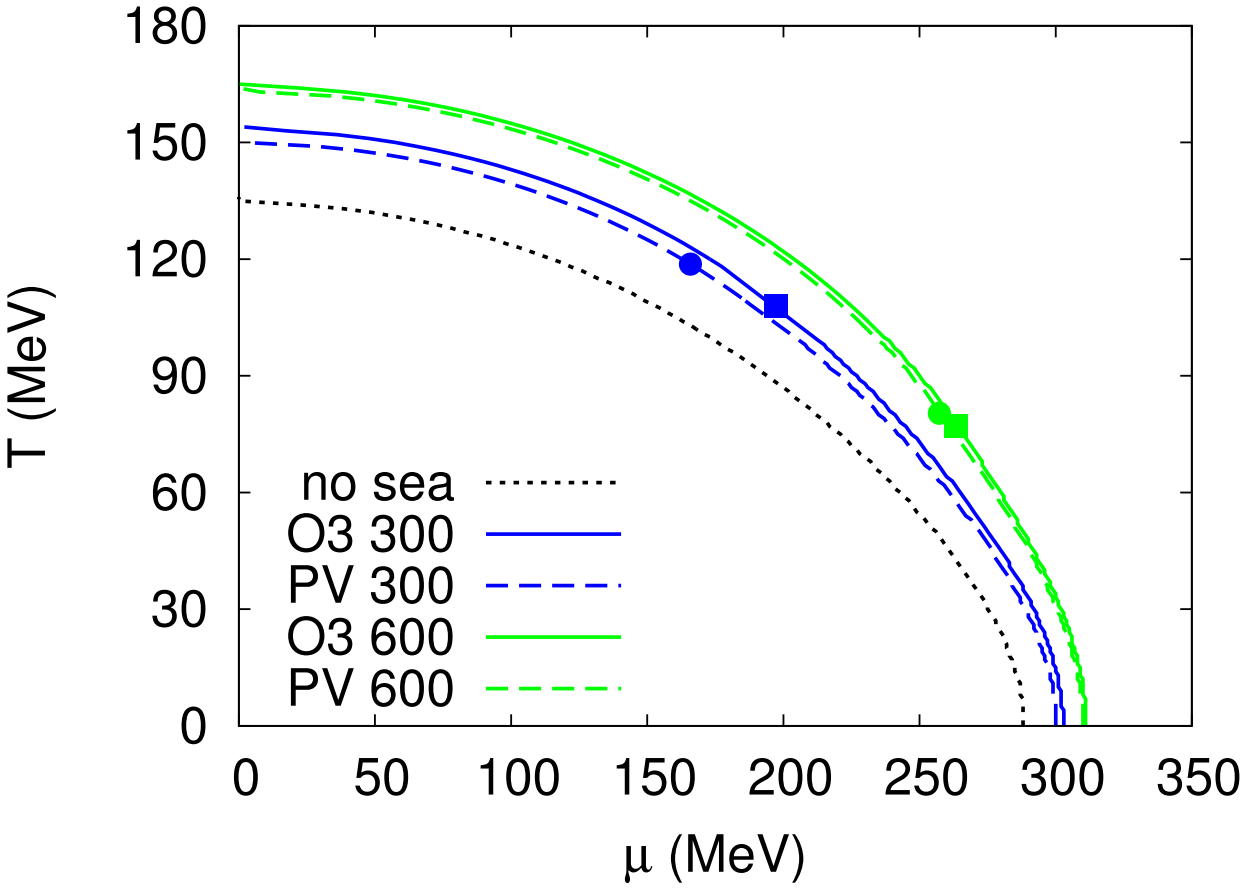

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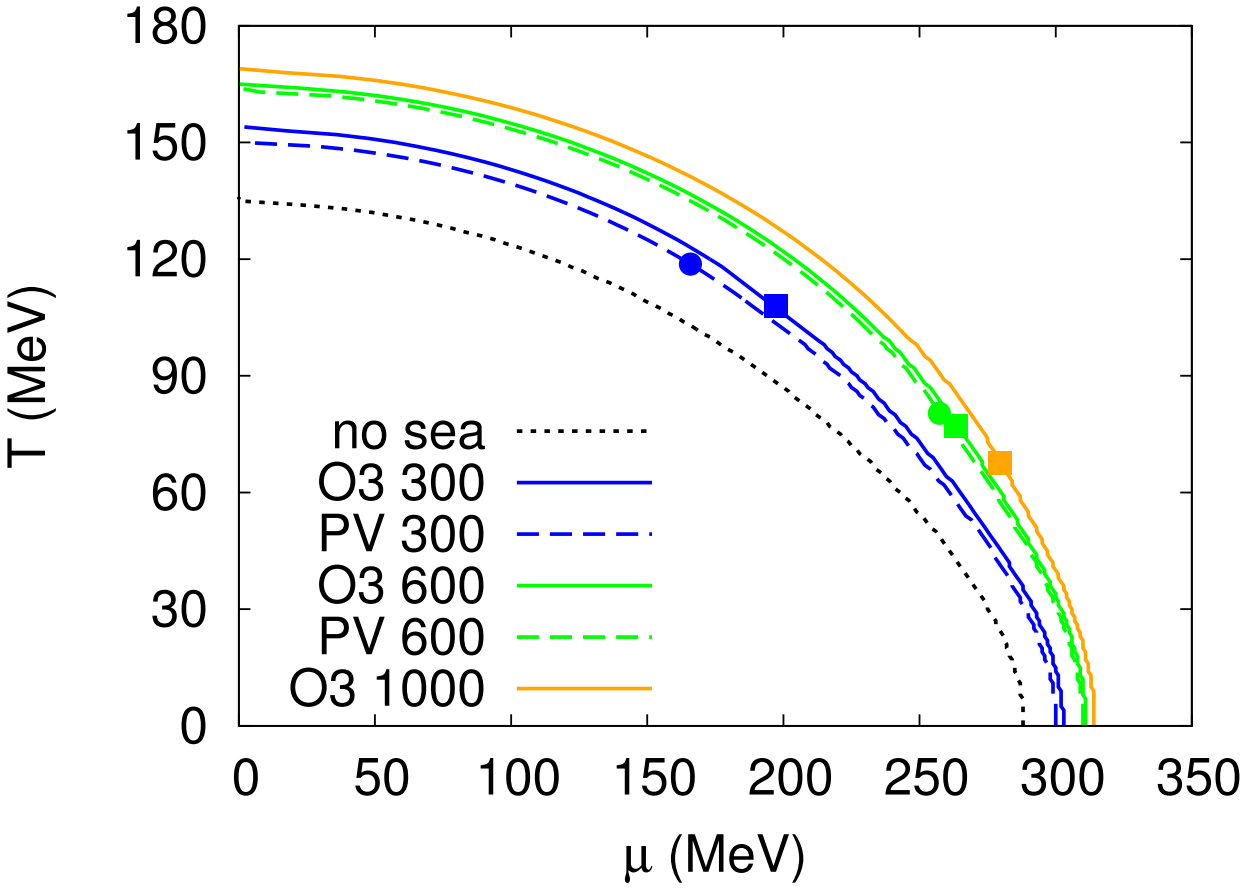

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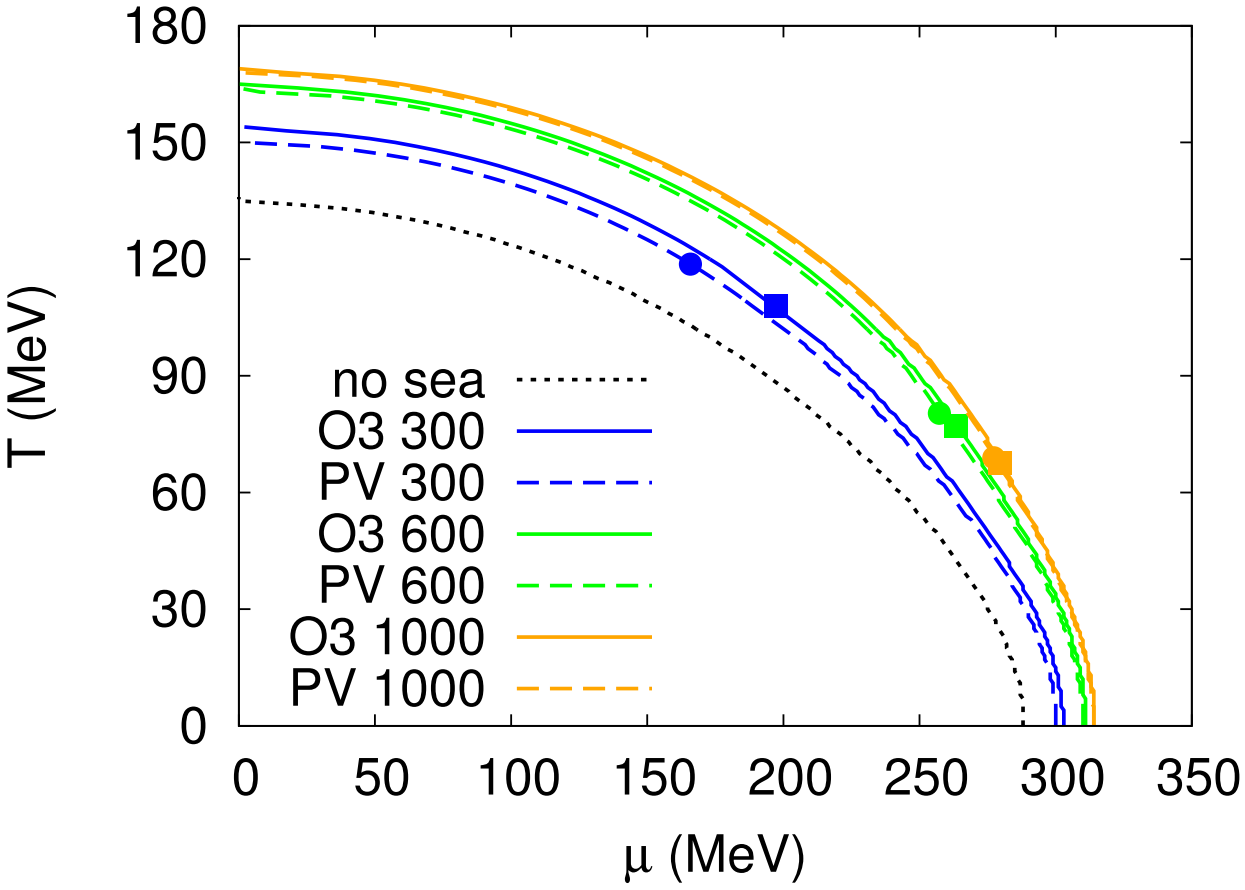

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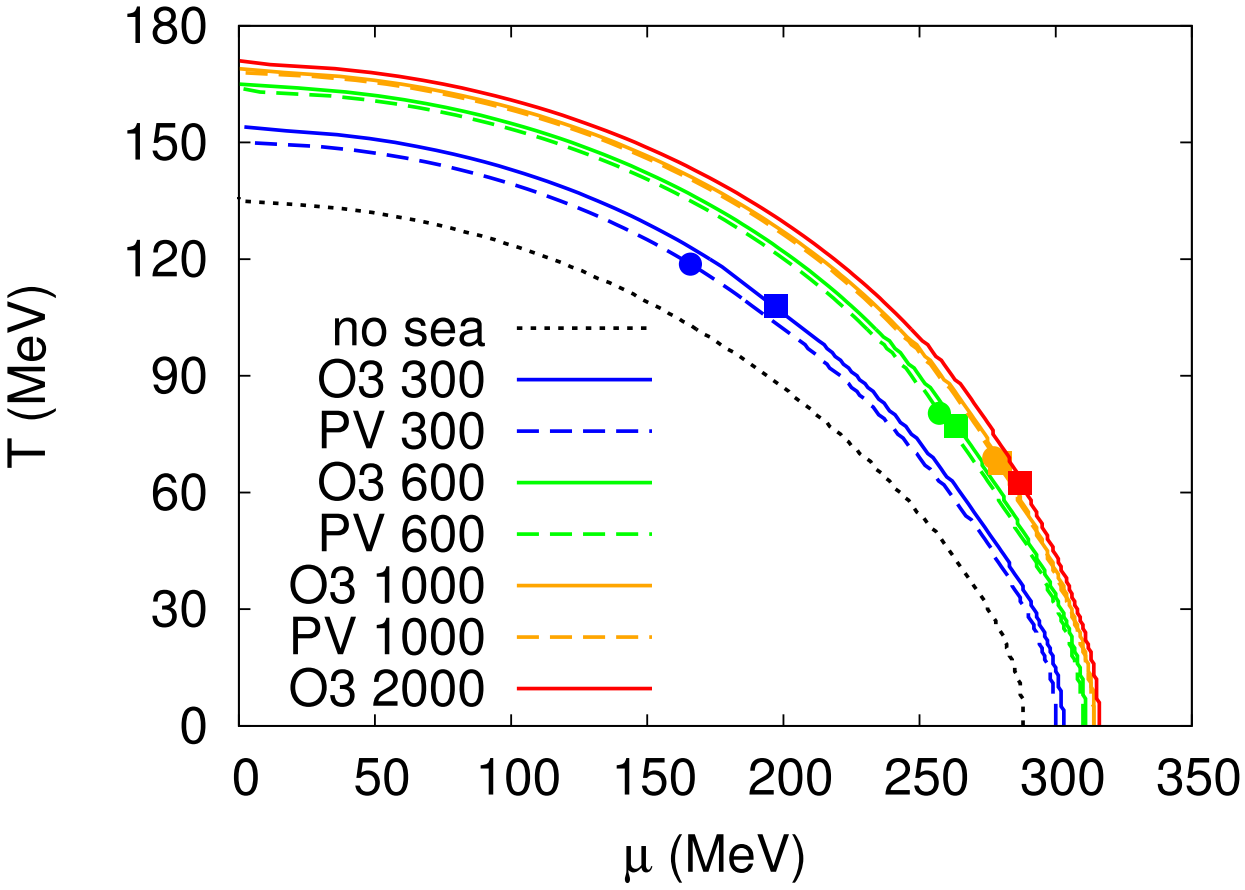

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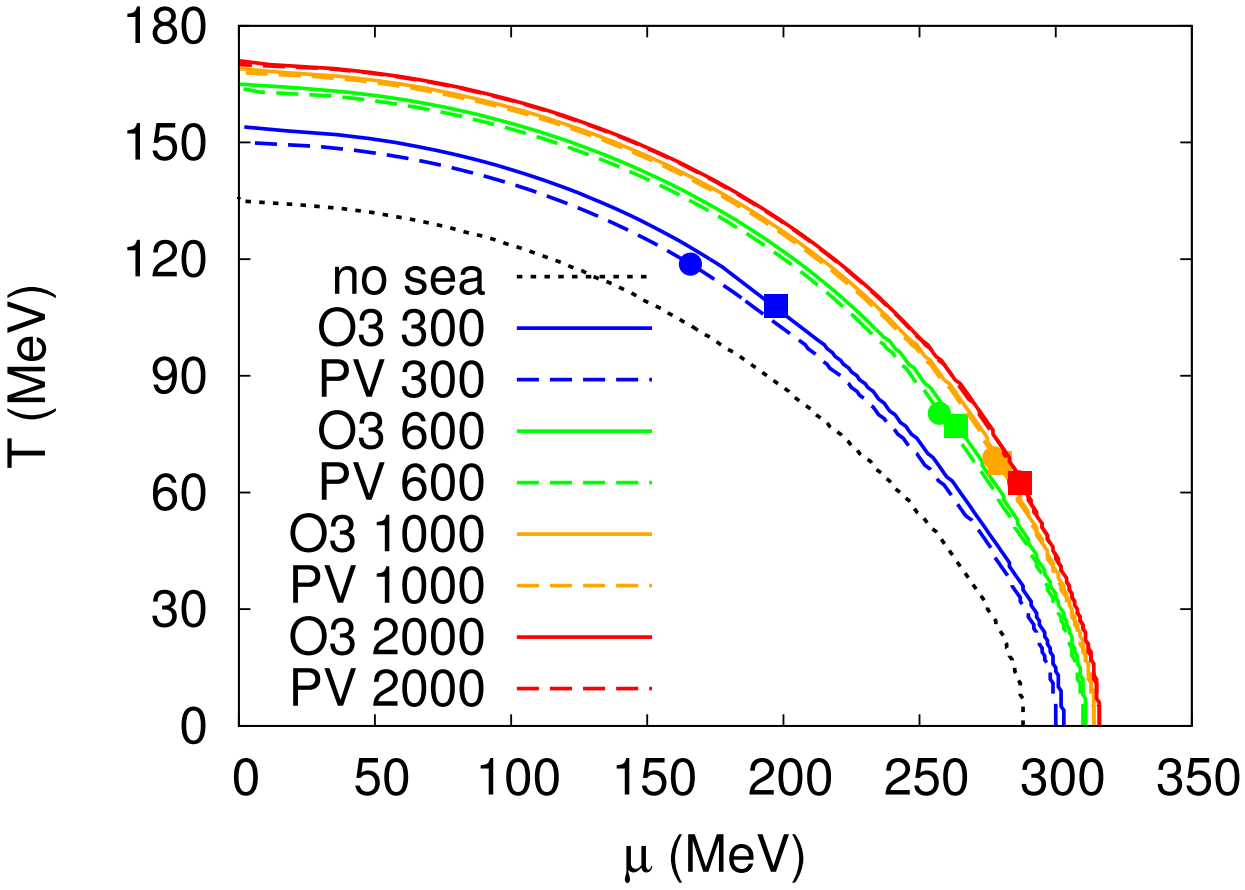

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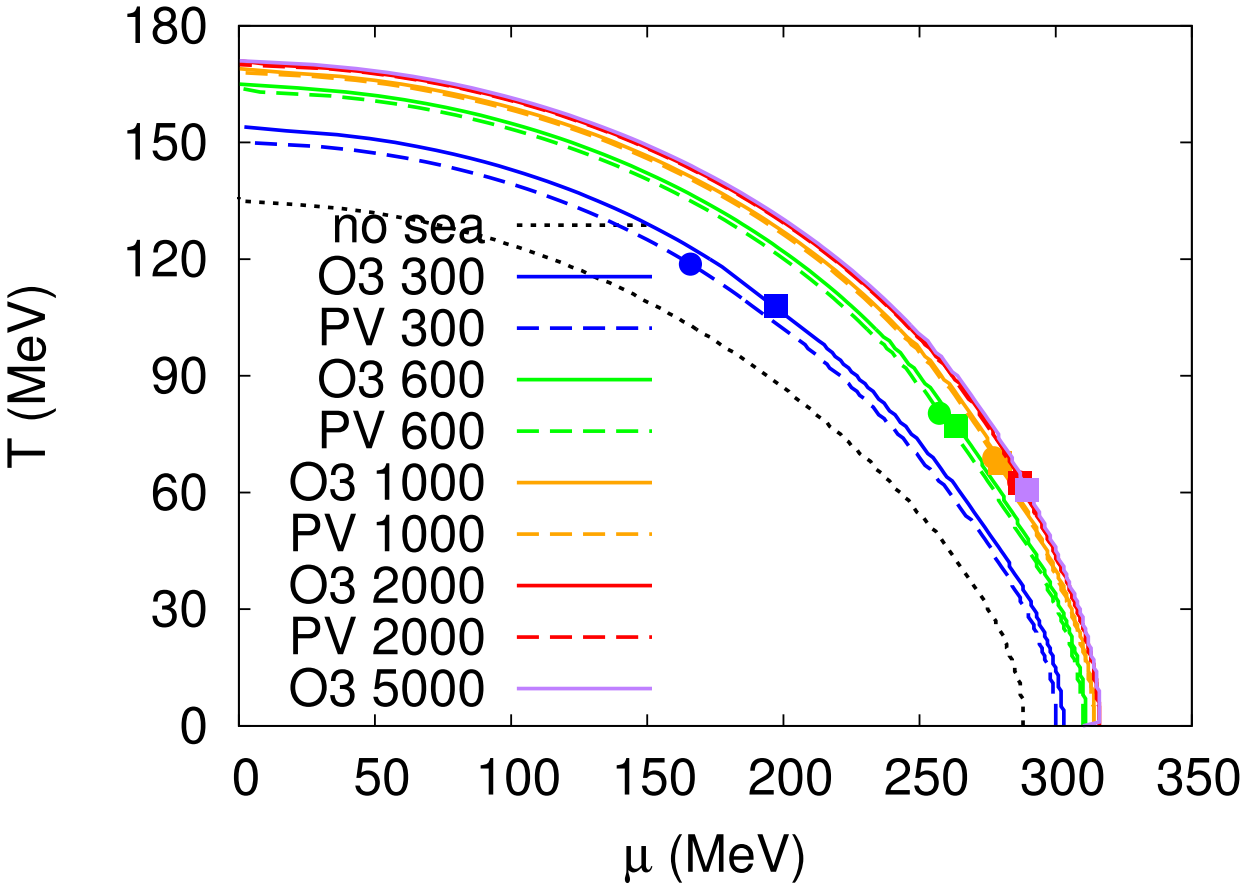

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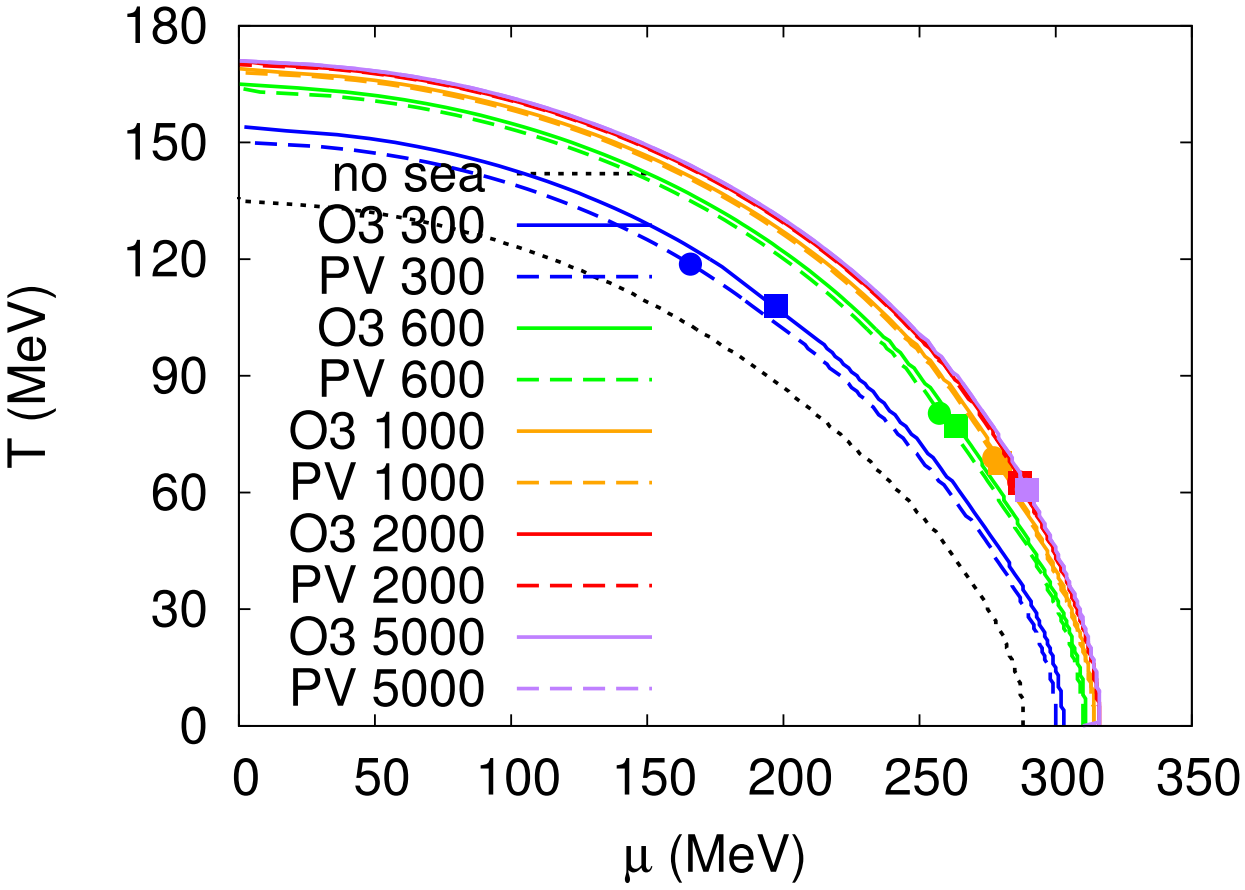

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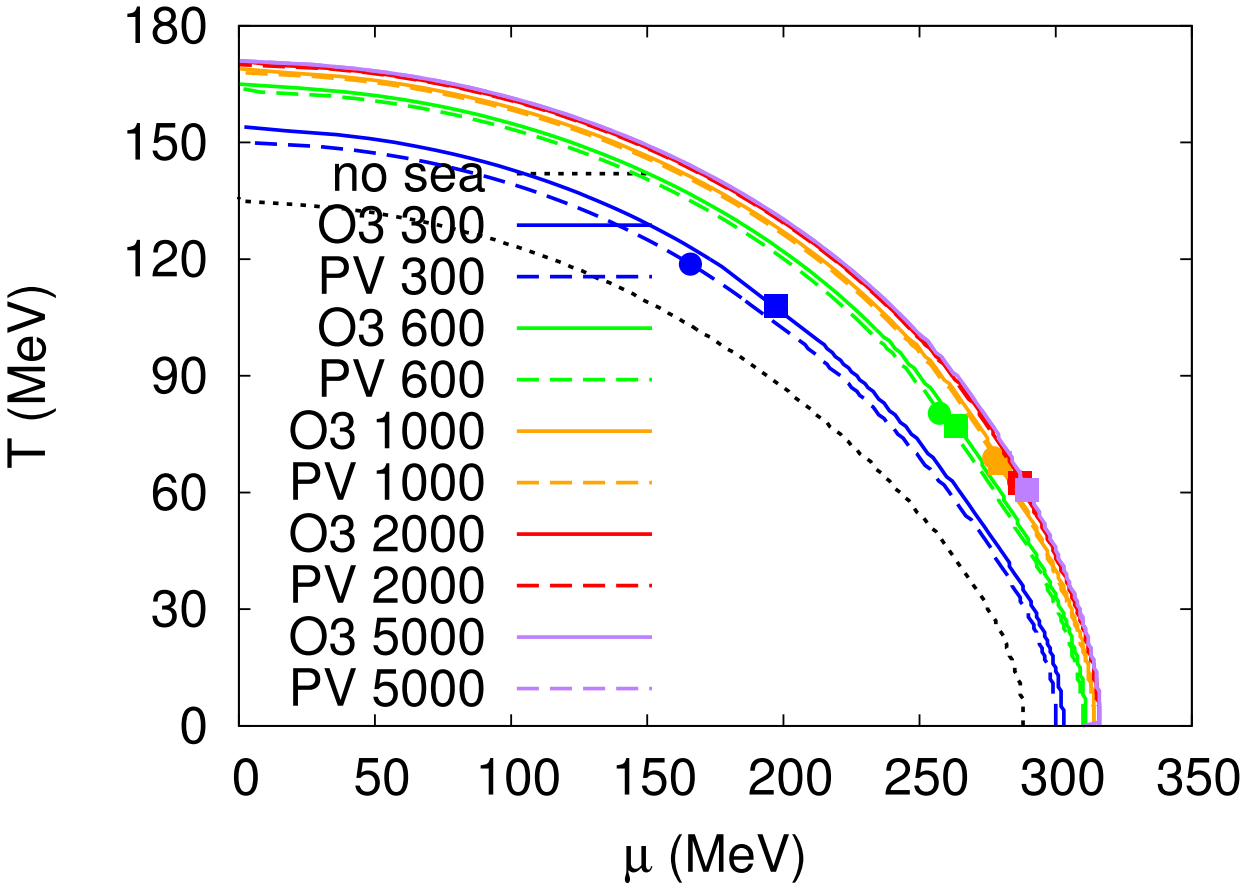

- Convergence reached at $\Lambda \approx 2$ GeV.

Phase diagram for inhomogeneous matter

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- CDW: $\sigma(z) = f \cos qz, \quad \pi_3(z) = f \sin qz$

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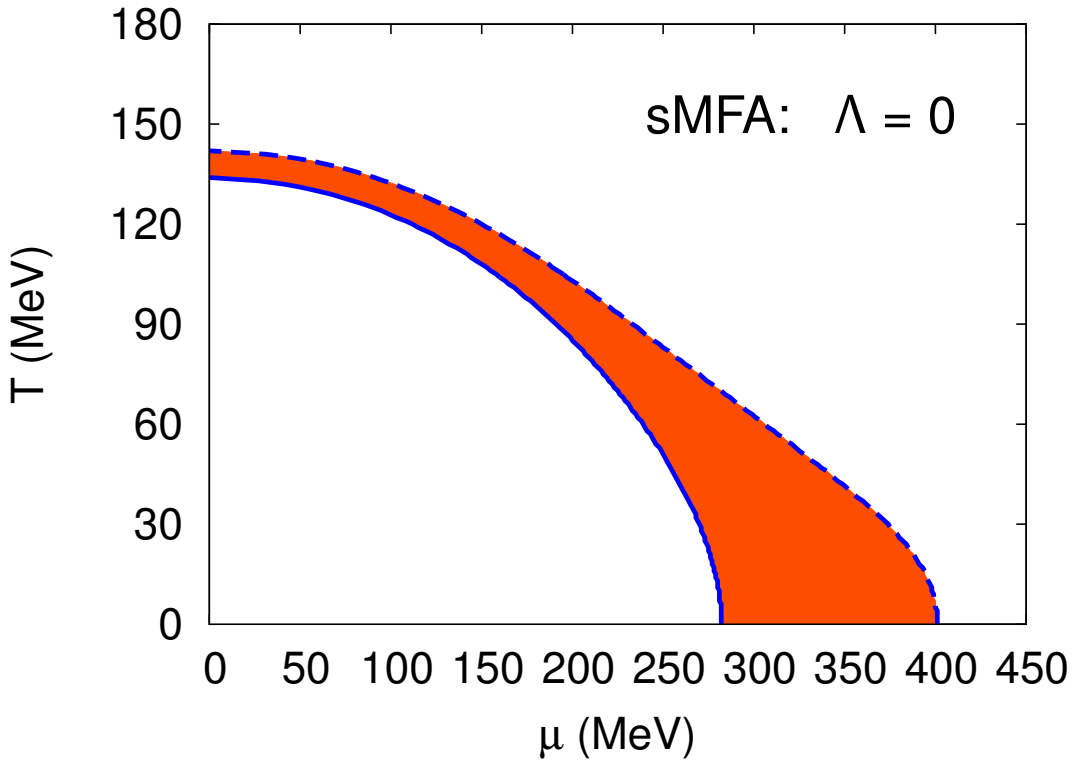

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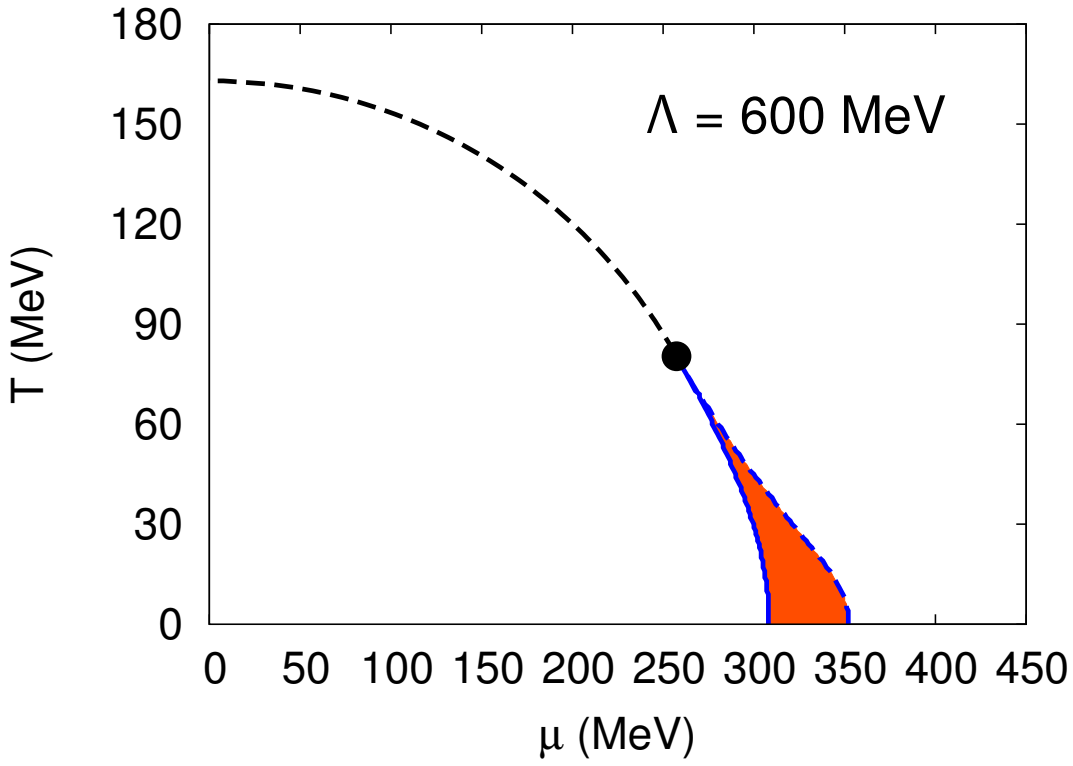

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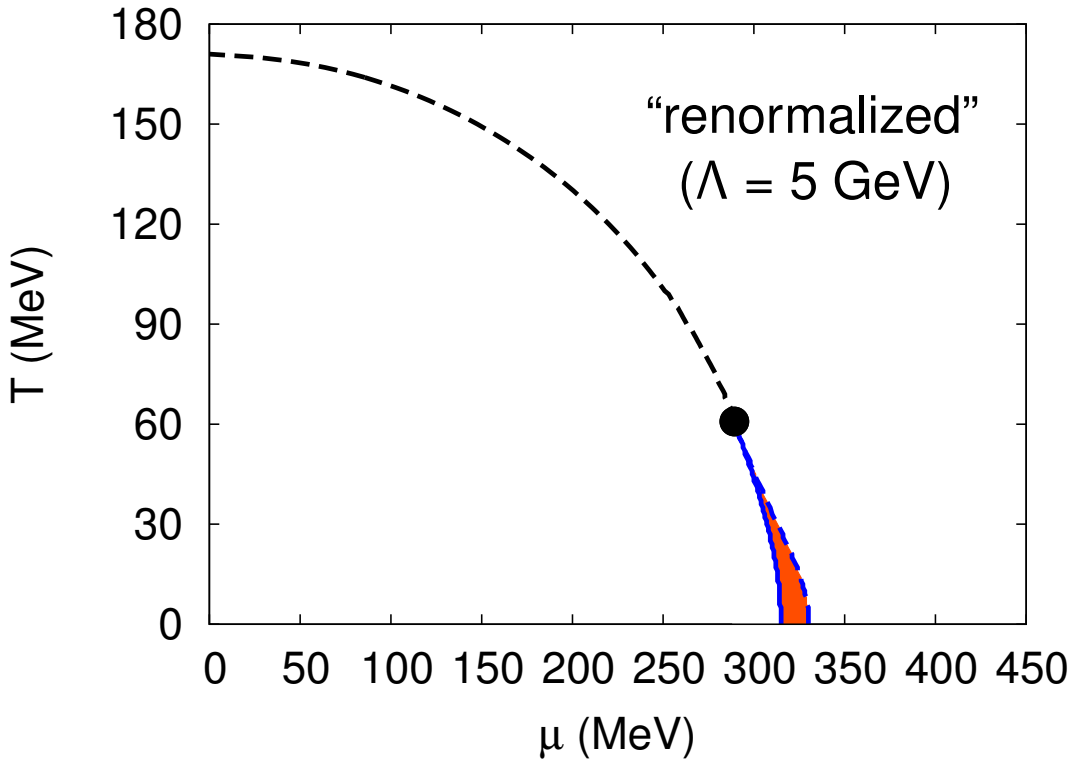

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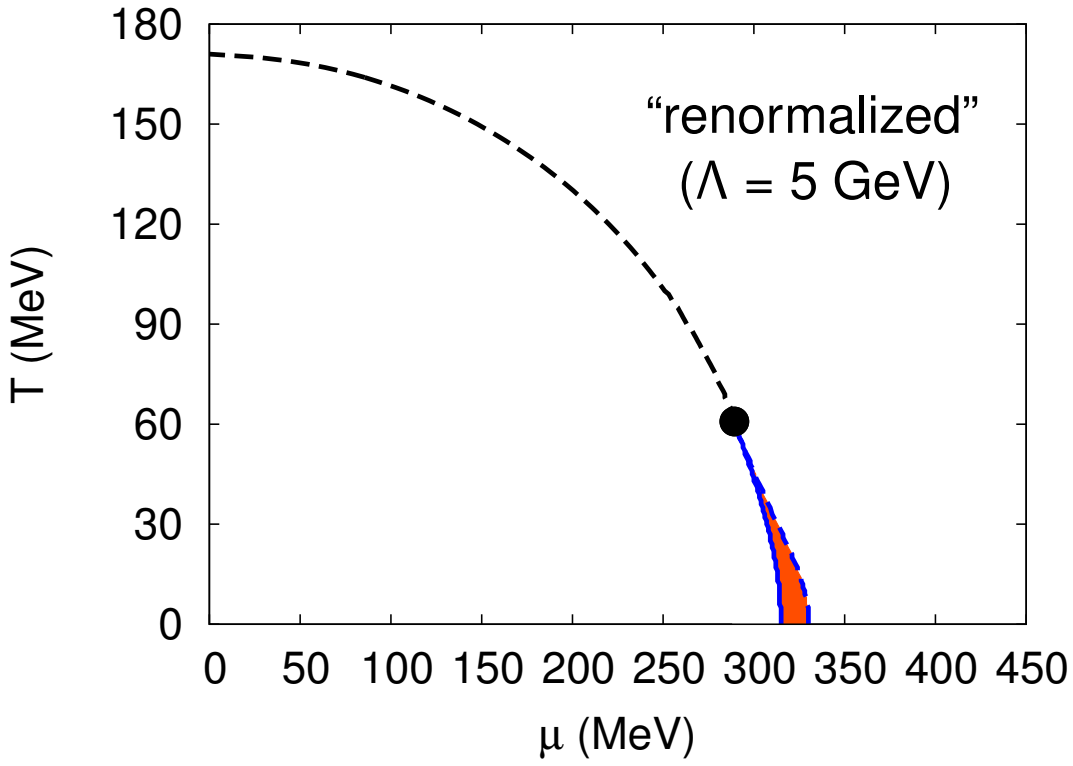

- inhomogeneous island gets smaller but survives.
- LP = CP

Ginzburg-Landau analysis

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- as before:

$$\Omega(M) = \Omega(0) + \frac{1}{V}\int d^3x \, \left\{ \frac{1}{2}\gamma_2 |M(\vec{x})|^2 + \frac{1}{4}\gamma_{4,a}|M(\vec{x})|^4 + \frac{1}{4}\gamma_{4,b}|\nabla M(\vec{x})|^2 + \ldots \right\}$$

  - CP: $\gamma_2 = \gamma_{4,a} = 0$
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- decomposition: $\gamma_i = \alpha_i + \beta_i$
  - $\Omega_q \quad \rightarrow \quad \beta_i$, identical with the NJL coefficients
  - $\Omega_{mes} \quad \rightarrow \quad \alpha_2 = -\frac{\lambda v^2}{g^2}, \quad \alpha_{4,a} = \frac{\lambda}{g^4}, \quad \alpha_{4,b} = \frac{2}{g^2},$

$$\Rightarrow \quad \gamma_{4,a} - \gamma_{4,b} = 2\frac{f_\pi^2}{M_{vac}^2}\left(\frac{m_\sigma^2}{4M_{vac}^2} - 1\right)$$

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$$\Rightarrow \quad \gamma_{4,a} - \gamma_{4,b} = 2\frac{f_\pi^2}{M_{vac}^2}\left(\frac{m_\sigma^2}{4M_{vac}^2} - 1\right)$$

- interesting result:

  $\gamma_{4,a} = \gamma_{4,b} \Leftrightarrow$ CP = LP if $m_\sigma = 2M_{vac}$ (as always in NJL!) ,

  but in general CP and LP do not coincide.

Sigma-mass dependence

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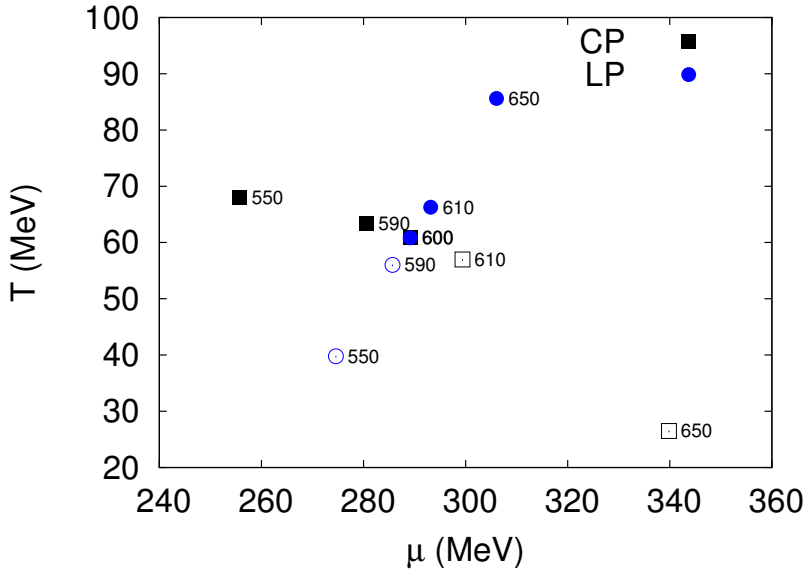
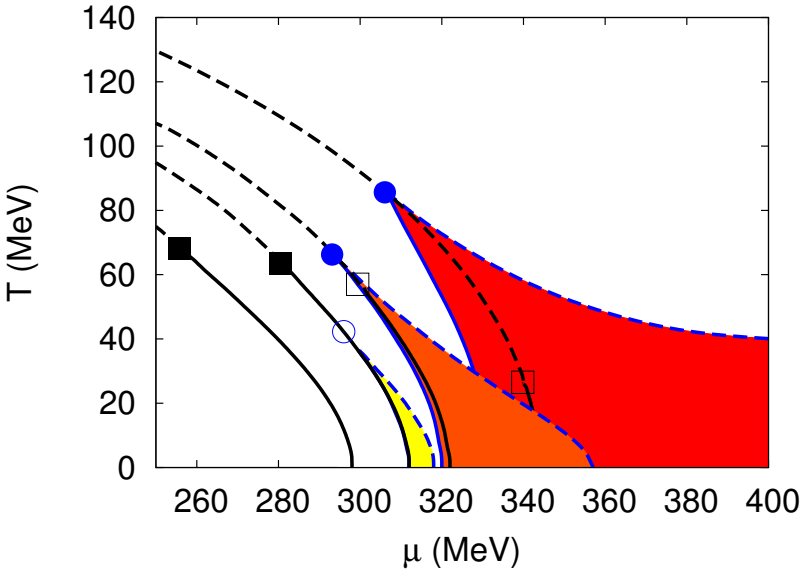
- GL results and phase diagrams for m_σ = 550, 590, 610, 650 MeV:





- Size of the inhomogeneous phase very sensitive to m_σ!

Conclusions

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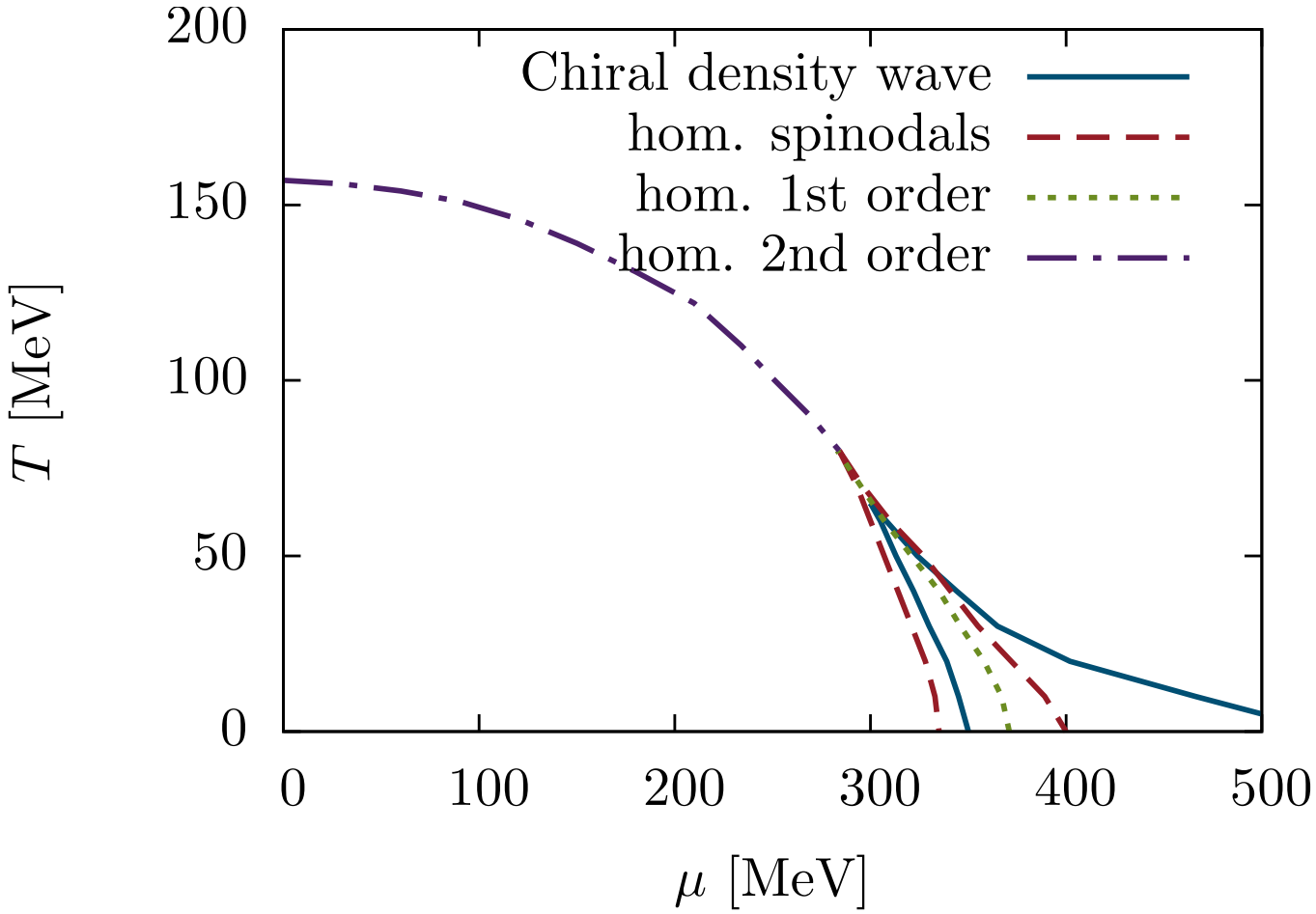
- First-order chiral phase transition or inhomogeneous phase?
- NJL model: inhomogeneous phase rather robust
  - standard NJL: CP $\rightarrow$ LP
  - vector interactions: CP covered by inhomogeneous phase
  - PNJL: CP slightly outside inhomogeneous phase
- QM model: size of inhomogeneous phase very sensitive to m_σ / m_{vac}

  $\rightarrow$ Fit to more physical observables?
- QCD?

  $\rightarrow$ Ginzburg-Landau for DSE calculations?
  difficulty: effective action not just a functional of the chiral condensate
- Role of fluctuations?
  - 1d modulations thermally unstable

    [Baym, Friman, Grinstein, NPB (1982); Lee et al., PRD (2015), Hidaka et al. PRD (2015)]

Dyson-Schwinger QCD [D. Müller et al. (2013)]



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Different parameter fixing

TECHNISCHE UNIVERSITÄT DARMSTADT

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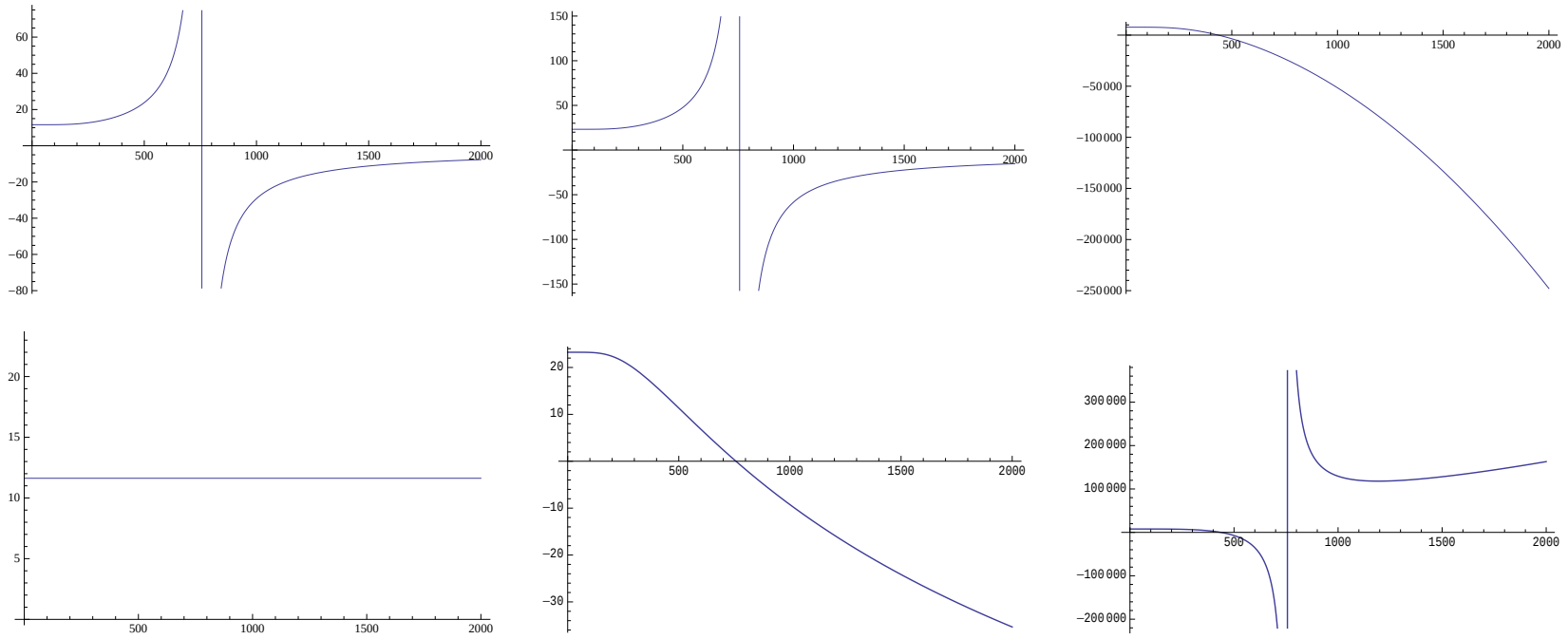
g^2 λ v^2



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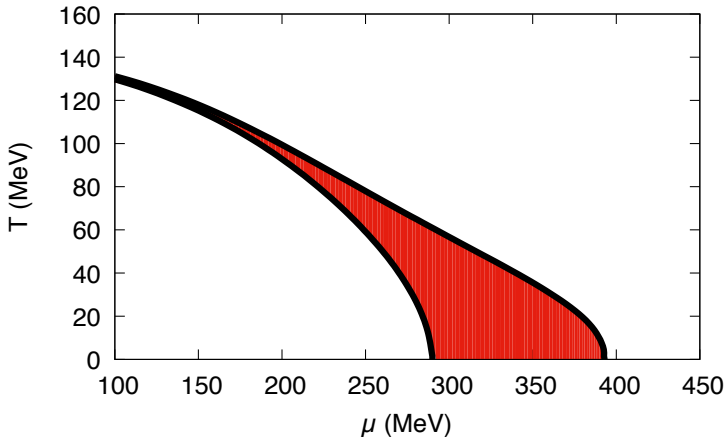
- Inhomogeneous phase diagram: $\Lambda = 200$ MeV

"correct"



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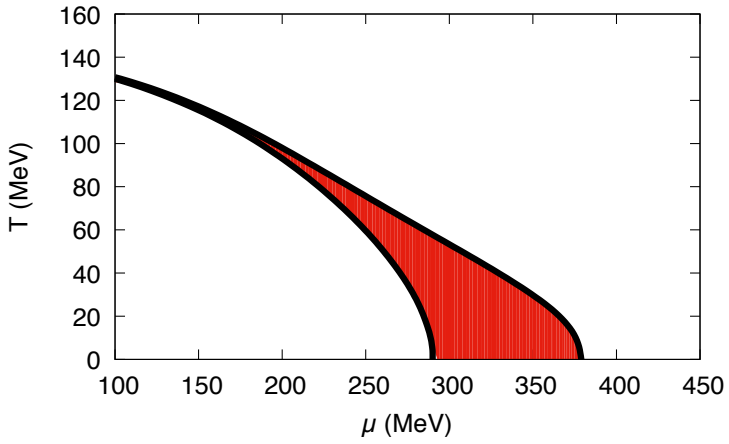

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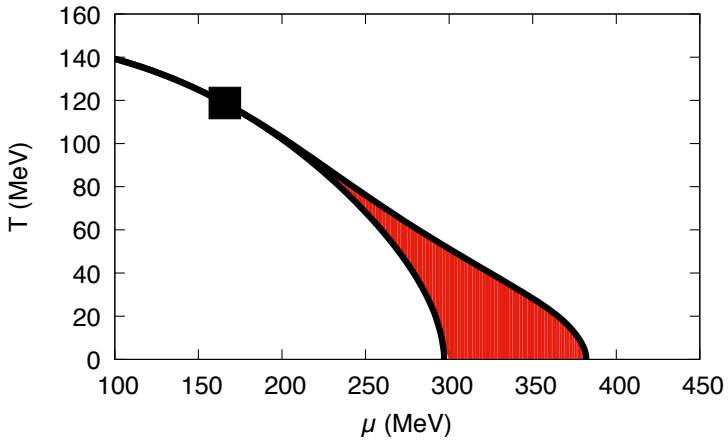
"correct"



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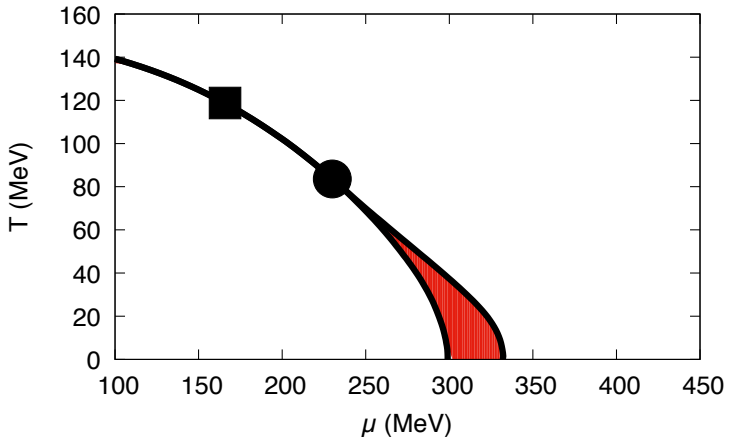

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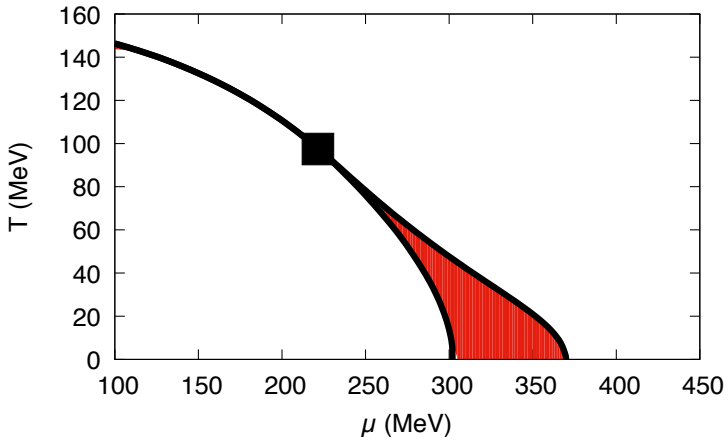
"correct"



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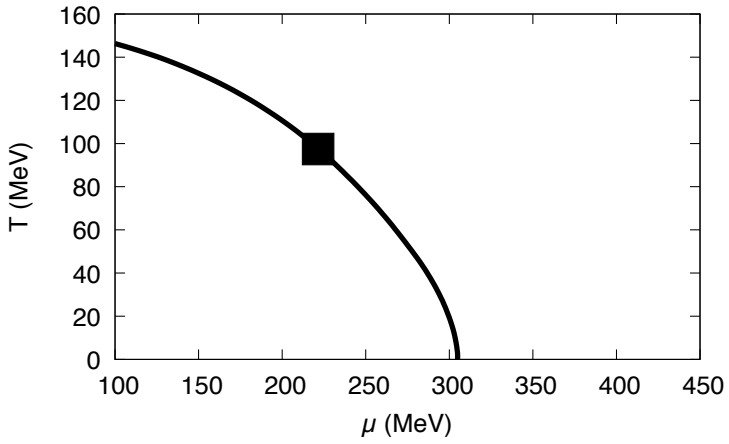

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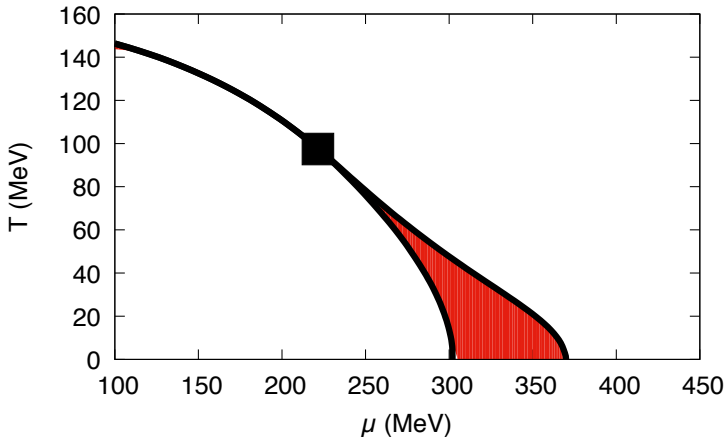

"wrong"