Effective dynamical models for fluctuations at the QCD phase transition

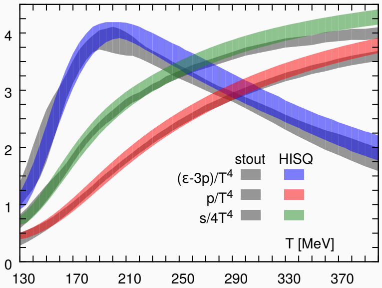
Marlene Nahrgang Duke University

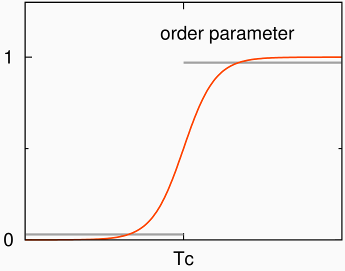
June 2, 2016

Critical Point and Onset of Deconfinement 2016, Wroclaw, Poland

What are fluctuation measures?

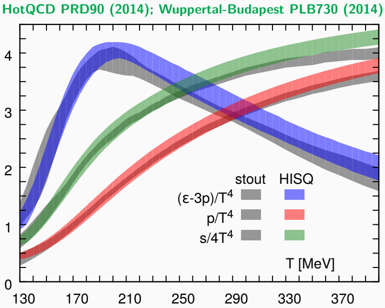
HotQCD PRD90 (2014); Wuppertal-Budapest PLB730 (2014)

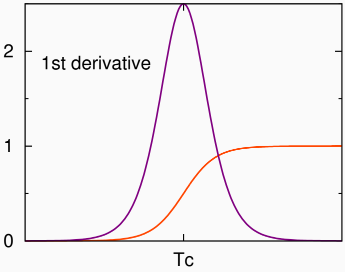




- Thermodynamic quantities change characteristically at the phase transition.
- Speed of sound $c_s^2 = (\partial p / \partial e)_S \rightarrow$ minimum at the phase transition
- Compressibility $\kappa_S = -1/V(\partial V/\partial p)_S \rightarrow$ maximum at the phase transition

What are fluctuation measures?



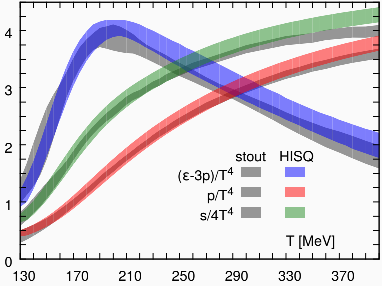


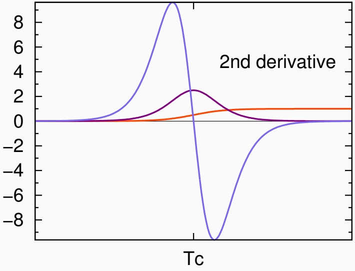
- Susceptibilities $\chi_n = \left.\frac{\partial^n(P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

What are fluctuation measures?

HotQCD PRD90 (2014); Wuppertal-Budapest PLB730 (2014)



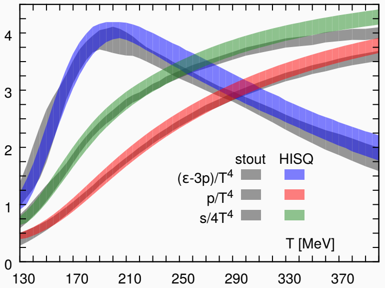


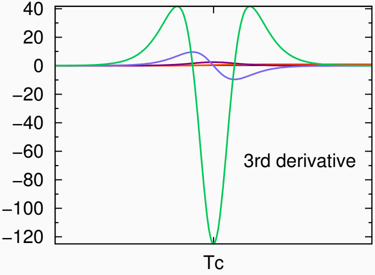
- Susceptibilities $\chi_n = \left.\frac{\partial^n (P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

What are fluctuation measures?

HotQCD PRD90 (2014); Wuppertal-Budapest PLB730 (2014)





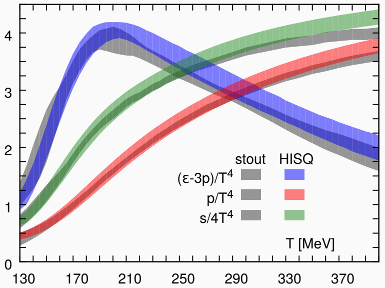
derivatives reveal more details!

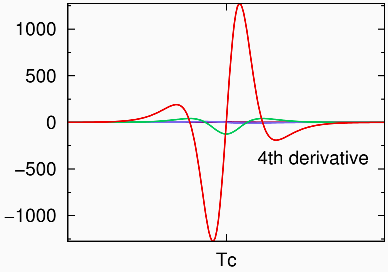
- Susceptibilities $\chi_n = \left.\frac{\partial^n (P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

What are fluctuation measures?

HotQCD PRD90 (2014); Wuppertal-Budapest PLB730 (2014)



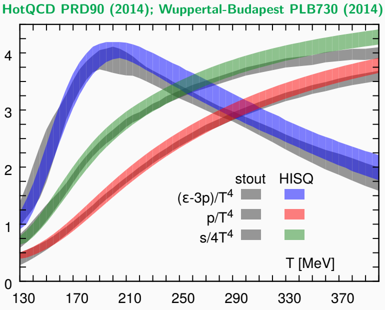


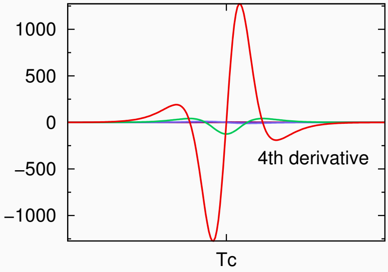
derivatives reveal more details!

- Susceptibilities $\chi_n = \left.\frac{\partial^n (P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

What are fluctuation measures?





derivatives reveal more details!

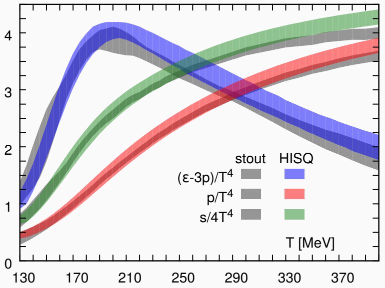
- Susceptibilities $\chi_n = \left.\frac{\partial^n (P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

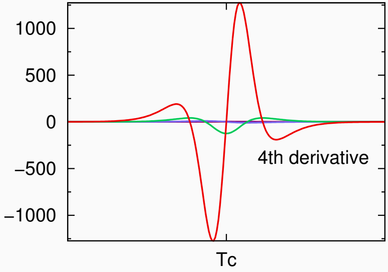
$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

variance

What are fluctuation measures?

HotQCD PRD90 (2014); Wuppertal-Budapest PLB730 (2014)





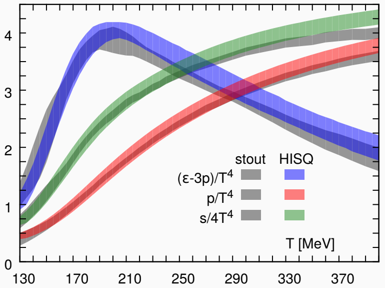
derivatives reveal more details!

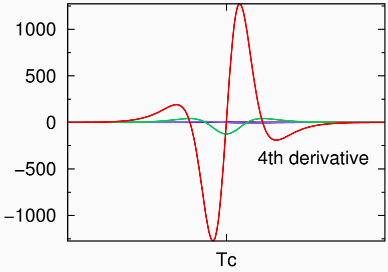
- Susceptibilities $\chi_n = \left. \frac{\partial^n (P/T^4)}{\partial(\mu/T)^n} \right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

variance   skewness

What are fluctuation measures?





derivatives reveal more details!

- Susceptibilities $\chi_n = \left.\frac{\partial^n (P/T^4)}{\partial(\mu/T)^n}\right|_T$ relate to fluctuations in multiplicity
- To zeroth-order in volume fluctuations:

$$\frac{\chi_2}{\chi_1} = \frac{\sigma^2}{M} \qquad \frac{\chi_3}{\chi_2} = S\sigma \qquad \frac{\chi_4}{\chi_2} = \kappa\sigma^2$$

variance skewness kurtosis

Fluctuations at a CP vs first-order

at a critical point:

- Correlation length diverges $\xi \to \infty \Rightarrow$ Fluctuations of the critical mode σ diverge.
- Higher moments more sensitive to ξ:

$$\langle \Delta\sigma^2 \rangle \propto \xi^2 , \quad \langle \Delta\sigma^3 \rangle \propto \xi^{9/2} , \quad \langle \Delta\sigma^4 \rangle_c \propto \xi^7 .$$

- Relaxation time $\tau_{\mathrm{rel}} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!

at a first-order phase transition:

- Coexistence of two stable thermodynamic phases at $T = T_c$.
- Metastable states above and below $T_c \Rightarrow$ supercooling and -heating.
- Nucleation & spinodal decomposition.

$\Rightarrow$ Domain formation and large inhomogeneities.

P. Hohenberg, B. Halperin, RMP49 (1977); T. Hatsuda, T. Kunihiro, PRL55 (1985); L Csernai, I Mishustin, PRL74 (1995); M. Stephanov, K. Rajagopal, E. Shuryak, PRL81 (1998), PRD60 (1999); S. Jeon, V. Koch, PRL83 (1999); B. Berdnikov and K. Rajagopal, PRD61 (2000); Y. Hatta, T. Ikeda, PRD67 (2003); M. Stephanov, PRL102 (2009); J. Randrup, PRC79 (2009), PRC82 (2010); M. Stephanov, PRL107 (2011)

Fluctuations at a CP vs first-order

at a critical point:

- Correlation length diverges $\xi \to \infty \Rightarrow$ Fluctuations of the critical mode σ diverge.
- Higher moments more sensitive to ξ:

Large ensemble fluctuations in equilibrated systems!

$$\langle \Delta\sigma^2 \rangle \propto \xi^2\,, \quad \langle \Delta\sigma^3 \rangle \propto \xi^{9/2}\,, \quad \langle \Delta\sigma^4 \rangle_c \propto \xi^7\,.$$

- Relaxation time $\tau_{\rm rel} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!

at a first-order phase transition:

- Coexistence of two stable thermodynamic phases at $T = T_c$.
- Metastable states above and below $T_c \Rightarrow$ supercooling and -heating.
- Nucleation & spinodal decomposition.

$\Rightarrow$ Domain formation and large inhomogeneities.

P. Hohenberg, B. Halperin, RMP49 (1977); T. Hatsuda, T. Kunihiro, PRL55 (1985); L Csernai, I Mishustin, PRL74 (1995); M. Stephanov, K. Rajagopal, E. Shuryak, PRL81 (1998), PRD60 (1999); S. Jeon, V. Koch, PRL83 (1999); B. Berdnikov and K. Rajagopal, PRD61 (2000); Y. Hatta, T. Ikeda, PRD67 (2003); M. Stephanov, PRL102 (2009); J. Randrup, PRC79 (2009), PRC82 (2010); M. Stephanov, PRL107 (2011)

Fluctuations at a CP vs first-order

at a critical point:

- Correlation length diverges $\xi \to \infty \Rightarrow$ Fluctuations of the critical mode σ diverge.
- Higher moments more sensitive to ξ:

$$\langle \Delta\sigma^2 \rangle \propto \xi^2 \,, \quad \langle \Delta\sigma^3 \rangle \propto \xi^{9/2} \,, \quad \langle \Delta\sigma^4 \rangle_c \propto \xi^7 \,.$$

Large ensemble fluctuations in equilibrated systems!

- Relaxation time $\tau_{\mathrm{rel}} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!

at a first-order phase transition:

- Coexistence of two stable thermodynamic phases at $T = T_c$.
- Metastable states above and below $T_c \Rightarrow$ supercooling and -heating.
- Nucleation & spinodal decomposition.

$\Rightarrow$ Domain formation and large inhomogeneities.

Large inhomogeneities/fluctuations in nonequilibrium systems!

P. Hohenberg, B. Halperin, RMP49 (1977); T. Hatsuda, T. Kunihiro, PRL55 (1985); L Csernai, I Mishustin, PRL74 (1995); M. Stephanov, K. Rajagopal, E. Shuryak, PRL81 (1998), PRD60 (1999); S. Jeon, V. Koch, PRL83 (1999); B. Berdnikov and K. Rajagopal, PRD61 (2000); Y. Hatta, T. Ikeda, PRD67 (2003); M. Stephanov, PRL102 (2009); J. Randrup, PRC79 (2009), PRC82 (2010); M. Stephanov, PRL107 (2011)

From chiral to net-proton fluctuations

couple order parameter to measurable particles: $g_p \bar{p} \sigma p$

M. Stephanov, K. Rajagopal, E. Shuryak, PRL81 (1998), PRD60 (1999); C. Athanasiou, K. Rajagopal, M. Stephanov, PRD82 (2010)

- Finite expectation value of σ in the chirally broken phase contributes to the mass of the proton
- Fluctuations $\Delta\sigma$ lead to fluctuations in the proton mass $m_p \to m_p + g\Delta\sigma$,
- Modification of flucs (statistical + critical) in the distrib. function:

$$\delta f = \delta f^0 + g \frac{\partial f^0}{\partial m_p} \Delta\sigma$$

From chiral to net-proton fluctuations

couple order parameter to measurable particles: $g_p \bar{p} \sigma p$

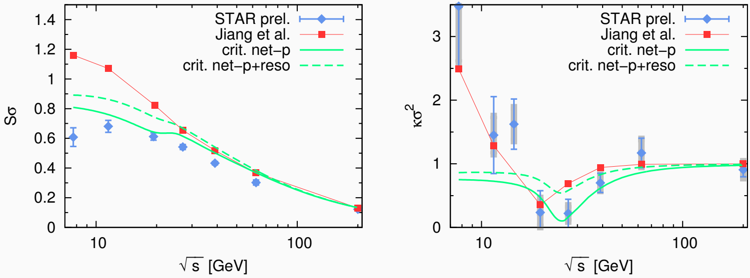
M. Stephanov, K. Rajagopal, E. Shuryak, PRL81 (1998), PRD60 (1999); C. Athanasiou, K. Rajagopal, M. Stephanov, PRD82 (2010)

- Finite expectation value of σ in the chirally broken phase contributes to the mass of the proton
- Fluctuations $\Delta\sigma$ lead to fluctuations in the proton mass $m_p \to m_p + g\Delta\sigma$,
- Modification of flucs (statistical + critical) in the distrib. function:

$$\delta f = \delta f^0 + g \frac{\partial f^0}{\partial m_p} \Delta\sigma$$

toy model! $\langle \Delta\sigma \rangle = 0 \Rightarrow$ no change in actual proton mass!

Critical net-proton fluctuations - phenomenology



MN, QM2015 proceedings, 1601.07437

- Equilibrium 3d Ising model assumptions for $\Delta\sigma$
- Fluctuations in net-protons at chemical freeze-out
- Critical fluctuations are reduced but survive when resonance decays are included

  M. Bluhm, MN, S. Bass, T. Schaefer work in progress
- Particle emission during Cooper-Frye freeze-out over a hypersurface from fluid dynamical evolution

  L. Jiang, P. Li and H. Song, arXiv:1512.06164

Still no dynamical fluctuations...

Non-critical effects on fluctuation observables

- Limited acceptance & detector efficiency. A. Bzdak, V. Koch, PRC86 (2012); PRC91 (2015)
- Isospin randomization. M. Kitazawa, M. Asakawa, PRC85, PRC86 (2012)
- Volume fluctuations V. Skokov, B. Friman, K. Redlich, PRC88 (2013)
  (→ strongly intensive measures).
  E. Sangaline, arxiv:1505.00261; M. Gorenstein, M. Gazdzicki, PRC84 (2011)
- Global net-baryon number conservation.
  MN, T. Schuster, M. Mitrovski, R. Stock, M. Bleicher, EPJC72 (2012); A. Bzdak, V. Koch, V. Skokov, PRC87 (2013)

⇒ These effects are or can be included in microscopic transport models, e.g. UrQMD, (P)HSD, or hybrid models = valuable baseline studies!

- Initial fluctuations due to baryon stopping.

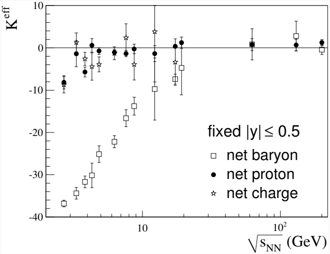
⇒ Need to be well understood!

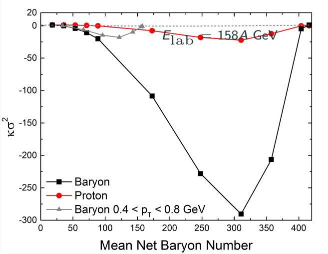
Non-critical effects on fluctuation observables

- Global net-baryon number conservation.

MN, T. Schuster, M. Mitrovski, R. Stock, M. Bleicher, EPJC72 (2012); A. Bzdak, V. Koch, V. Skokov, PRC87 (2013)

- In a microscopic transport model the microcanonical nature of individual scatterings is preserved.
- Strongly negative kurtosis of net-baryon number due to global conservation and volume fluctuations.
- Net-proton fluctuations follow this trend slightly.





by J. Steinheimer

MN, T. Schuster, M. Mitrovski, R. Stock, M. Bleicher, EPJC72 (2012)

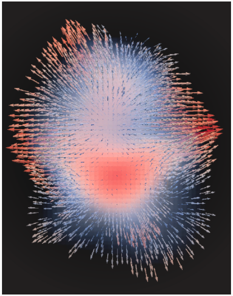
Why is dynamical modeling important?

In a grand-canonical ensemble, the system is

- in thermal equilibrium (= long-lived),
- in equilibrium with a particle heat bath,
- spatially infinite
- and static.

Systems created in heavy-ion collisions are

- short-lived,
- spatially small,
- inhomogeneous,
- and highly dynamical!



plot by H. Petersen, madai.us

Nonequilibrium chiral fluid dynamics (NχFD)

Propagate the critical mode σ coupled to a fluid dynamical expansion

- Relaxational equation for the critical mode: damping and noise from the interaction with the fermions

$$\partial_\mu \partial^\mu \sigma + \frac{\delta U}{\delta \sigma} + g \rho_s + \eta \partial_t \sigma = \xi$$

- Phenomenological dynamics for the Polyakov-loop

$$\eta_\ell \partial_t \ell T^2 + \frac{\partial V_{\rm eff}}{\partial \ell} = \xi_\ell$$

- Fluid dynamical expansion of the fermion fluid = heat bath, including energy-momentum exchange

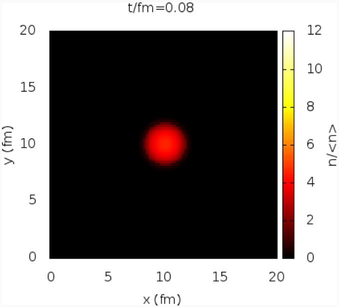
$$\partial_\mu T^{\mu\nu}_{\rm q} = S^\nu = -\partial_\mu T^{\mu\nu}_\sigma \,, \quad \partial_\mu N^\mu_{\rm q} = 0$$

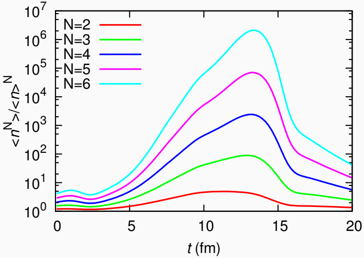
$\Rightarrow$ includes a stochastic source term!

MN, S. Leupold, I. Mishustin, C. Herold, M. Bleicher, PRC 84 (2011); PLB 711 (2012); JPG 40 (2013);
C. Herold, MN, I. Mishustin, M. Bleicher PRC 87 (2013); NPA925 (2014), C. Herold, MN, Y. Yan, C. Kobdaj
JPG 41 (2014); MN and C. Herold, 1602.07223; C. Herold, MN, Y. Yan and C. Kobdaj, PRC93 (2016) no.2

Domain formation & decay at the QH phase transition

- use a chiral effective model with correct low-temperature degrees of freedom in NχFD! **V. Dexheimer, S. Schramm, PRC81 (2010); M. Hempel, V. Dexheimer, S. Schramm, I. Iosilevskiy PRC88 (2013)**





- droplets of quark density decay in the hadronic phase due to non-vanishing large pressure (cf. also **J. Steinheimer, J. Randrup, V. Koch PRC89 (2014)**)
- future: combine initial and dynamical fluctuations, include particlization and late hadronic interactions

MN and C. Herold, 1602.07223

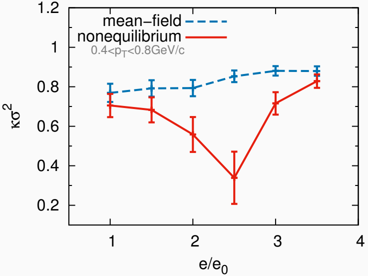
See talk by C. Herold, Fri 14:30!

Net-Proton fluctuations near the critical point

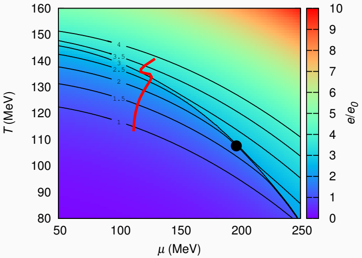
- From densities to particle via Cooper-Frye particlization:

$$E\frac{\mathrm{d}N_i}{\mathrm{d}^3p} = \int \mathrm{d}\sigma^\mu p_\mu (f_i^{\mathrm{eq}}(p) + \delta f)$$

- Here: couple the densities of the order parameter field with the fluid dynamical densities



C. Herold, MN, Y. Yan and C. Kobdaj, PRC93 (2016) no.2



- Future: include δf in the particlization and perform calculations for BES!

See talk by C. Herold, Fri 14:30!

About the critical mode

- At $\mu_B \neq 0$ σ mixes with the net-baryon density n (and e and $\vec{m}$)
- In a Ginzburg-Landau formalism:

$$V(\sigma, n) = \int \mathrm{d}^3x (\sum_m (a_m \sigma^m + b_m n^m) + \sum_{m,l} c_{m,l} \sigma^m n^l) - h\sigma - jn$$

- $V(\sigma, n)$ has a flat direction: $(a\sigma, bn)$ with vanishing curvature $D \to 0$ at the CP
- Equations of motion (including symmetries in $V(\sigma, n)$):

$$\partial_t^2 \sigma = -\Gamma \delta V / \delta \sigma + \ldots \qquad \partial_t n = \gamma \vec{\nabla}^2 \delta V / \delta n + \ldots$$

- eigenfrequencies

$$\omega_1 \propto -i\Gamma a \qquad \to \text{short time scale}$$
$$\omega_2 \propto -i\gamma D/a\, \vec{q}^2 \qquad \to \text{long time scale}$$

- The diffusive mode becomes the critical mode in the long-time dynamics. These fluctuations need to be included at the critical point!

H. Fuji, M. Ohtani PRD70 (2004); M. Stephanov, D. Son PRD70 (2004)

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- ... already in equilibrium there are thermal fluctuations
- ... the fast processes, which lead to local equilibration also lead to noise!

Conventional ideal fluid dynamics:

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}}$$
$$N^{\mu} = N^{\mu}_{\mathrm{eq}}$$

Y. Minami, T. Kunihiro, PTP122 (2010); P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); J. Kapusta, B. Müller, M. Stephanov PRC85 (2012); C. Chafin, T. Schäfer, PRA87 (2013); J. Kapusta, C. Young, PRC90 (2014); P. Romatschke, R. Young, PRA87 (2013); P. Kovtun, G. Moore, P. Romatschke, JHEP1407 (2014); C. Young, J. Kapusta, C. Gale, S. Jeon, B. Schenke, PRC91 (2015)

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- ... already in equilibrium there are thermal fluctuations
- ... the fast processes, which lead to local equilibration also lead to noise!

Conventional viscous fluid dynamics:

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Delta T^{\mu\nu}_{\rm visc}$$
$$N^{\mu} = N^{\mu}_{\rm eq} + \Delta N^{\mu}_{\rm visc}$$

Y. Minami, T. Kunihiro, PTP122 (2010); P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); J. Kapusta, B. Müller, M. Stephanov PRC85 (2012); C. Chafin, T. Schäfer, PRA87 (2013); J. Kapusta, C. Young, PRC90 (2014); P. Romatschke, R. Young, PRA87 (2013); P. Kovtun, G. Moore, P. Romatschke, JHEP1407 (2014); C. Young, J. Kapusta, C. Gale, S. Jeon, B. Schenke, PRC91 (2015)

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- ... already in equilibrium there are thermal fluctuations
- ... the fast processes, which lead to local equilibration also lead to noise!

Fluctuating viscous fluid dynamics:

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \Delta T^{\mu\nu}_{\mathrm{visc}} + \Xi^{\mu\nu}$$
$$N^{\mu} = N^{\mu}_{\mathrm{eq}} + \Delta N^{\mu}_{\mathrm{visc}} + \mathrm{I}^{\mu}$$

Y. Minami, T. Kunihiro, PTP122 (2010); P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); J. Kapusta, B. Müller, M. Stephanov PRC85 (2012); C. Chafin, T. Schäfer, PRA87 (2013); J. Kapusta, C. Young, PRC90 (2014); P. Romatschke, R. Young, PRA87 (2013); P. Kovtun, G. Moore, P. Romatschke, JHEP1407 (2014); C. Young, J. Kapusta, C. Gale, S. Jeon, B. Schenke, PRC91 (2015)

Fluid dynamical fluctuations

Conventional fluid dynamics propagates thermal averages of the energy density, pressure, velocities, charge densities, etc.

However, ...

- ... already in equilibrium there are thermal fluctuations
- ... the fast processes, which lead to local equilibration also lead to noise!

Fluctuating viscous fluid dynamics:

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \Delta T^{\mu\nu}_{\mathrm{visc}} + \Xi^{\mu\nu}$$
$$N^{\mu} = N^{\mu}_{\mathrm{eq}} + \Delta N^{\mu}_{\mathrm{visc}} + \mathrm{I}^{\mu}$$

- $\langle T^{\mu\nu} T^{\mu\nu} \rangle$ give viscosities (Kubo-formula), consistently with dissipation-fluctuation theorem fluctuations need to be included as well!
- This is especially important at the critical point, because the true critical mode is the net-baryon density!

Y. Minami, T. Kunihiro, PTP122 (2010); P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); J. Kapusta, B. Müller, M. Stephanov PRC85 (2012); C. Chafin, T. Schäfer, PRA87 (2013); J. Kapusta, C. Young, PRC90 (2014); P. Romatschke, R. Young, PRA87 (2013); P. Kovtun, G. Moore, P. Romatschke, JHEP1407 (2014); C. Young, J. Kapusta, C. Gale, S. Jeon, B. Schenke, PRC91 (2015)

Linearized fluid dynamical fluctuations

- Linearized fluid dynamical equations: small fluctuations $\bar{e}+\delta e$, $\bar{p}+\delta p$ and δv^i with: $\delta T^{00}=\delta e$ and $\delta T^{ij}=m^i=(\bar{e}+\bar{p})v^i=\bar{w}v^i$

$$\begin{aligned}\partial_t \mathbf{m}_\perp + \eta/\bar{w}\mathbf{k}^2\mathbf{m}_\perp &= 0\\ \partial_t \delta e + i\mathbf{k}\cdot\mathbf{m}_{||} &= 0\\ \partial_t \mathbf{m}_{||} + i v_s^2 \mathbf{k}\delta e + \gamma_v \mathbf{k}^2 \mathbf{m}_{||} &= 0\end{aligned}$$

- retarded Green's function for δe and $\mathbf{m}_{||}$:

$$G^{\mathrm{ret}}_{ab}(\omega,\mathbf{k}) = \frac{\bar{w}}{\omega^2 - v_s^2\mathbf{k}^2 + i\omega\gamma_s\mathbf{k}^2}\begin{pmatrix}\mathbf{k}^2 & \omega|\mathbf{k}|\\ \omega|\mathbf{k}| & v_s^2\mathbf{k}^2 - i\omega\gamma_s\mathbf{k}^2\end{pmatrix}$$

- including the transverse momentum density:

$$G^{\mathrm{ret}}_{m_i,m_j}(\omega,\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\frac{\eta\mathbf{k}^2}{i\omega - \gamma_\eta\mathbf{k}^2} + \frac{k_i k_j}{\mathbf{k}^2}\frac{\bar{w}(v_s^2\mathbf{k}^2 - i\omega\gamma_s\mathbf{k}^2)}{\omega^2 - v_s^2\mathbf{k}^2 + i\omega\gamma_s\mathbf{k}^2}$$

- Kubo-formulas for viscosities:

$$\begin{aligned}\eta &= -\frac{\omega}{2\mathbf{k}^2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\Im G^{\mathrm{ret}}_{m_i m_j}(\omega,\mathbf{k}\to 0)\\ \zeta + \frac{4}{3}\eta &= -\frac{\omega^3}{\mathbf{k}^4}\Im G^{\mathrm{ret}}_{ee}(\omega,\mathbf{k}\to 0)\end{aligned}$$

Linearized fluid dynamical fluctuations

$$\frac{\partial}{\partial_x^\mu}\frac{\partial}{\partial_{x'}^\mu}\langle \Xi^{\mu 0}(x)\Xi^{\mu 0}(x')\rangle^S = -\frac{\partial}{\partial_x^\mu}\frac{\partial}{\partial_{x'}^\mu}\langle T^{\mu 0}(x)T^{\mu 0}(x')\rangle^S$$

$$= \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{-i\omega(t-t')}\times$$

$$\times \left(\omega^2 \underbrace{G^S_{ee}(\omega,\mathbf{k})}_{\text{FDT}} - 2\omega|\mathbf{k}| \underbrace{G^S_{em_{||}}(\omega,\mathbf{k})}_{\text{FDT}} + \mathbf{k}^2 \underbrace{G^S_{m_{||}m_{||}}(\omega,\mathbf{k})}_{\text{FDT}} \right)$$

$$G^S_{ab}(\omega,\mathbf{k}) = -\frac{2T}{\omega}\Im G^{\mathrm{ret}}_{ab}(\omega,\mathbf{k})$$

$$= 0$$

$$\frac{\partial}{\partial_x^\mu}\frac{\partial}{\partial_{x'}^\mu}\langle \Xi^{\mu i}(x)\Xi^{\mu j}(x')\rangle^S = -\frac{\partial}{\partial_x^\mu}\frac{\partial}{\partial_{x'}^\mu}\langle T^{\mu i}(x)T^{\mu j}(x')\rangle^S$$

$$= 2T\left[\left(\zeta + \frac{4}{3}\eta\right)\partial_i\partial_j + \eta(\delta_{ij}\nabla^2 - \partial_i\partial_j)\right]\delta^4(x-x')$$

finally boost to arbitrary frame

Fluid dynamical equations with fluctuations

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq}} + \Delta T^{\mu\nu}_{\mathrm{visc}} + \Xi^{\mu\nu}$$
$$N^{\mu} = N^{\mu}_{\mathrm{eq}} + \Delta N^{\mu}_{\mathrm{visc}} + \mathrm{I}^{\mu}$$

with the noise correlator in linear response theory:

$$\langle \Xi^{\mu\nu}(x)\Xi^{\alpha\beta}(x')\rangle = 2T[\eta(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + (\zeta - 2/3\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}]\delta^4(x - x')$$

- In second-order fluid dynamics there are relaxation equations for $\Xi^{\mu\nu}$:

$$u^{\gamma}\partial_{\gamma}\Xi^{\langle\mu\nu\rangle} = -\frac{\Xi^{\mu\nu} - \xi^{\mu\nu}_{\mathrm{gauss}}}{\tau_{\pi}}$$

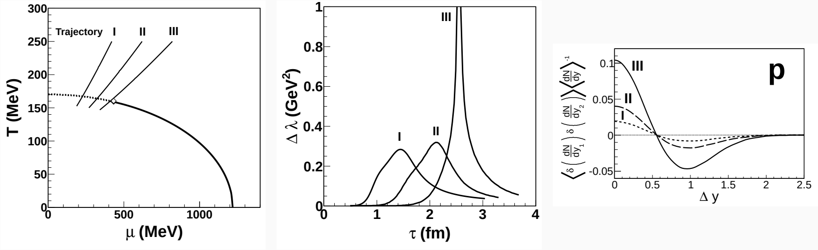
- In white noise approximation and ignoring bulk viscosity ($\zeta = 0$):

$$\langle \xi^{\mu\nu}_{\mathrm{gauss}}(x)\xi^{\alpha\beta}_{\mathrm{gauss}}(x')\rangle = 4T\eta\Delta^{\mu\nu\alpha\beta}\delta^{(4)}(x - x')$$

Fluid dynamical fluctuations in Bjorken expansion

Bjorken expansion example with a critical point:

J. Kapusta, J. Torres-Rincon PRC86 (2012)



- near the CP the thermal conductivity is enhanced ⇒ enhancement of the rapidity correlator of protons
- how to implement in a $3 + 1$d relativistic causal fluid dynamical evolution?

Fluid dynamical fluctuations - nonlinearities

- correlation functions from linearized fluctuations describe noninteracting modes
- if nonlinearities are included $\to$ interaction of modes
  $\to$ modification of correlations
  $\to$ contributions to transport coefficients
- symmetrized correlator:

$$G_S^{xyxy}(\omega,\mathbf{0}) = \int \mathrm{d}^3x\mathrm{d}t\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \left\langle \frac{1}{2}\{T^{xy}(t,\mathbf{x}), T^{xy}(0,\mathbf{0})\} \right\rangle$$

- for the shear-shear contribution $\Rightarrow$

$$G_{R,\mathrm{shear-shear}}^{xyxy}(\omega,\mathbf{0}) = -\frac{7T}{90\pi^2}\Lambda^3 - i\omega\frac{7T}{60\pi^2}\frac{\Lambda}{\gamma_\eta} + (i+1)\omega^{3/2}\frac{7T}{90\pi^2}\frac{1}{\gamma_\eta^{3/2}}$$

P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); C. Chafin, T. Schfer, PRA87 (2013); P. Romatschke, R. Young, PRA87 (2013)

Fluid dynamical fluctuations - nonlinearities

- correlation functions from linearized fluctuations describe noninteracting modes
- if nonlinearities are included $\rightarrow$ interaction of modes
  $\rightarrow$ modification of correlations
  $\rightarrow$ contributions to transport coefficients
- symmetrized correlator:

$$G_S^{xyxy}(\omega, \mathbf{0}) = \int \mathrm{d}^3x\mathrm{d}t\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \left\langle \frac{1}{2}\{T^{xy}(t,\mathbf{x}), T^{xy}(0,\mathbf{0})\} \right\rangle$$

- for the shear-shear contribution $\Rightarrow$

$$G^{xyxy}_{R,\mathrm{shear-shear}}(\omega, \mathbf{0}) = -\frac{7T}{90\pi^2}\Lambda^3 - i\omega\frac{7T}{60\pi^2}\frac{\Lambda}{\gamma_\eta} + (i+1)\omega^{3/2}\frac{7T}{90\pi^2}\frac{1}{\gamma_\eta^{3/2}}$$

cutoff-dependent
fluctuation contribution
to the pressure

P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); C. Chafin, T. Schfer, PRA87 (2013); P. Romatschke, R. Young, PRA87 (2013)

Fluid dynamical fluctuations - nonlinearities

- correlation functions from linearized fluctuations describe noninteracting modes
- if nonlinearities are included $\rightarrow$ interaction of modes
  $\rightarrow$ modification of correlations
  $\rightarrow$ contributions to transport coefficients
- symmetrized correlator:

$$G_S^{xyxy}(\omega, \mathbf{0}) = \int \mathrm{d}^3x\mathrm{d}t\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \left\langle \frac{1}{2}\{T^{xy}(t,\mathbf{x}), T^{xy}(0,\mathbf{0})\} \right\rangle$$

- for the shear-shear contribution $\Rightarrow$

$$G_{R,\mathrm{shear-shear}}^{xyxy}(\omega, \mathbf{0}) = -\frac{7T}{90\pi^2}\Lambda^3 - i\omega\frac{7T}{60\pi^2}\frac{\Lambda}{\gamma_\eta} + (i+1)\omega^{3/2}\frac{7T}{90\pi^2}\frac{1}{\gamma_\eta^{3/2}}$$

cutoff-dependent fluctuation contribution to the pressure

cutoff-dependent correction to η

P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); C. Chafin, T. Schfer, PRA87 (2013); P. Romatschke, R. Young, PRA87 (2013)

Fluid dynamical fluctuations - nonlinearities

- correlation functions from linearized fluctuations describe noninteracting modes
- if nonlinearities are included $\rightarrow$ interaction of modes
  $\rightarrow$ modification of correlations
  $\rightarrow$ contributions to transport coefficients
- symmetrized correlator:

$$G_S^{xyxy}(\omega,\mathbf{0}) = \int \mathrm{d}^3x\mathrm{d}t\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \left\langle \frac{1}{2}\{T^{xy}(t,\mathbf{x}), T^{xy}(0,\mathbf{0})\} \right\rangle$$

- for the shear-shear contribution $\Rightarrow$

$$G_{R,\mathrm{shear-shear}}^{xyxy}(\omega,\mathbf{0}) = -\frac{7T}{90\pi^2}\Lambda^3 - i\omega\frac{7T}{60\pi^2}\frac{\Lambda}{\gamma_\eta} + (i+1)\omega^{3/2}\frac{7T}{90\pi^2}\frac{1}{\gamma_\eta^{3/2}}$$

cutoff-dependent fluctuation contribution to the pressure

cutoff-dependent correction to η

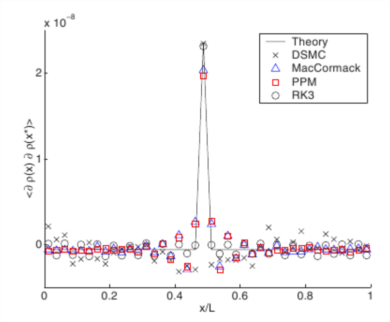
frequency-dependent contribution to η and τ_π

P. Kovtun, G. Moore, P. Romatschke, PRD84 (2011); C. Chafin, T. Schfer, PRA87 (2013); P. Romatschke, R. Young, PRA87 (2013)

Non-relativistic dynamical fluctuations

- Fluctuations depend on the cut-off scale of fluid dynamics: $\Lambda \sim 1/\Delta x$.
- In a numerical treatment $\rightarrow$ discretization: $\langle \xi^2 \rangle \propto \frac{1}{\Delta V}$.

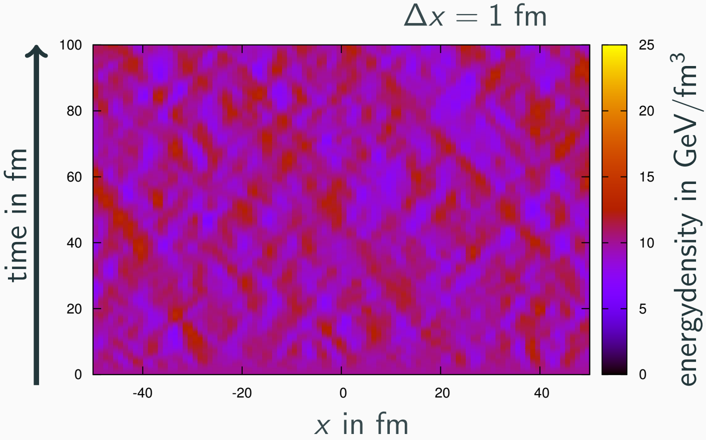
- Example: non-relativistic Navier-Stokes + fluctuations
- 1d, dilute gas, periodic boundary conditions



J. Bell, A. Garcia, S. Williams, PRE76 (2007)

- Different algorithms treat fluctuations differently, third-order methods seem to work best.

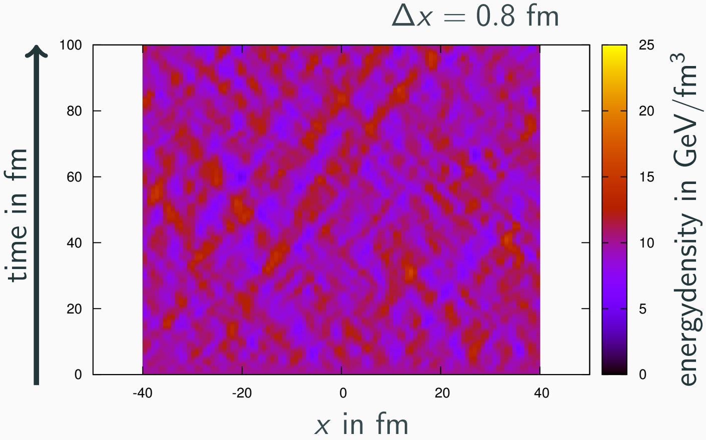
- Static box with periodic boundary conditions in relativistic $1+1$d fluid dynamics
- Initialized at $e_0 = 10$ GeV/fm^3 (without fluctuations nothing would happen)



MN, M. Bluhm, Y. Karpenko, S. Bass, T. Schaefer, work in progress

Fluid dynamical fluctuations - 1+1d

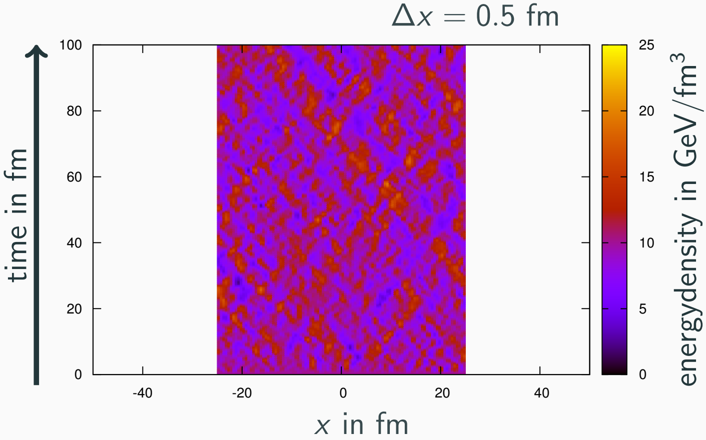
- Static box with periodic boundary conditions in relativistic $1+1$d fluid dynamics
- Initialized at $e_0 = 10$ GeV/fm^3 (without fluctuations nothing would happen)



MN, M. Bluhm, Y. Karpenko, S. Bass, T. Schaefer, work in progress

Fluid dynamical fluctuations - 1+1d

- Static box with periodic boundary conditions in relativistic $1+1$d fluid dynamics
- Initialized at $e_0 = 10$ GeV/fm^3 (without fluctuations nothing would happen)



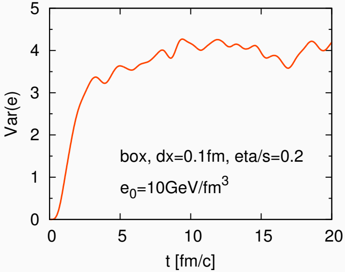
MN, M. Bluhm, Y. Karpenko, S. Bass, T. Schaefer, work in progress

Fluid dynamical fluctuations - 3+1d

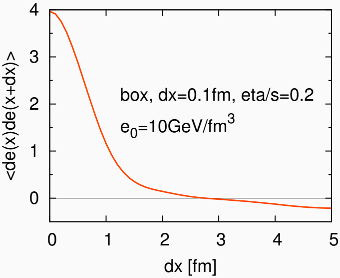
- Static box with periodic boundary conditions in relativistic $3+1$d fluid dynamics

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(T^{\mu\nu}_{\text{eq}} + \Delta T^{\mu\nu}_{\text{visc}} + \Xi^{\mu\nu} \right) = 0$$

time evolution of the variance $\langle \Delta e^2 \rangle$:



$\langle \delta e(x) \delta e(x+dx) \rangle$ correlation function:

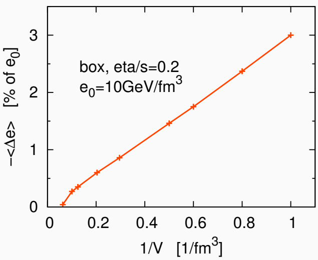


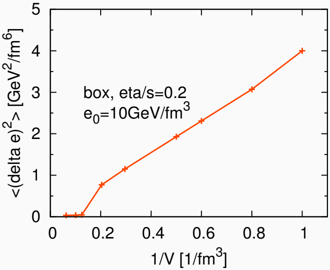
- The variance $\langle \Delta e^2 \rangle$ saturates after ~ 5 fm.
- Fluctuations are large in the computational cell of fluid dynamics $\Rightarrow$ noise correlated over ~ 1 fm^3 - reproduced.

MN, M. Bluhm, Y. Karpenko, S. Bass, T. Schaefer, work in progress

Fluid dynamical fluctuations - 3+1d nonlinearities

- Important check: equilibrium expectations for fluctuations and nonlinear effects.





- Proportionality to $1/V$ reproduced for the correction to the average and the variance of energy density in the local rest frame.
- Implementing fluid dynamical fluctuations is important,

  but requires a sustained and systematic effort!

MN, M. Bluhm, Y. Karpenko, S. Bass, T. Schaefer, work in progress

Toward the discovery of the critical point

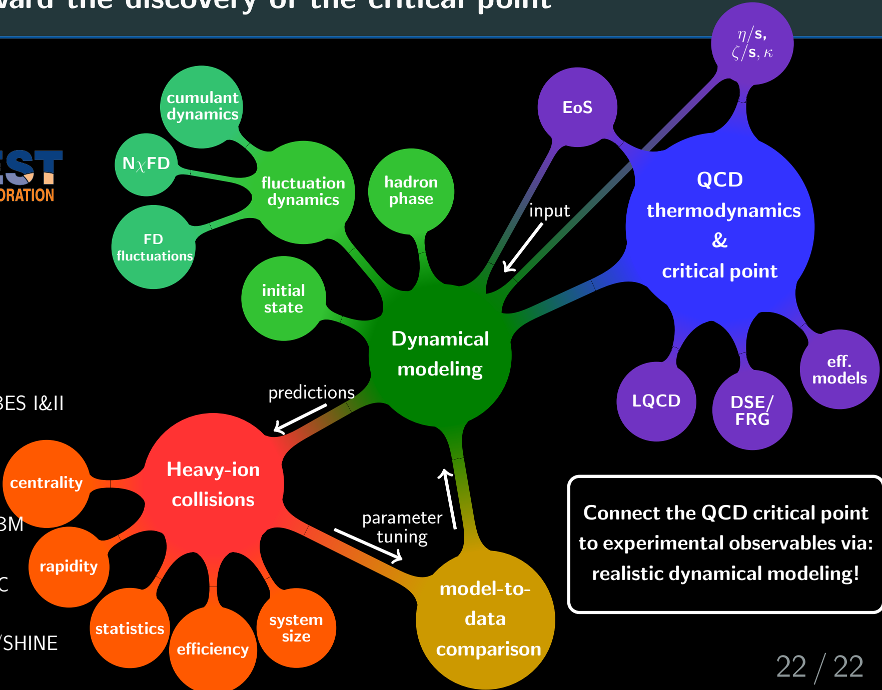
RHIC BES I&II

NICA

FAIR CBM

JPARC

NA61/SHINE



Connect the QCD critical point to experimental observables via: realistic dynamical modeling!