Factorization and resummation of non-global observables within SCET

Ding Yu Shao
University of Bern

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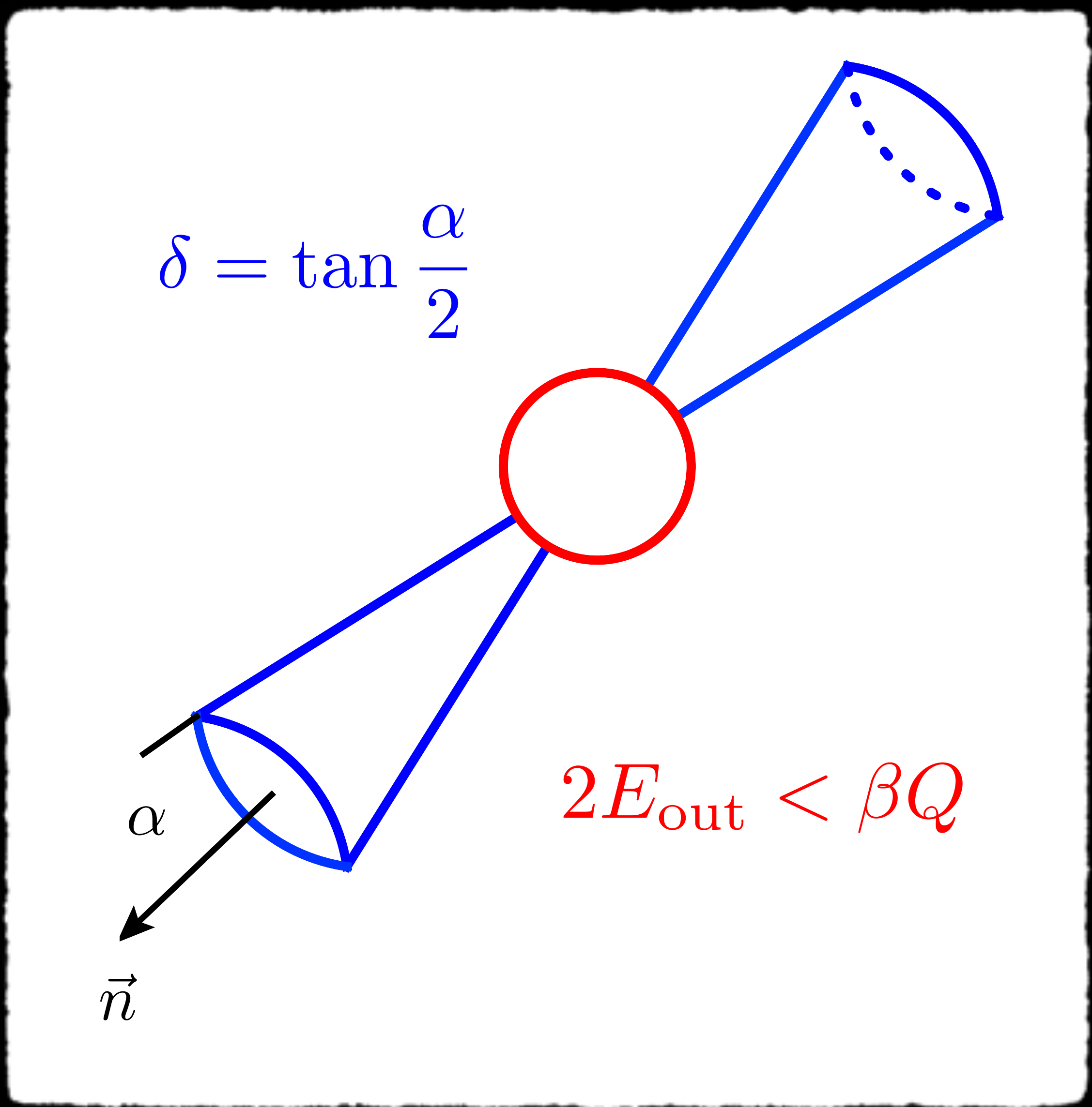
In collaboration with T. Becher, M. Neubert & L. Rothen
(PRL116(2016)192001, arXiv:1605.02737, work in progress)

Sterman-Weinberg dijets

(Sterman & Weinberg 1977)



$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{3\pi}\left[-16\ln\delta\ln\beta - 12\ln\delta + 10 - \frac{4\pi^2}{3}\right]$$

IR finite, but problems for small β, δ

- Large log can spoil perturbative expansion
- Scale choice?

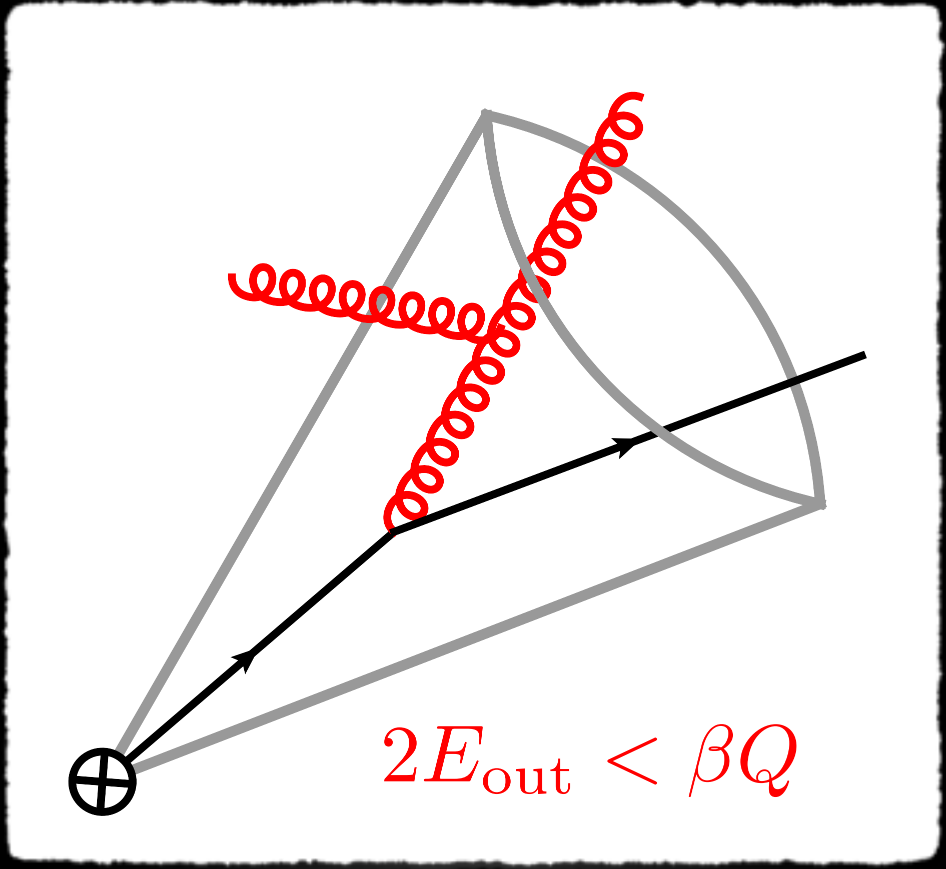
$$\mu = Q,\ Q\beta,\ Q\delta,\ Q\beta\delta\ ?$$

Non-global logarithms (NGLs)

(Dasgupta & Salam 2001)

Observables which are insensitive to emissions into certain regions of phase space involve additional NGLs **not captured** by the usual resummation formula

$$\sigma \sim \mathcal{H} \cdot \mathcal{J}_1 \otimes \mathcal{J}_2 \otimes \mathcal{S}$$



Jet observables involve NGLs because they are insensitive to emissions inside the cone

$$\alpha_s^2 C_F C_A \pi^2 \ln^2 \beta$$

These types of logarithm do not exponentiate in the usual way

Leading-Log resummation

Banfi, Marchesini & Smye 2002

- The leading logarithms arise from configuration in which the emitted gluons are strongly ordered

$$E_1 \gg E_2 \gg \cdots \gg E_m$$

- In the large-Nc limit, multi-gluon emission amplitudes become simple:

$$N_c^m g^{2m} \sum_{(1\cdots m)} \frac{p_a \cdot p_b}{(p_a \cdot p_1)(p_1 \cdot p_2)\cdots(p_m \cdot p_b)}$$

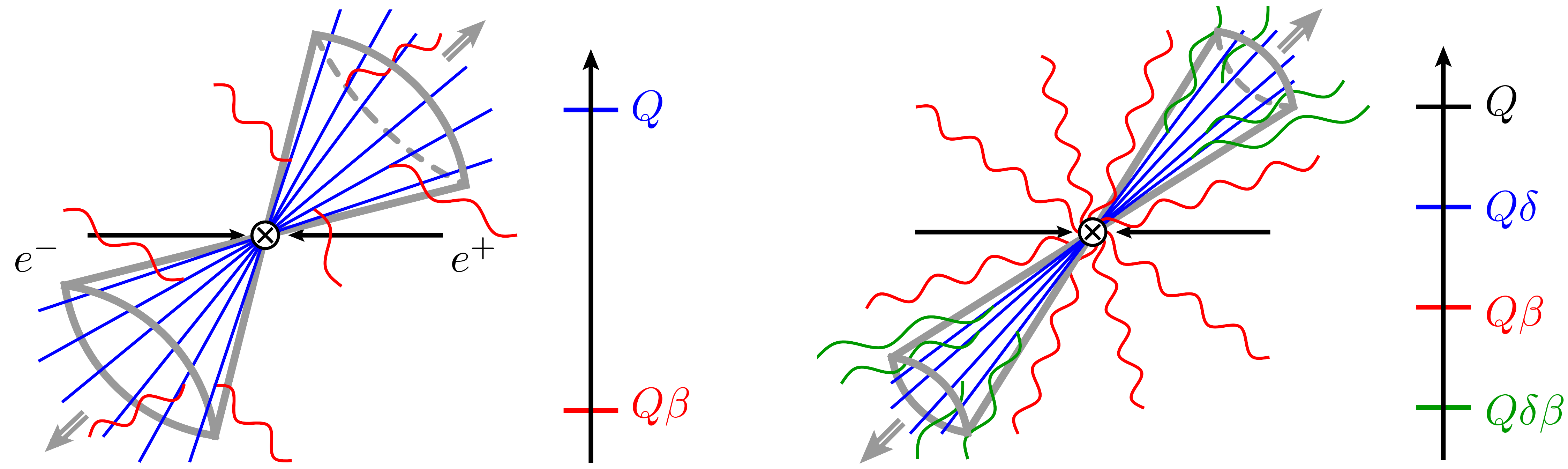
- Based on this structure, Banfi, Marchesini & Smye derive an integral-differential equation for resuming NG logarithms at LL level in the large-Nc limit:

BMS equation: $$\partial_L G_{ab}(L) = \int \frac{d\Omega_j}{4\pi} W_{ab}^j \left[\Theta_{\rm in}^{n\bar{n}}(j) G_{aj}(L) G_{jb}(L) - G_{ab}(L)\right]$$

Some recent progress

- Resummation of LL NGLs beyond large Nc Weigert ’03; Hatta Ueda ’13 + Hagiwara ’15; Caron-Huot ‘15
- Fixed-order results
  - two-loop hemisphere soft function Kelley, Schwartz, Schabinger & Zhu ’11; Horning, Lee, Stewart, Walsh & Zuberi ’11
  - with jet-cone Kelley, Schwartz, Schabinger & Zhu ’11; von Manteuffel, Schabinger & Zhu ’13
  - LL NGLs 5-loops (BMS eq & finite Nc) Schwartz, Zhu ’14; Delenda, Khelifa-Kerfa ’15
- Expansion in soft sub-jets Larkoski, Moult & Neill ’15; Neill ’15; Laroski, Moult ’15
- Avoid NGLs Dasgupta, Fregoso, Marzani & Powling ’13; Dasgupta, Fregoso, Marzani & Salam ’13; Larkoski, Marzani, Soyez & Thaler ’14; Frye, Larkoski, Matthew & Yan ‘16

Non-Global Observables



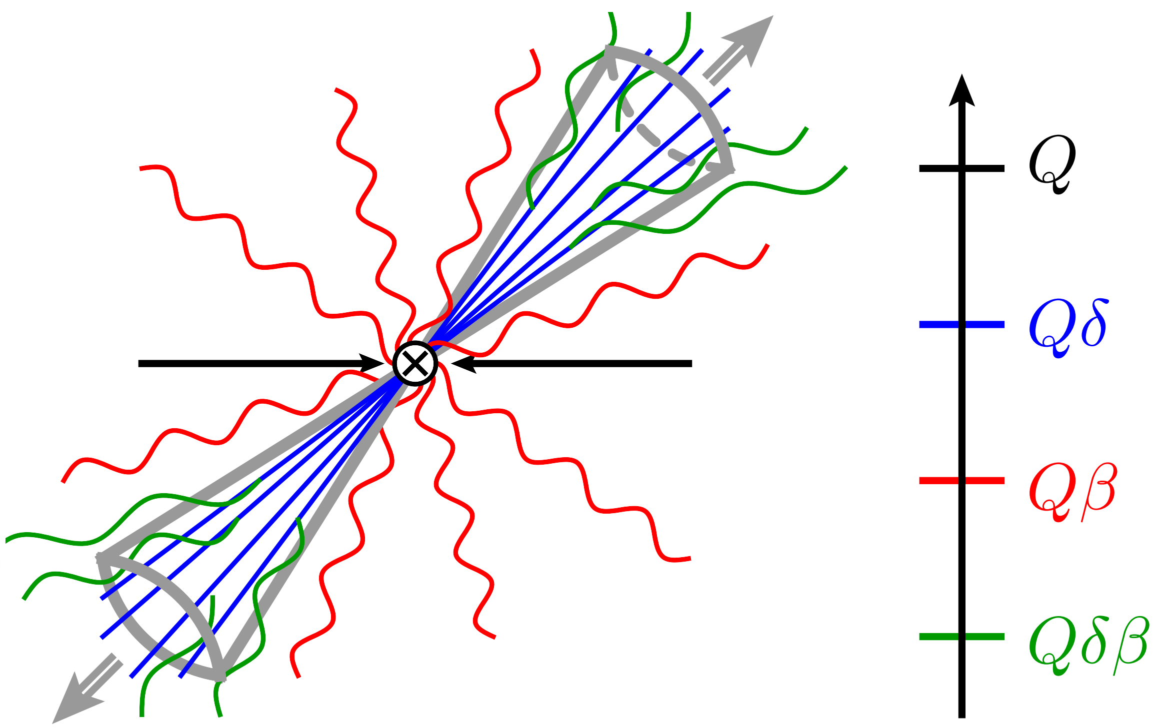
From SCET to Jet Effective Theory

Becher, Neubert, Rothen & DYS, PRL116(2016)192001

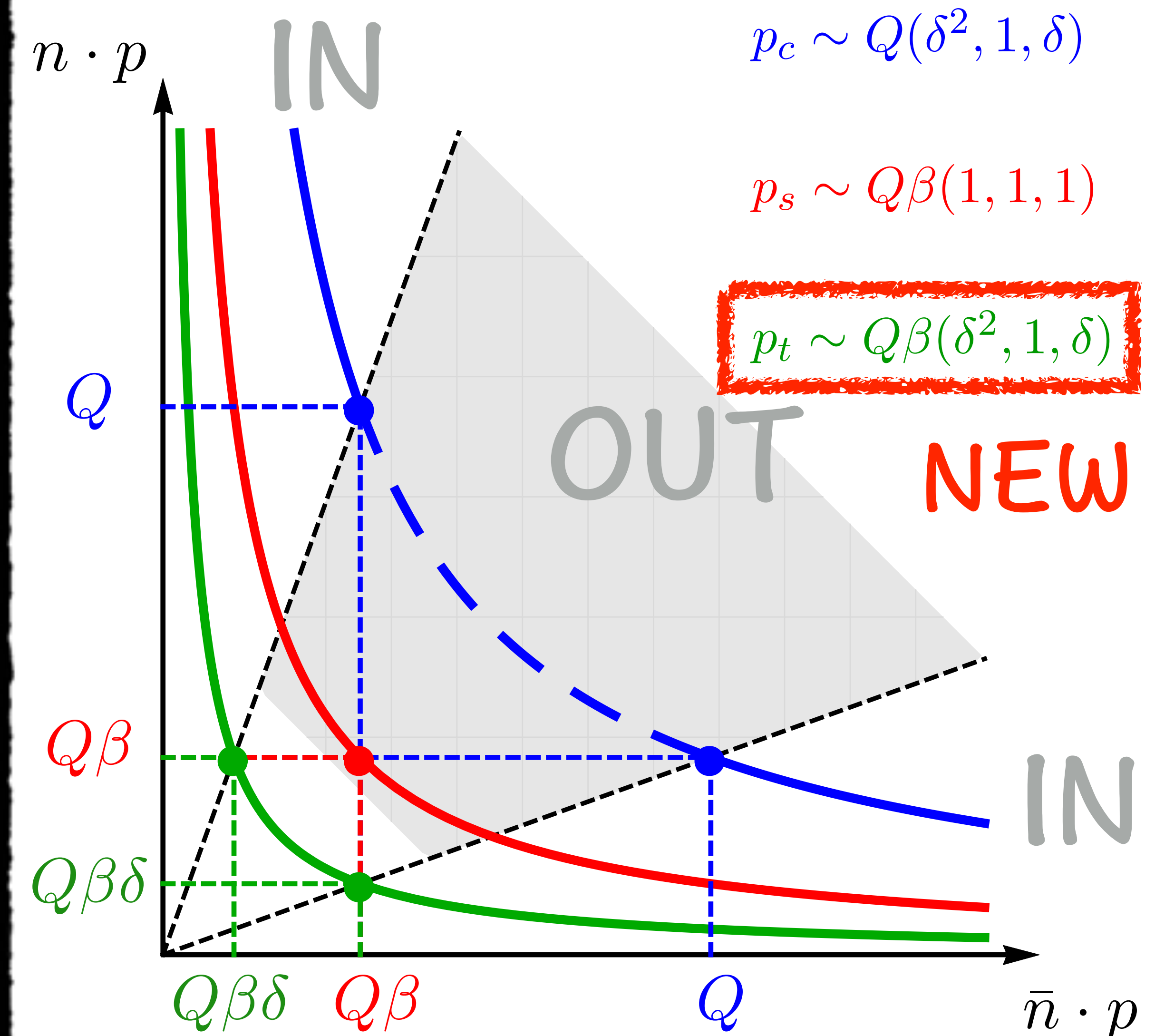
EFT for narrow-cone jets

$p \sim (n \cdot p, \bar{n} \cdot p, \vec{p}_\perp)$

$\delta \sim \frac{\alpha}{2} \ll 1$



$2E_{\text{out}} < \beta Q \ll Q$



One-loop Region Analysis

Hard $$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}+\frac{7\pi^2}{3}-16\right)$$

Collinear $$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}+\frac{6}{\epsilon}+c_0\right)$$

"Soft" $$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{8}{\epsilon}\ln\delta - 8\ln^2\delta - \frac{2\pi^2}{3}\right)$$

(Cheung, Luke, Zuberi 2009......)

$$\Delta\sigma^{\text{tot}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(-16\ln\delta\ln\beta - 12\ln\delta + c_0 + \frac{5\pi^2}{3} - 16\right)$$

Constant c_0 depends on the definition of jet axis:

$c_0 = -3\pi^2 + 26$ (Sterman-Weinberg)

$c_0 = -5\pi^2/3 + 14 + 12\ln 2$ (thrust axis)

One-loop Region Analysis

Hard	$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}+\frac{7\pi^2}{3}-16\right)$
Collinear	$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}+\frac{6}{\epsilon}+c_0\right)$
Soft	$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}-\pi^2\right)$
Coft	$\Delta\sigma_{t+\bar{t}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta\beta}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}+\frac{\pi^2}{3}\right)$

$$\Delta\sigma^{\text{tot}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(-16\ln\delta\ln\beta - 12\ln\delta + c_0 + \frac{5\pi^2}{3} - 16\right)$$

Constant c_0 depends on the definition of jet axis:

$c_0 = -3\pi^2 + 26$ (Sterman-Weinberg)

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Factorization for two-jet cross section

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q)\, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m(Q\delta) \otimes \widetilde{\mathcal{U}}_m(Q\delta\tau) \right\rangle \right]^2$$

color trace

integration over angles

hard function

soft function

jet function with m patrons at fixed direction

coft function with m Wilson lines

First all-order factorization theorem for non-global observable.

Achieves full scale separation!

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon) \Big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon)$$

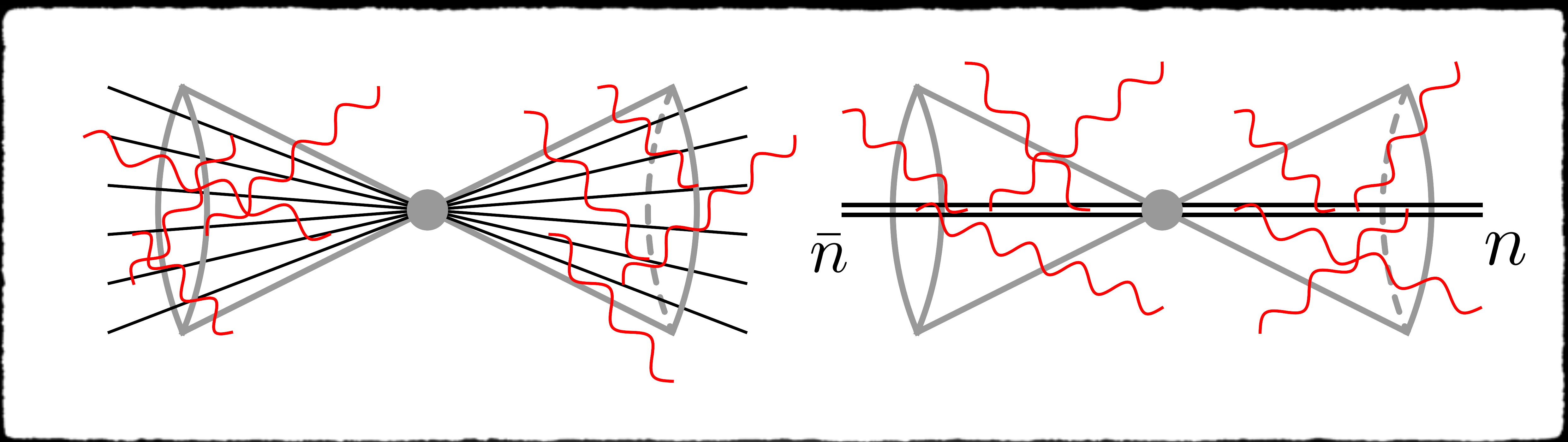
$$+ \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \Big\langle \mathcal{J}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon)$$
$$+ \mathcal{J}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \mathcal{J}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$

Soft function:

$$S(Q\beta)\,\mathbf{1} = \sum\!\!\!\!\!\!\int_{X_s} \langle 0|\, S^\dagger(\bar{n})\, S(n)\, |X_s\rangle\langle X_s|\, S^\dagger(n)\, S(\bar{n})\, |0\rangle \theta(Q\beta - 2E_{X_s})$$



Soft Radiation

Large-angle soft radiation off a jet of collinear particles does not resolve individual energetic patrons

$$\sum_i Q_i \frac{p_i \cdot \epsilon}{p_i \cdot k} \approx Q_{\text{tot}} \frac{n \cdot \epsilon}{n \cdot k}$$

This approximation breaks down for soft radiation collinear to the jet!!!

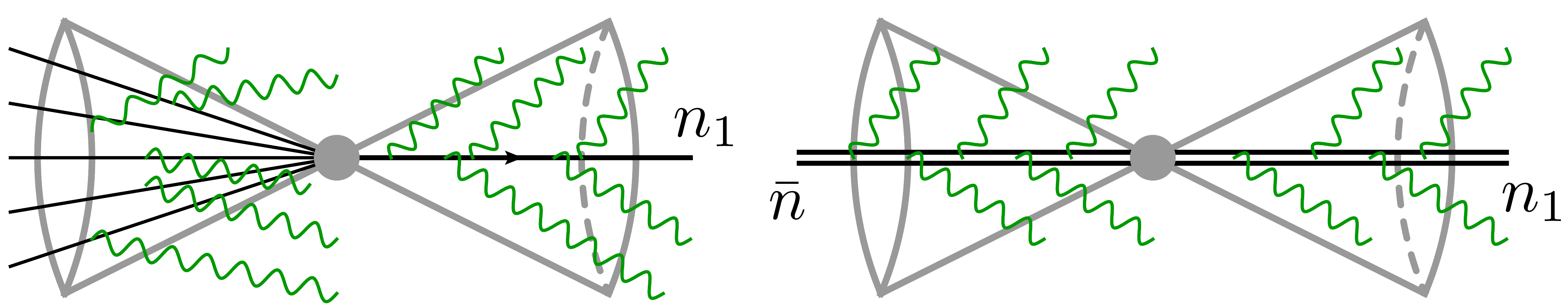
$$k^\mu = \omega n^\mu$$

Typically this small region of phase space does not give an $\mathcal{O}(1)$ contribution.

However it does in the non-global observable!

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \Big\langle \underline{\boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon)}$$
$$+ \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$



NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \Big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon)$$
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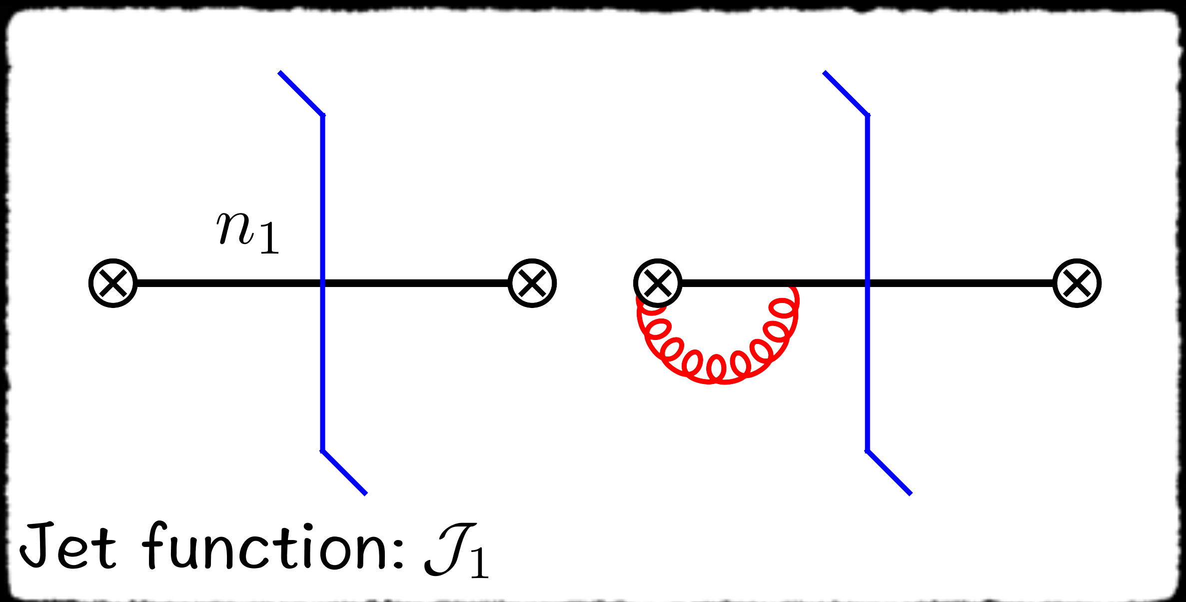
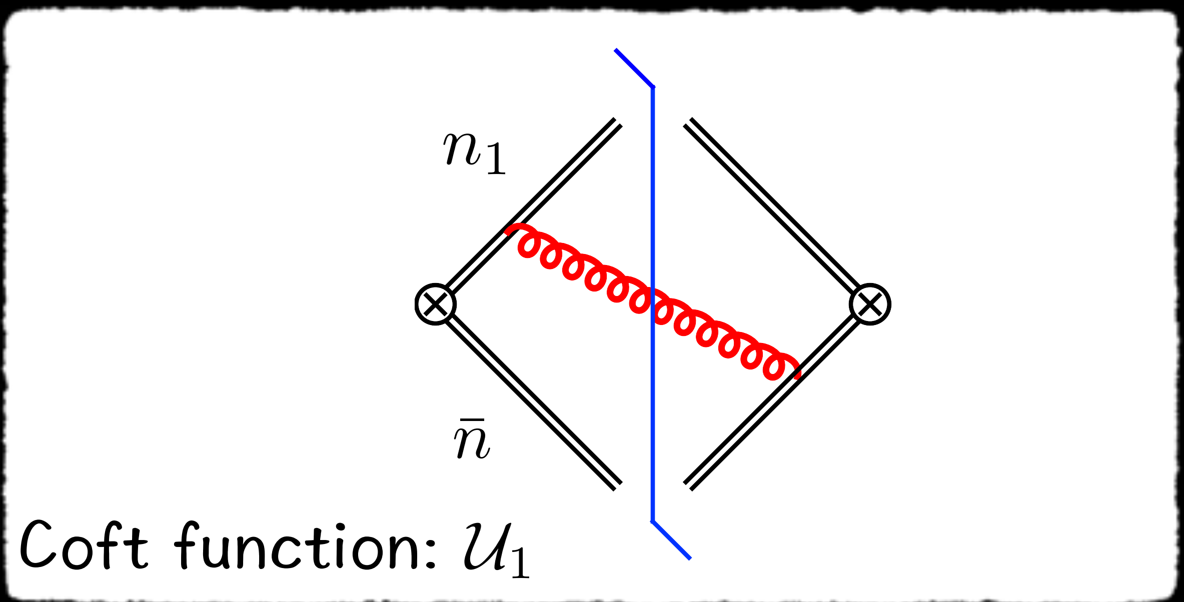




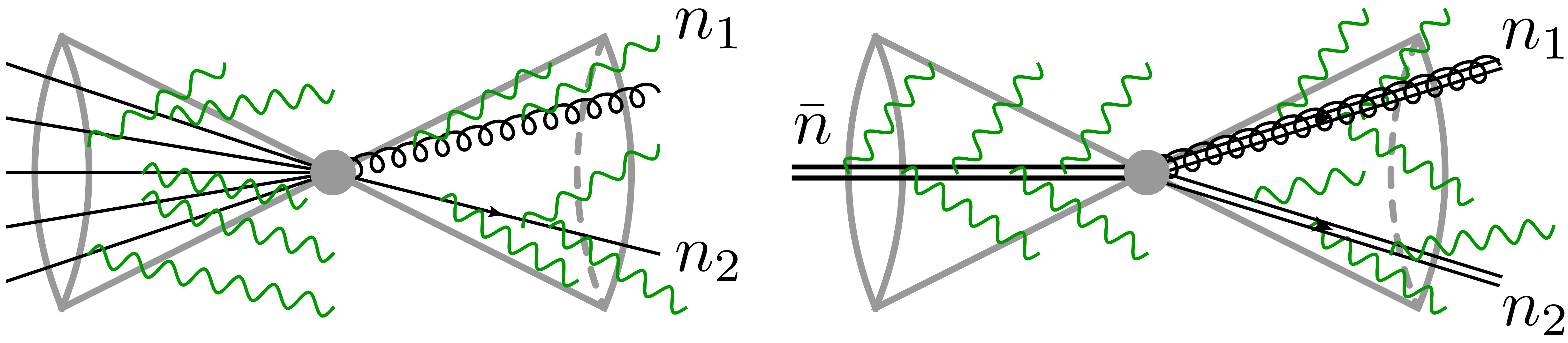
$$\boldsymbol{\mathcal{J}}_1(\hat{\theta}_1,Q\delta,\epsilon) = \delta(\hat{\theta}_1)\,\mathbf{1}$$

$$\widetilde{\boldsymbol{\mathcal{U}}}_1(\hat{\theta}_1,Q\tau\delta,\epsilon) = \mathbf{1} + \frac{C_F\alpha_0}{4\pi}\, e^{-2\,\epsilon L_t}\, u_F(\hat{\theta}_1)\,\mathbf{1}$$

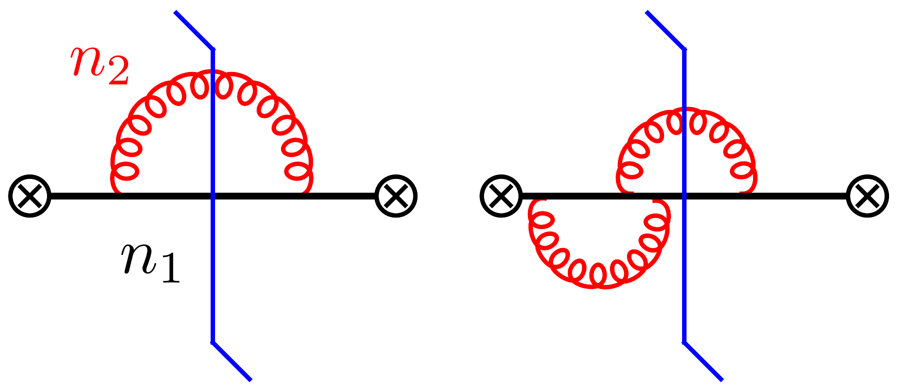
$$\big\langle \boldsymbol{\mathcal{J}}_1 \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1 \big\rangle = \big\langle \widetilde{\boldsymbol{\mathcal{U}}}_1(0,Q\delta\tau,\epsilon) \big\rangle$$

NNLO check

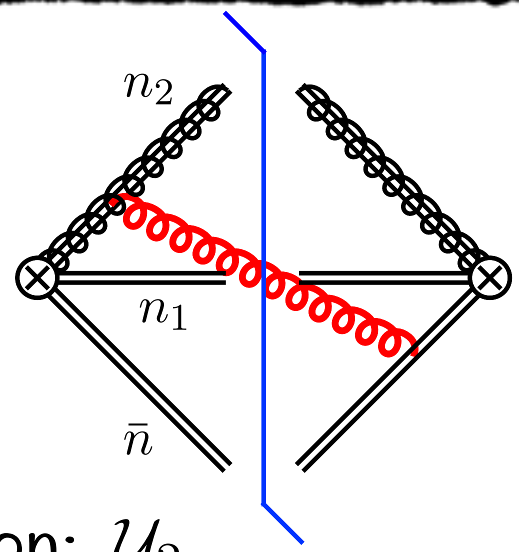
$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon) \left\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon) \right.$$
$$\left. + \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \right\rangle^2$$



NNLO check



Jet function: $\mathcal{J}_2$



Coft function: $\mathcal{U}_2$

$$\boldsymbol{\mathcal{J}}_2^{(1)}(\hat{\theta}_1, \hat{\theta}_2, \phi_2, Q\delta, \epsilon) = C_F\, \delta(\phi_2 - \pi)\, e^{-2\epsilon L_c}$$

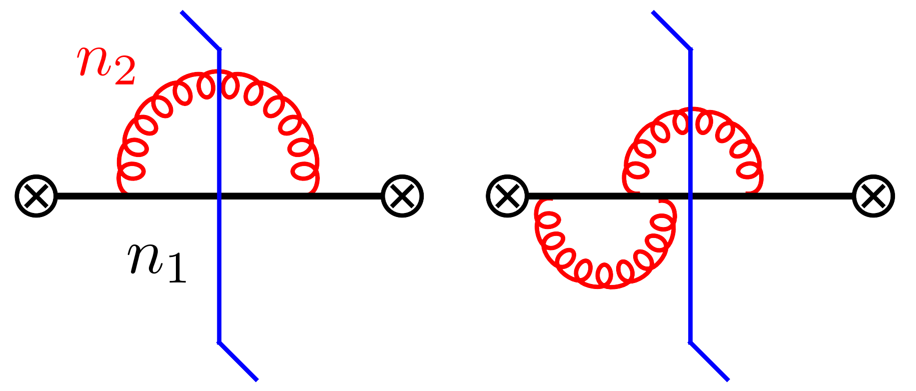
$$\times \Big\{ \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 7 - \frac{5\pi^2}{6} + 6\ln 2 \right) \delta(\hat{\theta}_1)\, \delta(\hat{\theta}_2) - \frac{4}{\epsilon}\, \delta(\hat{\theta}_1) \left[\frac{1}{\hat{\theta}_2} \right]_+ + 8\, \delta(\hat{\theta}_1) \left[\frac{\ln \hat{\theta}_2}{\hat{\theta}_2} \right]_+$$

$$+ 4\, \frac{dy}{d\hat{\theta}_2} \left[\frac{1}{\hat{\theta}_1} \right]_+ \frac{1 + 2y + 2y^2}{(1+y)^3}\, \theta(\hat{\theta}_1 - \hat{\theta}_2)$$

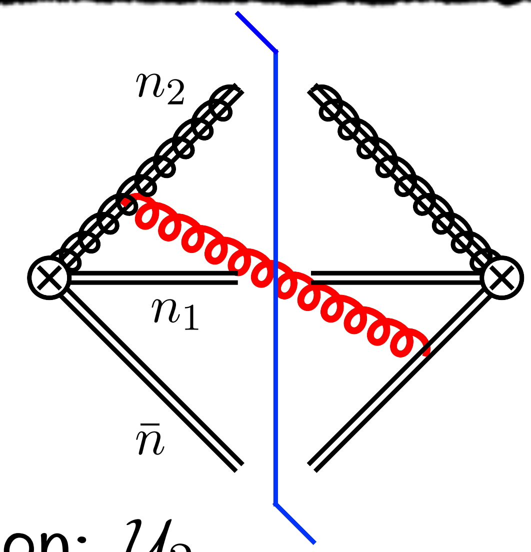
$$+ 4\, \frac{dy}{d\hat{\theta}_1} \left[\frac{1}{\hat{\theta}_2} \right]_+ \left(2 \left[\frac{1}{y} \right]_+ - \frac{4 + 5y + 2y^2}{(1+y)^3} \right) \theta(\hat{\theta}_2 - \hat{\theta}_1) + \mathcal{O}(\epsilon) \Big\} \mathbf{1}$$

$$\widetilde{\boldsymbol{\mathcal{U}}}_2(\hat{\theta}_1, \hat{\theta}_2, \phi_2, Q\tau\delta, \epsilon) = \mathbf{1} + \frac{\alpha_0}{4\pi}\, e^{-2\,\epsilon L_t} \Big[C_F\, u_F(\hat{\theta}_1) + C_A\, u_A(\hat{\theta}_1, \hat{\theta}_2, \phi_2) \Big] \mathbf{1}$$

NNLO check



Jet function: $\mathcal{J}_2$



Coft function: $\mathcal{U}_2$

$$\left\langle \boldsymbol{\mathcal{J}}_2^{(1)} \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2^{(1)} \right\rangle = e^{-2\epsilon(L_c+L_t)} \left(C_F^2 M_F + C_F C_A M_A \right)$$

$$\begin{aligned} M_F = & -\frac{4}{\epsilon^4} - \frac{6}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-14 + \frac{2\pi^2}{3} - 12\ln 2\right) + \frac{1}{\epsilon}\left(-26 - \pi^2 + 10\,\zeta_3 - 32\ln 2\right) \\ & - 52 - \frac{10\pi^2}{3} - 27\zeta_3 + \frac{11\pi^4}{30} - \frac{4}{3}\ln^4 2 - 8\ln^3 2 - 4\ln^2 2 + \frac{4\,\pi^2}{3}\ln^2 2 \\ & - 52\ln 2 + 4\pi^2 \ln 2 - 28\zeta_3 \ln 2 - 32\,\mathrm{Li}_4\left(\frac{1}{2}\right), \end{aligned}$$

$$\begin{aligned} M_A = & \frac{2\pi^2}{3\epsilon^2} + \frac{1}{\epsilon}\left(-2 + \frac{\pi^2}{2} + 12\,\zeta_3 + 6\ln^2 2 + 4\ln 2\right) - 4 + \frac{7\pi^2}{6} - 24\zeta_3 - \frac{\pi^4}{6} + \frac{8}{3}\ln^4 2 \\ & - 4\ln^3 2 + 6\ln^2 2 - \frac{8\pi^2}{3}\ln^2 2 - 4\ln 2 + 9\pi^2\ln 2 + 56\zeta_3 \ln 2 + 64\,\mathrm{Li}_4\left(\frac{1}{2}\right) \end{aligned}$$

NNLO check

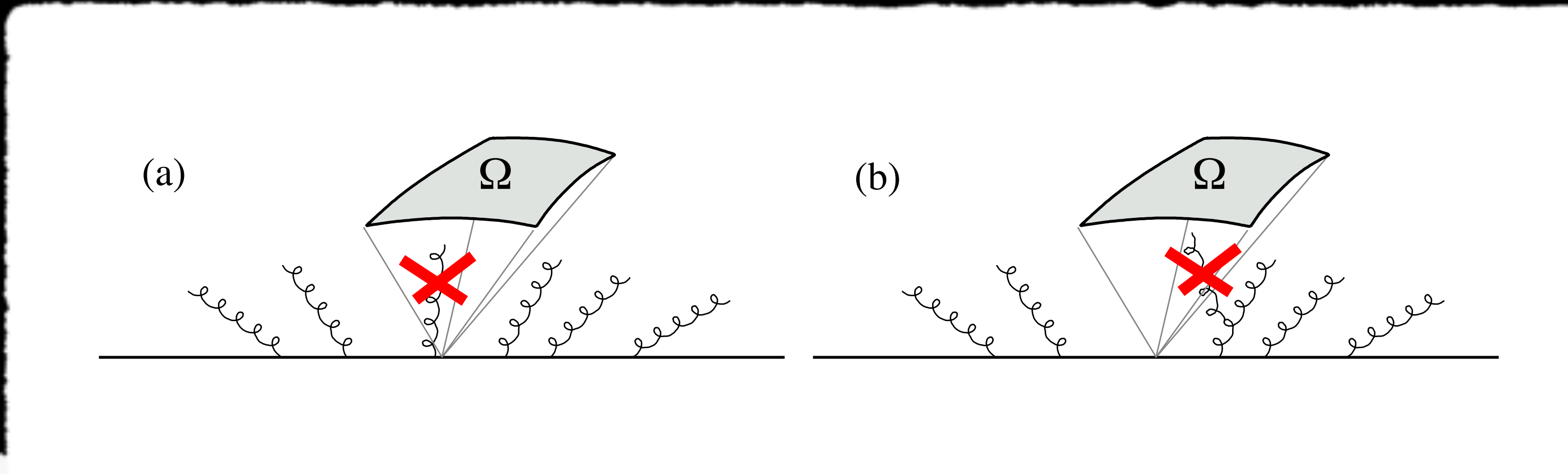
$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi}\, A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta) + \ldots$$

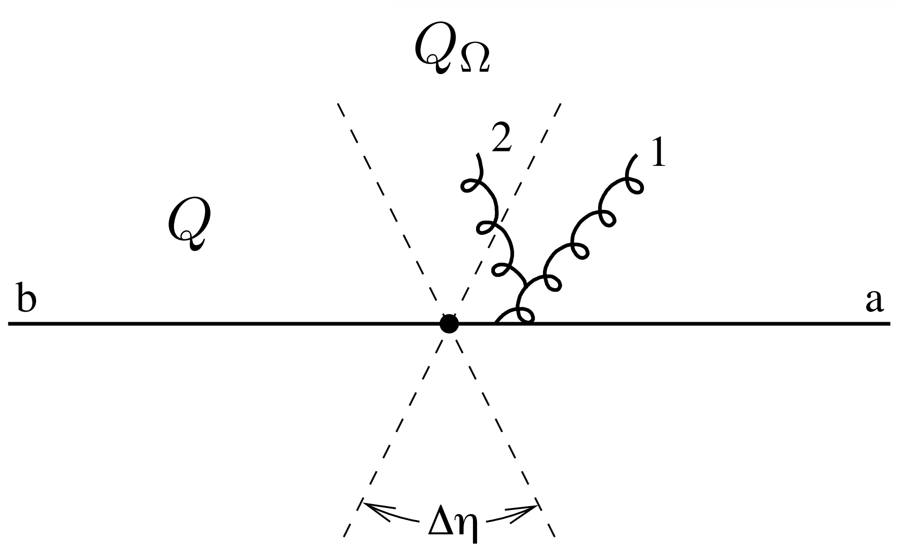
$$\begin{aligned}
B(\beta,\delta) = C_F^2 \Bigg[& \left(32\ln^2\beta + 48\ln\beta + 18 - \frac{16\pi^2}{3}\right)\ln^2\delta + \left(-2 + 10\zeta_3 - 12\ln^2 2 + 4\ln 2\right)\ln\beta \\
& + \left((8 - 48\ln 2)\ln\beta + \frac{9}{2} + 2\pi^2 - 24\zeta_3 - 36\ln 2\right)\ln\delta + c_2^F \Bigg] \\
+ C_F C_A \Bigg[& \left(\frac{44\ln\beta}{3} + 11\right)\ln^2\delta - \frac{2\pi^2}{3}\ln^2\beta + \left(\frac{8}{3} - \frac{31\pi^2}{18} - 4\zeta_3 - 6\ln^2 2 - 4\ln 2\right)\ln\beta \\
& + \left(\frac{44\ln^2\beta}{3} + \left(-\frac{268}{9} + \frac{4\pi^2}{3}\right)\ln\beta - \frac{57}{2} + 12\zeta_3 - 22\ln 2\right)\ln\delta + c_2^A \Bigg] \\
+ C_F T_F n_f \Bigg[& \left(-\frac{16\ln\beta}{3} - 4\right)\ln^2\delta + \left(-\frac{16}{3}\ln^2\beta + \frac{80\ln\beta}{9} + 10 + 8\ln 2\right)\ln\delta \\
& + \left(-\frac{4}{3} + \frac{4\pi^2}{9}\right)\ln\beta + c_2^f \Bigg].
\end{aligned}$$

- $\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}$ **divergences have cancelled!**

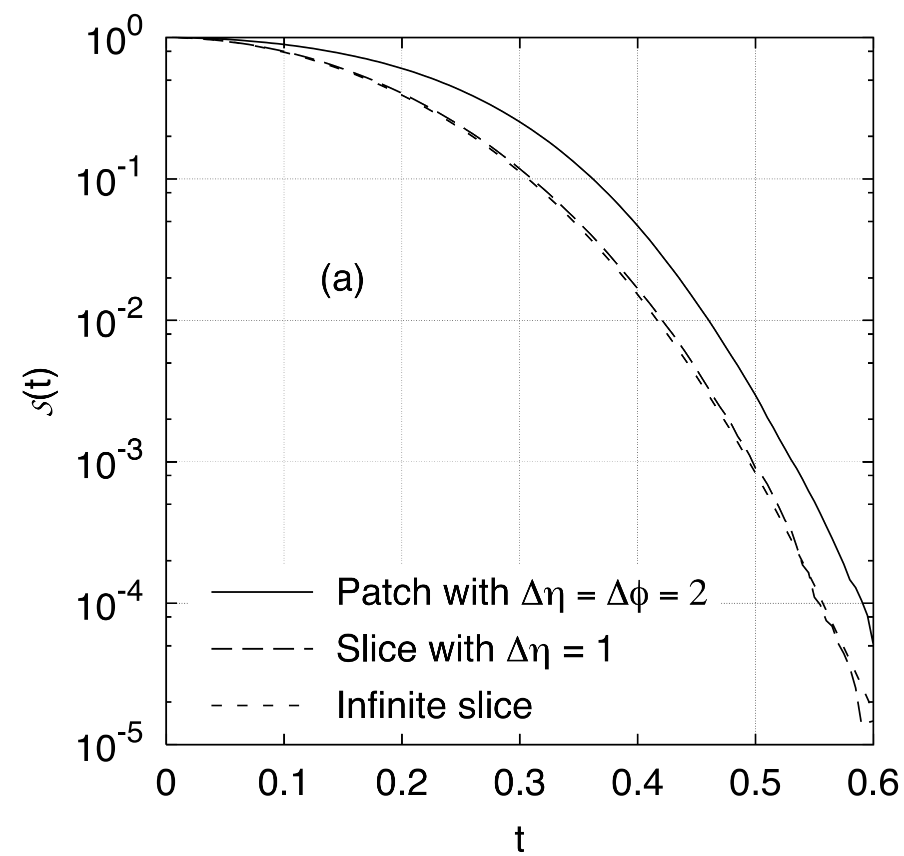
Energy flow in restricted angular regions

(Dasgupta & Salam '02)



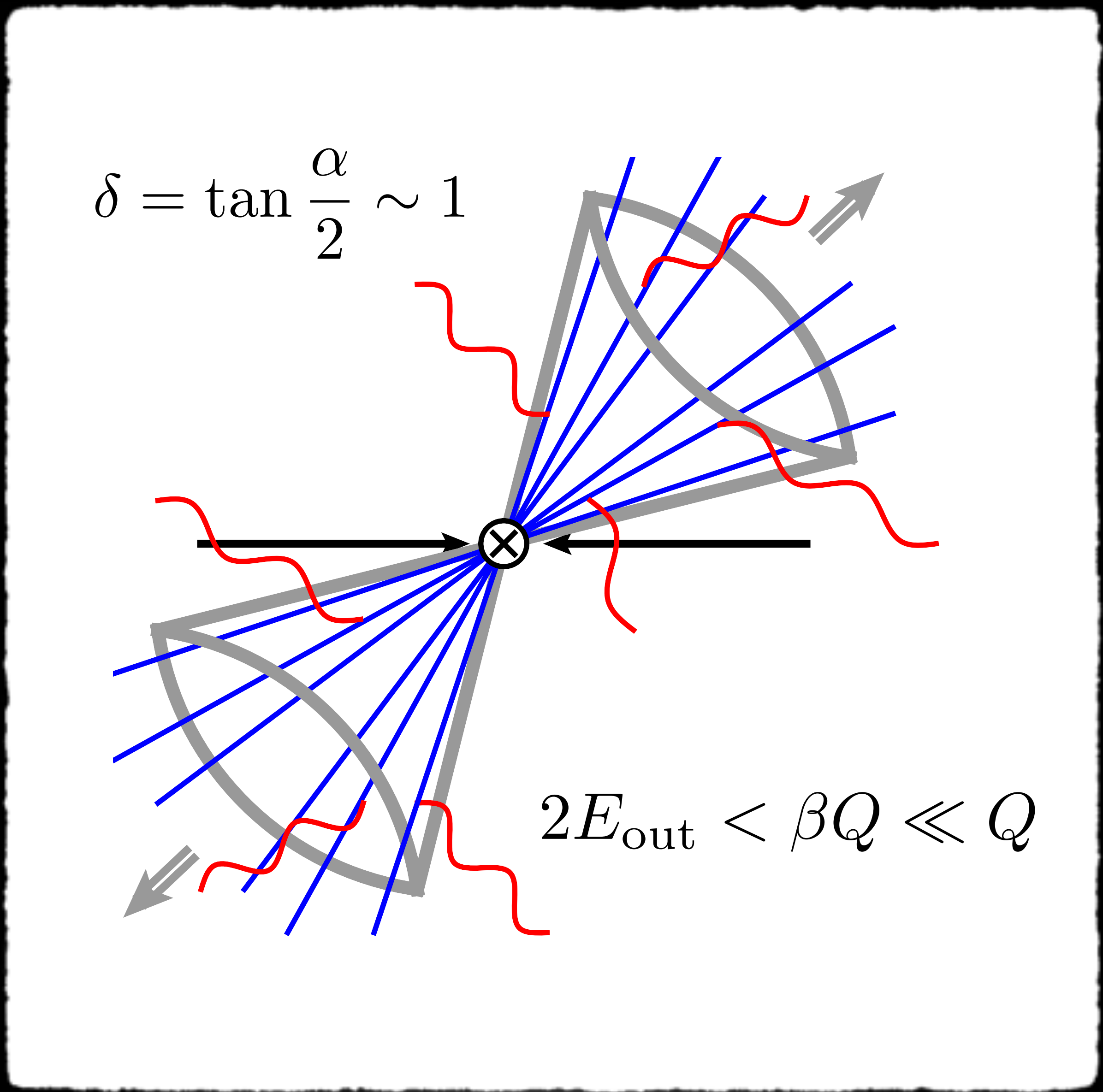


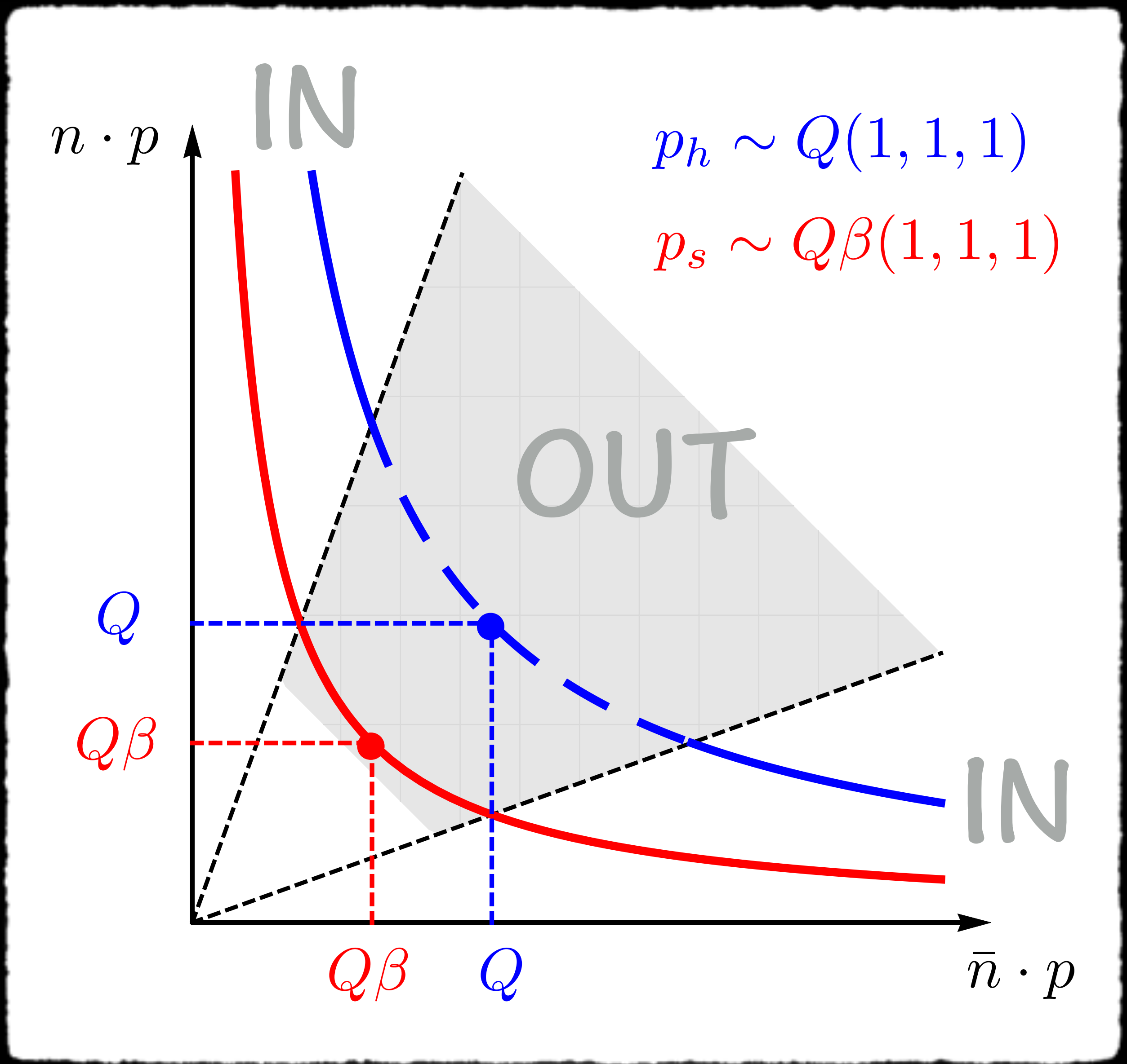
$$\mathcal{S}_2 = -4C_F C_A \left[\frac{\pi^2}{12} + (\Delta\eta)^2 - \Delta\eta \ln\left(e^{2\Delta\eta} - 1\right) - \frac{1}{2}\mathrm{Li}_2\left(e^{-2\Delta\eta}\right) - \frac{1}{2}\mathrm{Li}_2\left(1 - e^{2\Delta\eta}\right)\right]$$



EFT for NGOs with rapidity gap

(Becher, Neubert, Rothen, DYS 1605.02737)





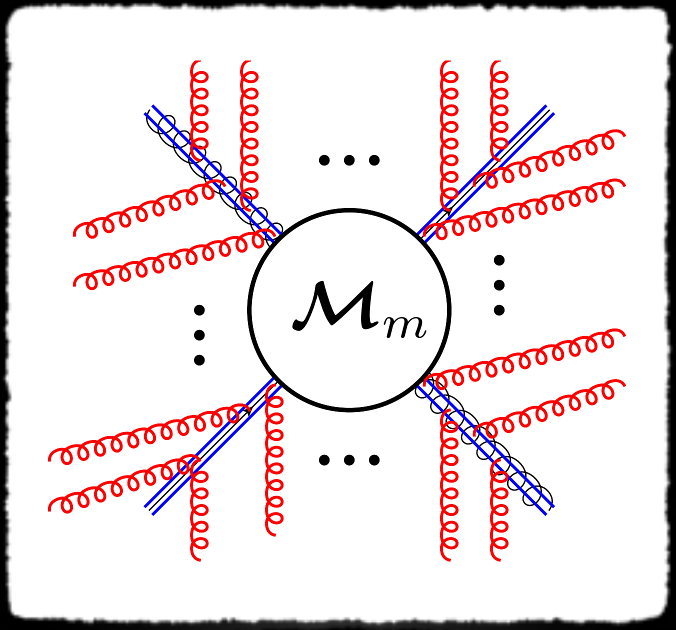
$$\Delta\eta = -2\ln\delta$$

Factorization

- Hard parton -> collinear fields $\Phi_i \in \{\chi_i, \bar{\chi}_i, \mathcal{A}^{\mu}_{i\perp}\}$ along $n_i^{\mu} = (1, \vec{n}_i)$
- performing SCET decoupling transformation: $\Phi_i = \boldsymbol{S}_i(n_i)\, \Phi_i^{(0)}$

$$\boldsymbol{S}_i(n_i) = \mathbf{P} \exp\left(i g_s \int_0^{\infty} ds\, n_i \cdot A_s^a(s n_i)\, \boldsymbol{T}_i^a \right)$$

- The operator for the emission from an amplitude with m hard partons



hard scattering amplitude with m particles (vector in color space)

$$S_1(n_1)\, S_2(n_2)\, \ldots\, S_m(n_m) |\mathcal{M}_m(\{\underline{p}\})\rangle$$

soft Wilson lines along the directions of the energetic particles (color matrices)

Factorization

- Then the cross section can be written in factorized form as,

$$\sigma(\beta,\delta) = \sum_{m=2}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\},Q,\delta) \otimes \mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) \rangle$$

- We define the squared matrix element of this operator as

$$\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \sum\!\!\!\!\!\!\int_X \langle 0|S_1^\dagger(n_1)\ \ldots\ S_m^\dagger(n_m)|X_s\rangle\langle X_s|S_1(n_1)\ \ldots\ S_m(n_m)|0\rangle\,\theta\,(Q\beta - 2E_{\text{out}})$$

- The hard functions are obtained by integrating over the energies of the hard particles, while keeping their direction fixed

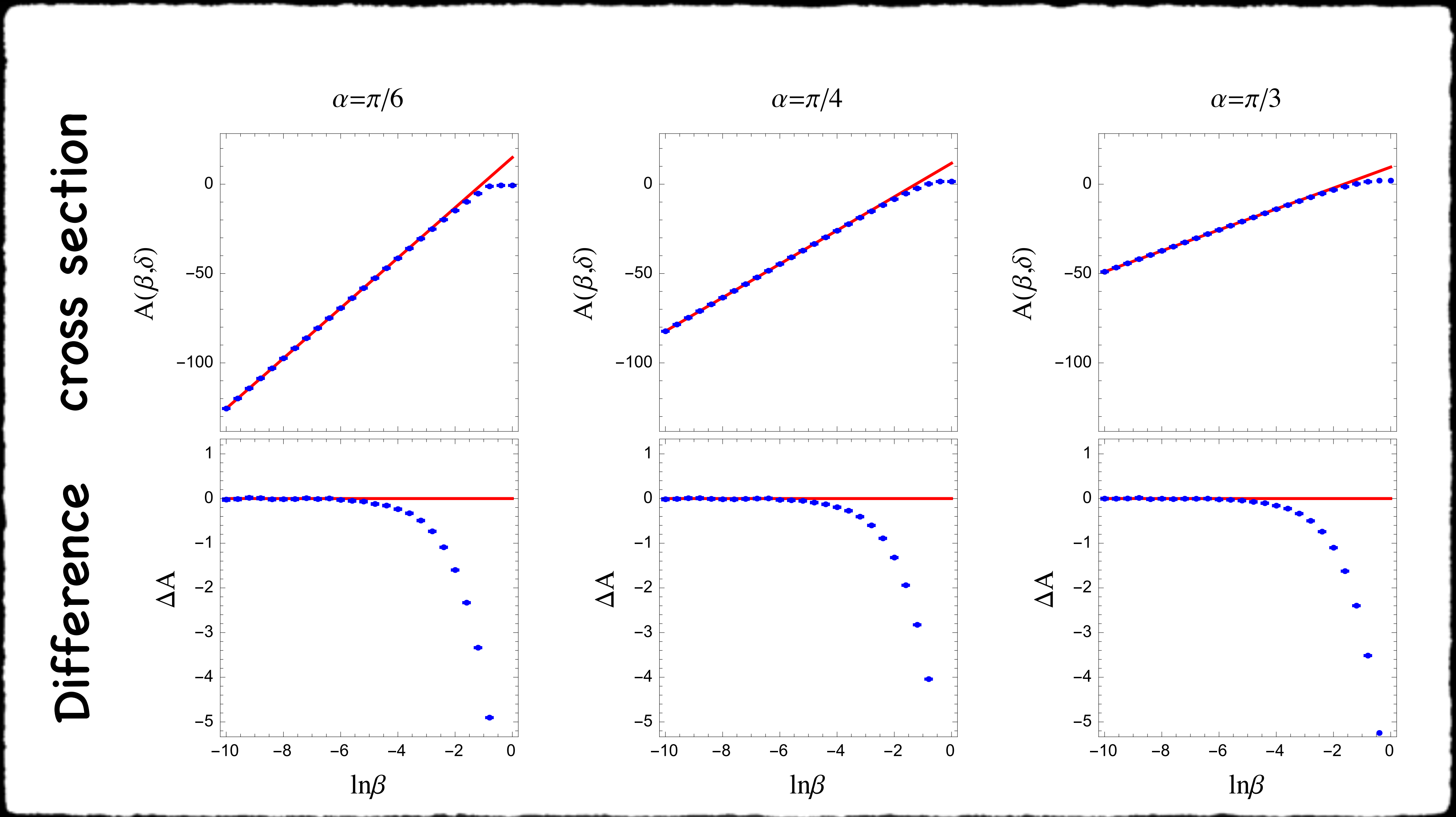
$$\mathcal{H}_m(\{\underline{n}\},Q,\delta) = \frac{1}{2Q^2}\sum_{\text{spins}}\prod_{i=1}^{m}\int\frac{d\omega_i\,\omega_i^{d-3}}{(2\pi)^{d-2}}|\mathcal{M}_m\rangle\langle\mathcal{M}_m|\delta\Big(Q-\sum_{i=1}^{m}\omega_i\Big)\delta^{d-1}(\vec{p}_{\text{tot}})\,\Theta_{\text{in}}^{n\bar{n}}(\{\underline{p}\})$$

- $\otimes$ indicates integration over the direction of the energetic partons

$$\mathcal{H}_m(\{\underline{n}\},Q,\delta)\otimes\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \prod_{i=1}^{m}\int\frac{d\Omega(n_i)}{4\pi}\,\mathcal{H}_m(\{\underline{n}\},Q,\delta)\,\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta)$$

One-loop coefficient v.s. EVENT2

$$A(\beta,\delta) = C_F\left[-8\ln\delta\ln\beta - 1 + 6\ln 2 - 6\ln\delta - 6\,\delta^2 + \left(\frac{9}{2} - 6\ln 2\right)\delta^4 - 4\,\mathrm{Li}_2(-\delta^2) + 4\,\mathrm{Li}_2(\delta^2)\right]$$



Two-loop coefficient v.s. EVENT2

$$B(\beta,\delta) = C_F^2 B_F + C_F C_A B_A + C_F T_F n_f B_f$$

$$\begin{aligned}
B_A = &\left[\frac{44}{3}\ln\delta - \frac{2\pi^2}{3} + 4\,\mathrm{Li}_2(\delta^4)\right]\ln^2\beta + \left[\frac{4}{3\,(1-\delta^4)} - \frac{16\ln\delta}{3\,(1-\delta^4)} + \frac{16\ln\delta}{3\,(1-\delta^4)^2}\right.\\
&-\frac{4}{3}\ln^3\left(1-\delta^2\right) - \frac{20}{3}\ln^3\left(1+\delta^2\right) + 32\ln\delta\ln^2\left(1-\delta^2\right) - 4\ln\left(1+\delta^2\right)\ln^2\left(1-\delta^2\right)\\
&-4\ln^2\left(1+\delta^2\right)\ln\left(1-\delta^2\right) + 64\ln\delta\ln^2\left(1+\delta^2\right) - 64\ln^2\delta\ln\left(1+\delta^2\right)\\
&+\frac{88}{3}\ln\delta\ln\left(1-\delta^2\right) - \frac{16}{3}\pi^2\ln\left(1-\delta^2\right) + 44\ln\delta\ln\left(1+\delta^2\right) + \frac{16}{3}\pi^2\ln\left(1+\delta^2\right)\\
&+\frac{44\ln^2\delta}{3} - \frac{16}{3}\pi^2\ln\delta - \frac{268\ln\delta}{9} + \frac{88\,\mathrm{Li}_2\left(\delta^4\right)}{3} - 4\,\mathrm{Li}_3\left(\delta^4\right) + 8\,\mathrm{Li}_3\left(-\frac{\delta^4}{1-\delta^4}\right)\\
&+8\ln 2\,\mathrm{Li}_2\left(\delta^4\right) - \frac{88\,\mathrm{Li}_2\left(\delta^2\right)}{3} - \frac{22}{3}\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) + \frac{22}{3}\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) + 32\,\mathrm{Li}_3\left(1-\delta^2\right)\\
&+32\,\mathrm{Li}_3\left(\frac{\delta^2}{1+\delta^2}\right) + 32\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) + 32\ln\delta\,\mathrm{Li}_2\left(\delta^2\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^2\right)\\
&+32\ln\delta\,\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\delta\,\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right)\\
&+32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) - 8\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) + 8\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) - 24\,\zeta_3\\
&\left.-\frac{2}{3} - \frac{4}{3}\pi^2\ln 2 - M_A^{[1]}(\delta)\right]\ln\beta + c_2^A(\delta),
\end{aligned}$$

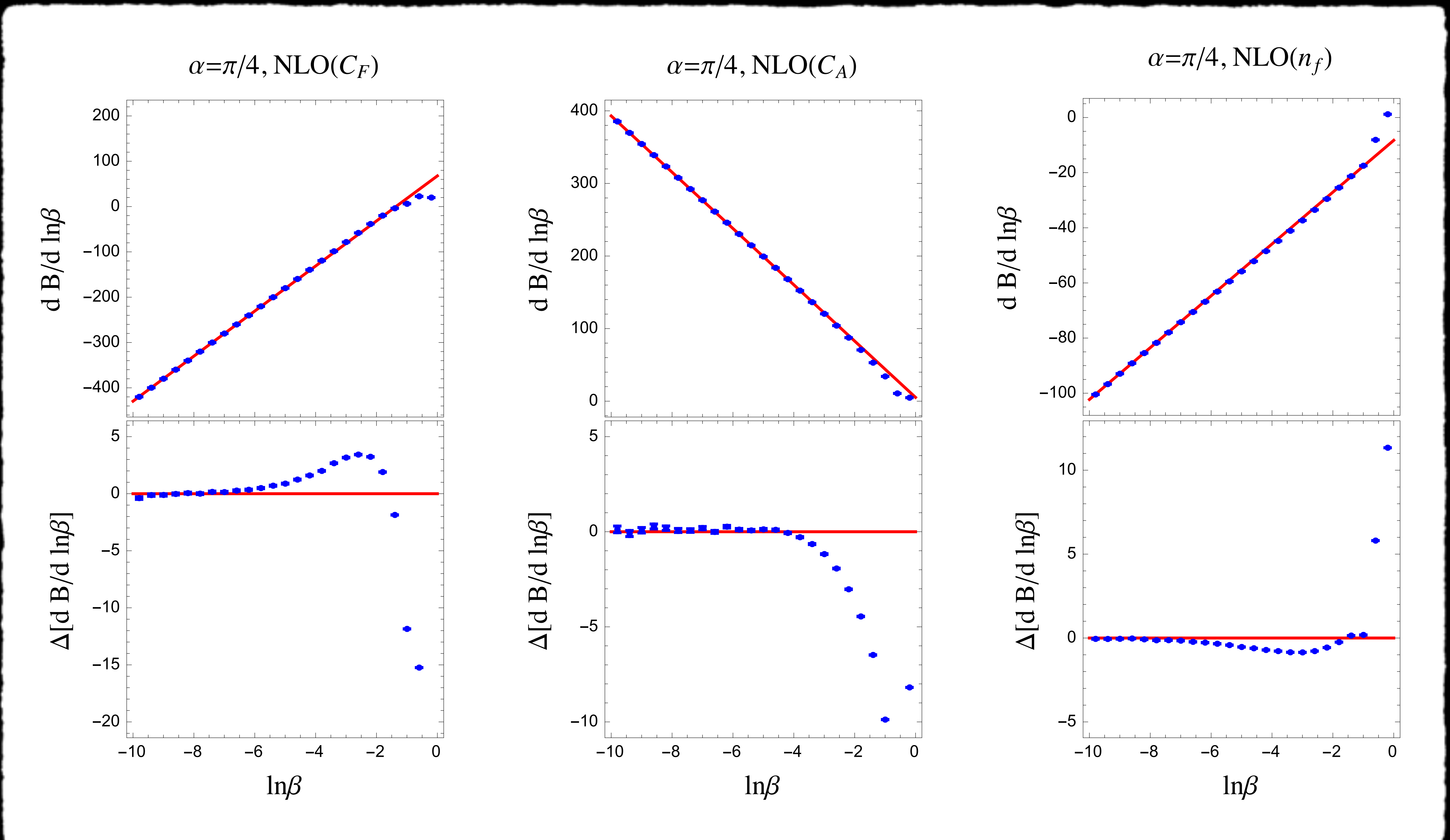
Two-loop coefficient v.s. EVENT2

Leading NGLs

$$B(\beta,\delta) = C_F^2 B_F + C_F C_A B_A + C_F T_F n_f B_f$$

$$\begin{aligned}
B_A = &\left[\frac{44}{3}\ln\delta - \frac{2\pi^2}{3} + 4\,\mathrm{Li}_2(\delta^4)\right]\ln^2\beta + \left[\frac{4}{3\,(1-\delta^4)} - \frac{16\ln\delta}{3\,(1-\delta^4)} + \frac{16\ln\delta}{3\,(1-\delta^4)^2}\right.\\
&-\frac{4}{3}\ln^3\left(1-\delta^2\right) - \frac{20}{3}\ln^3\left(1+\delta^2\right) + 32\ln\delta\ln^2\left(1-\delta^2\right) - 4\ln\left(1+\delta^2\right)\ln^2\left(1-\delta^2\right)\\
&-4\ln^2\left(1+\delta^2\right)\ln\left(1-\delta^2\right) + 64\ln\delta\ln^2\left(1+\delta^2\right) - 64\ln^2\delta\ln\left(1+\delta^2\right)\\
&+\frac{88}{3}\ln\delta\ln\left(1-\delta^2\right) - \frac{16}{3}\pi^2\ln\left(1-\delta^2\right) + 44\ln\delta\ln\left(1+\delta^2\right) + \frac{16}{3}\pi^2\ln\left(1+\delta^2\right)\\
&+\frac{44\ln^2\delta}{3} - \frac{16}{3}\pi^2\ln\delta - \frac{268\ln\delta}{9} + \frac{88\,\mathrm{Li}_2\left(\delta^4\right)}{3} - 4\,\mathrm{Li}_3\left(\delta^4\right) + 8\,\mathrm{Li}_3\left(-\frac{\delta^4}{1-\delta^4}\right)\\
&+8\ln 2\,\mathrm{Li}_2\left(\delta^4\right) - \frac{88\,\mathrm{Li}_2\left(\delta^2\right)}{3} - \frac{22}{3}\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) + \frac{22}{3}\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) + 32\,\mathrm{Li}_3\left(1-\delta^2\right)\\
&+32\,\mathrm{Li}_3\left(\frac{\delta^2}{1+\delta^2}\right) + 32\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) + 32\ln\delta\,\mathrm{Li}_2\left(\delta^2\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^2\right)\\
&+32\ln\delta\,\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\delta\,\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right)\\
&+32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) - 8\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) + 8\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) - 24\,\zeta_3\\
&\left.-\frac{2}{3} - \frac{4}{3}\pi^2\ln 2 - M_A^{[1]}(\delta)\right]\ln\beta + c_2^A(\delta),
\end{aligned}$$

Two-loop coefficient v.s. EVENT2



➢ **We reproduce ALL logs at two loops**

Renormalization

- We renormalise the bare hard function

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \delta, \epsilon) = \sum_{l=2}^{m} \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu)\, \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)$$

e.g. $\mathcal{H}_2(\epsilon) = \mathcal{H}_2(\mu)\boldsymbol{Z}^H_{22}(\epsilon, \mu)$

$$\mathcal{H}_3(\epsilon) = \mathcal{H}_2(\mu)\boldsymbol{Z}^H_{23}(\epsilon, \mu) + \mathcal{H}_3(\mu)\boldsymbol{Z}^H_{33}(\epsilon, \mu)$$

- The Z-factor has the form $Z^H(\{\underline{n}\}, \epsilon, \mu) \sim \begin{pmatrix} 1 & \alpha_s & \alpha_s^2 & \alpha_s^3 & \dots \\ 0 & 1 & \alpha_s & \alpha_s^2 & \dots \\ 0 & 0 & 1 & \alpha_s & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

- By consistency, matrix Z^H must render the soft function finite

$$\boldsymbol{\mathcal{S}}_l(\{\underline{n}\}, Q\beta, \delta, \mu) = \sum_{m=l}^{\infty} \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)\,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \epsilon)$$

Renormalization

- We verify that Z^H renormalises the two-loop soft function

$$\mathcal{S}_2(\mu) = Z^H_{22}\, \mathcal{S}_2(\epsilon) + Z^H_{23}\, \hat{\otimes}\, \mathcal{S}_3(\epsilon) + Z^H_{24}\, \hat{\otimes}\, 1 + \mathcal{O}(\alpha_s^3)$$

- and the general one-loop soft function

$$\frac{\alpha_s}{4\pi}\, \boldsymbol{z}^{(1)}_{m,m}(\{\underline{n}\}, Q, \delta, \epsilon, \mu) + \frac{\alpha_s}{4\pi} \int \frac{d\Omega(n_{m+1})}{4\pi}\, \boldsymbol{z}^{(1)}_{m,m+1}(\{\underline{n}, n_{m+1}\}, Q, \delta, \epsilon, \mu)$$
$$+ \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \epsilon) = \text{finite}$$

Resummation

Large logarithms in the soft function

$$\boldsymbol{\mathcal{S}}_l(\{\underline{n}\}, Q\beta, \delta, \mu_h) = \sum_{m \geq l} \boldsymbol{U}^S_{lm}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \mu_s)$$

with the formal evolution matrix

$$\boldsymbol{U}^S(\{\underline{n}\}, \delta, \mu_s, \mu_h) = \mathbf{P} \exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \, \boldsymbol{\Gamma}^H(\{\underline{n}\}, \delta, \mu) \right]$$

Therefore the resumed cross section

$$\sigma(\beta, \delta) = \sum_{l=2}^{\infty} \big\langle \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu_h) \otimes \sum_{m \geq l} \boldsymbol{U}^S_{lm}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \mu_s)$$

Resummation

At LL level,

$$\boldsymbol{\mathcal{S}}^T = (1, 1, \cdots, 1) \qquad \mathcal{H} = (\sigma_0, 0, \cdots, 0) \qquad \boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

V_m : div. of one-loop virtual correction to m-legs amplitude

R_m : div. from additional radiation

$$\sigma_{\text{LL}}(\delta, \beta) = \sigma_0 \left\langle \boldsymbol{\mathcal{S}}_2(\{n, \bar{n}\}, Q\beta, \delta, \mu_h) \right\rangle = \sigma_0 \sum_{m=2}^{\infty} \left\langle \boldsymbol{U}^S_{2m}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \mathbf{1} \right\rangle$$

The symbol $\hat{\otimes}$ indicates that one has to integrate over the additional directions present in the higher-multiplicity anomalous dimensions R_m and V_m

LL resummation

Expand RG equation order by order

$$W_{ij}^k = \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_j \cdot n_k}$$

$$\mathcal{S}_2^{(1)} = -\,(4N_c)\int_\Omega \mathbf{3}_{\rm Out}\, W_{12}^3\,,$$

$$\mathcal{S}_2^{(2)} = \frac{1}{2!}\,(4N_c)^2\int_\Omega \Big[-\,\mathbf{3}_{\rm In}\,\mathbf{4}_{\rm Out}\,\big(P_{12}^{34} - W_{12}^3\,W_{12}^4\big) + \mathbf{3}_{\rm Out}\,\mathbf{4}_{\rm Out}\,W_{12}^3\,W_{12}^4\Big]\,,$$

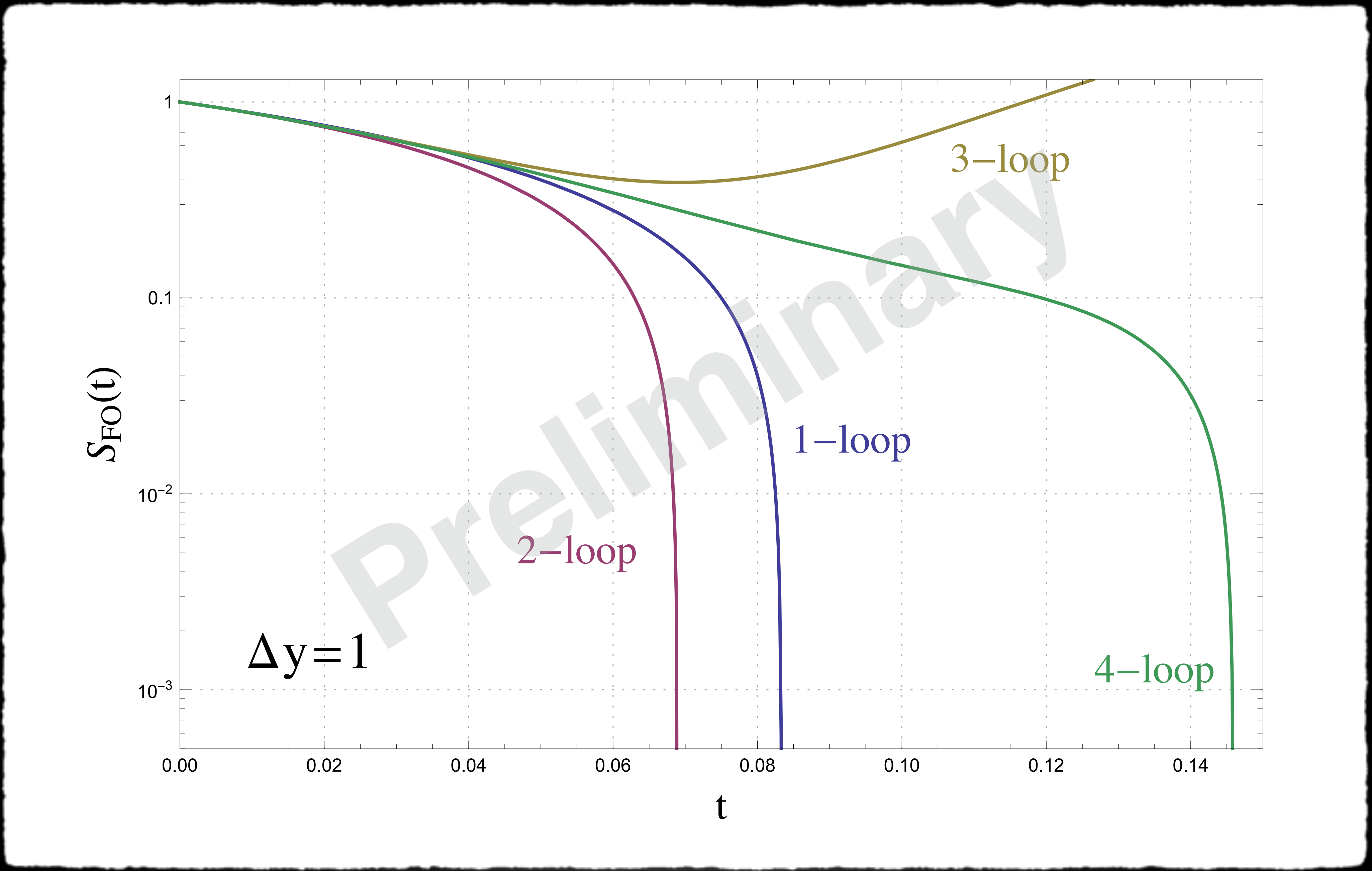
$$\begin{aligned}\mathcal{S}_2^{(3)} = \frac{1}{3!}\,(4N_c)^3\int_\Omega \Big[&\mathbf{3}_{\rm In}\,\mathbf{4}_{\rm Out}\,\mathbf{5}_{\rm Out}\,\big[P_{12}^{34}\,\big(W_{13}^5 + W_{32}^5 + W_{12}^5\big) - 2W_{12}^3\,W_{12}^4\,W_{12}^5\big] \\ &-\,\mathbf{3}_{\rm In}\,\mathbf{4}_{\rm In}\,\mathbf{5}_{\rm Out}\,W_{12}^3\,\big[\big(P_{13}^{45} - W_{13}^4\,W_{13}^5\big) + \big(P_{32}^{45} - W_{32}^4\,W_{32}^5\big) - \big(P_{12}^{45} - W_{12}^4\,W_{12}^5\big)\big] \\ &-\,\mathbf{3}_{\rm Out}\,\mathbf{4}_{\rm Out}\,\mathbf{5}_{\rm Out}\,W_{12}^3\,W_{12}^4\,W_{12}^5\Big]\end{aligned}$$

Agrees with order-by-order expansion of BMS equation

$$\partial_L G_{12}(L) = \int \frac{d\Omega_j}{4\pi} W_{12}^j \left[\Theta_{\rm in}^{n\bar{n}}(j) G_{1j}(L) G_{j2}(L) - G_{12}(L)\right]$$

Schwartz, Zhu '14

LL resummation



$$S(t) = -12t - 36.3399\, t^2 + 1186.55\, t^3 - 4768.05\, t^4$$

LL resummation

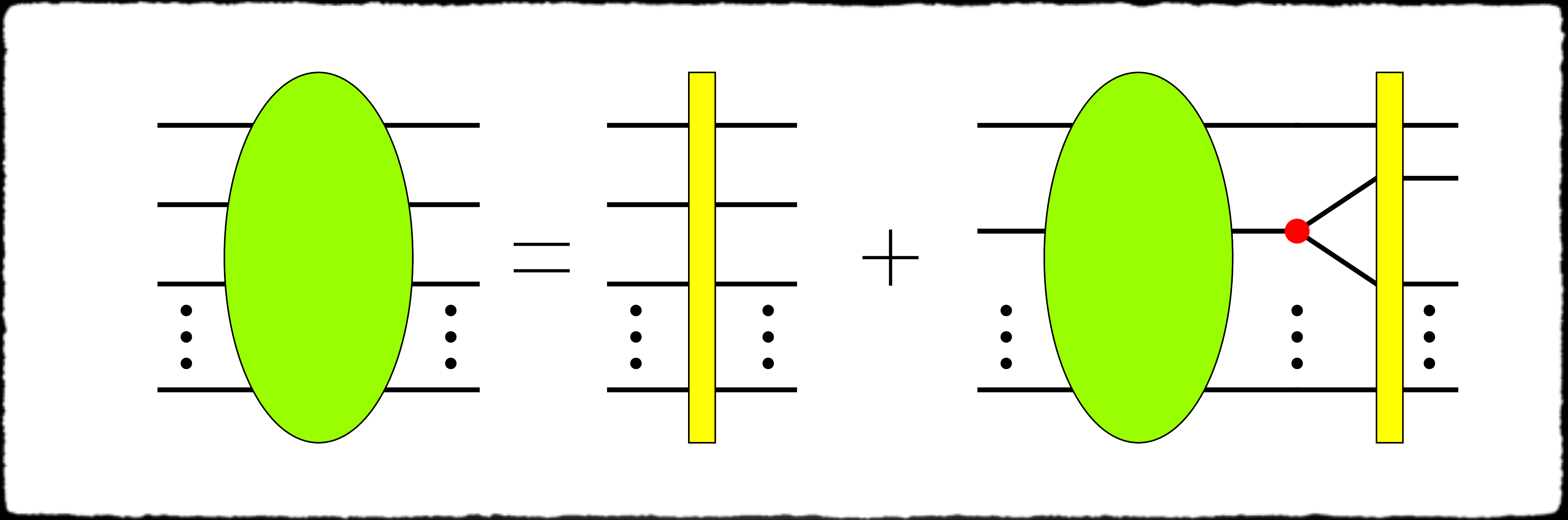
LL evolution equation: $\frac{d}{dt}\mathcal{H}_n(t) = \mathcal{H}_n(t)V_n + \mathcal{H}_{n-1}(t)R_{n-1}$

$$t = \int_{\alpha(\mu_h)}^{\alpha(\mu_s)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha}{4\pi}$$

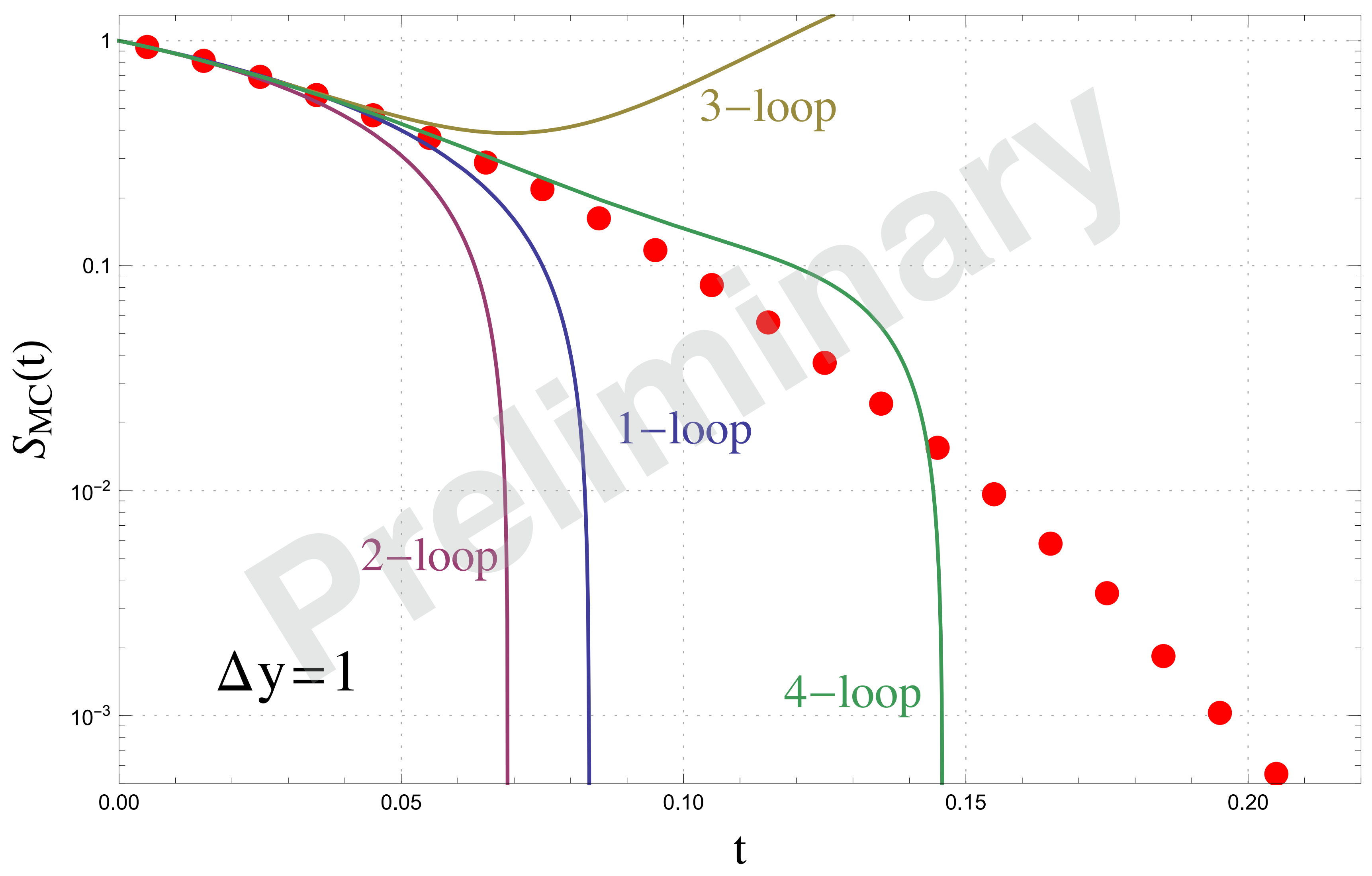
Solution:

$$\mathcal{H}_n(t) = \mathcal{H}_n(t_1)e^{(t-t_1)V_n} + \int_{t_1}^{t} dt'\mathcal{H}_{n-1}(t')R_{n-1}e^{(t-t')V_n}$$

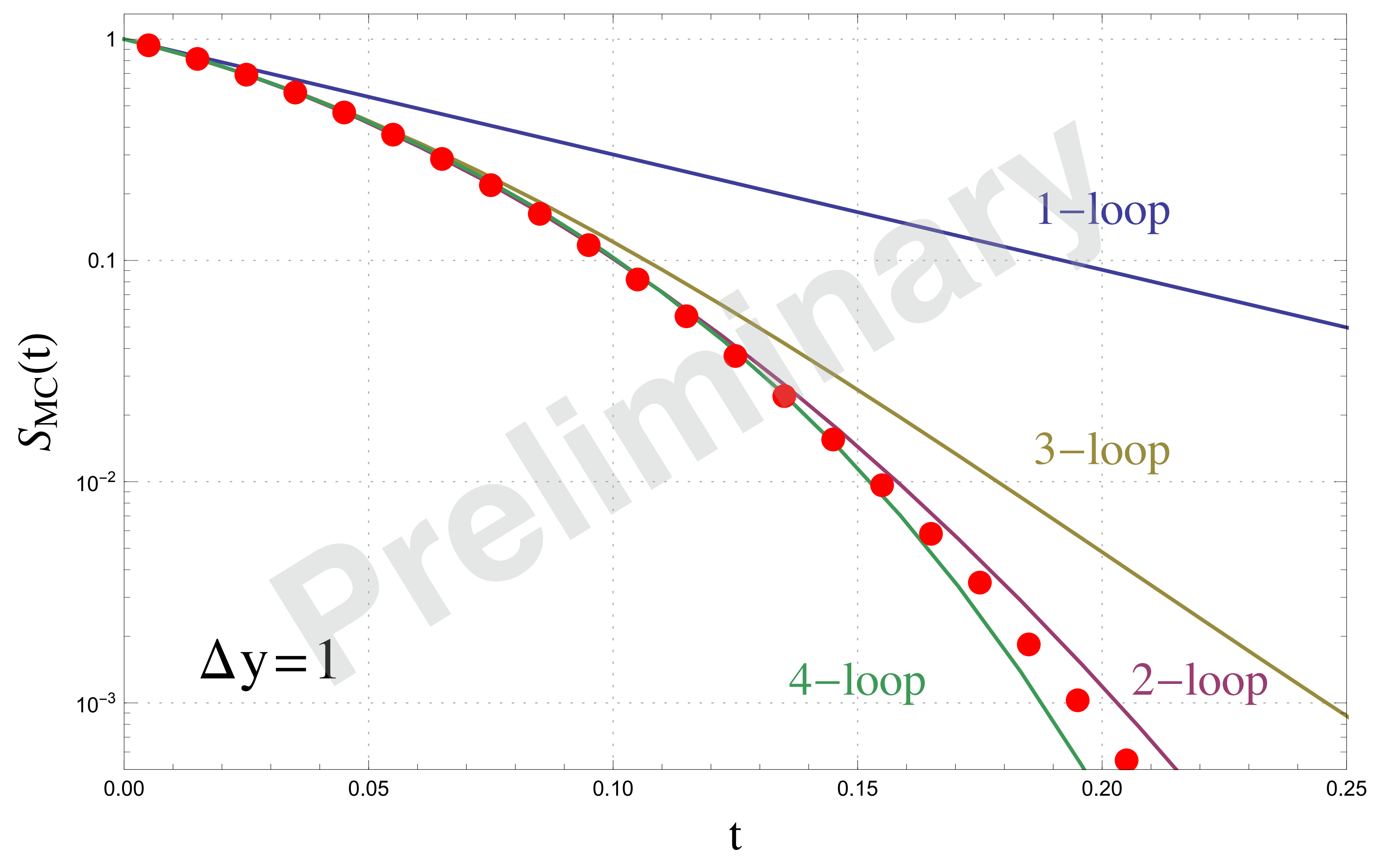
This form is exactly what is implemented in a standard parton shower MC



LL resummation



LL resummation



exponentiate the fix order soft function

Conclusion

- We have derived a factorization formula for a NG observable: cone-jet process

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

$$\widetilde{\sigma} = \sigma_0 \, H \, \widetilde{S} \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m \otimes \widetilde{\mathcal{U}}_m \right\rangle \right]^2$$

- In both case we have checked the factorization up to NNLO and reproduce full QCD results

- All the scales are separated ⟶ RG evolution can be used to resum all large logarithms

- We apply MC method to solve the associated RG equations at LL level (next step: NLL)

- Numerous possible applications: jet cross sections, jet substructure, jet veto,……

Thank you

Extra Slides

NNLO singular terms

- Up to NNLO,

$$\sigma(\beta,\delta) = \sigma_0 \langle \mathcal{H}_2 \mathcal{S}_2 + \mathcal{H}_3 \otimes \mathcal{S}_3 + \mathcal{H}_4 \otimes \mathbf{1} \rangle .$$

- the hard function $\mathcal{H}_m$ starts from $\mathcal{O}(\alpha_s^{m-2})$

$$\mathcal{H}_2 \sim 1 \qquad \mathcal{H}_3 \sim \alpha_s \qquad \mathcal{H}_4 \sim \alpha_s^2$$

- two-loop $\mathcal{S}_2$ (Kelley, Schwartz, Schabinger & Zhu '11)

$$\mathcal{S}_2(Q\beta,\epsilon) = \int_0^{Q\beta/2} d\,\lambda \int_0^{+\infty} d\,k_L \int_0^{+\infty} d\,k_R \, \boxed{S_R(k_L,k_R,\lambda,\mu)}$$

- Combining all the bare ingredients we obtain a finite result

$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta)$$

Comparison to BMS

Consider real and virtual together, all collinear divergences drop out. Leading soft divergence obtained by the soft approximation for the emitted (real or virtual) gluon

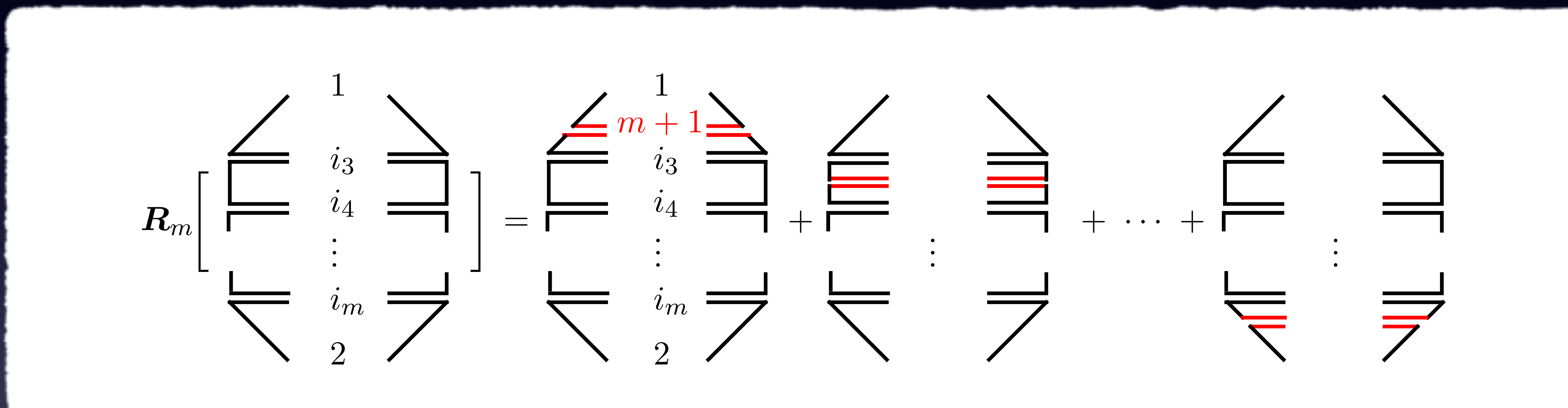
$$\boldsymbol{V}_m = \boldsymbol{\Gamma}^{(1)}_{m,m} = -4 \sum_{(ij)} \frac{1}{2} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int \frac{d\Omega(n_k)}{4\pi} W^k_{ij} \left[\Theta^{n\bar{n}}_{\text{in}}(k) + \Theta^{n\bar{n}}_{\text{out}}(k) \right] ,$$

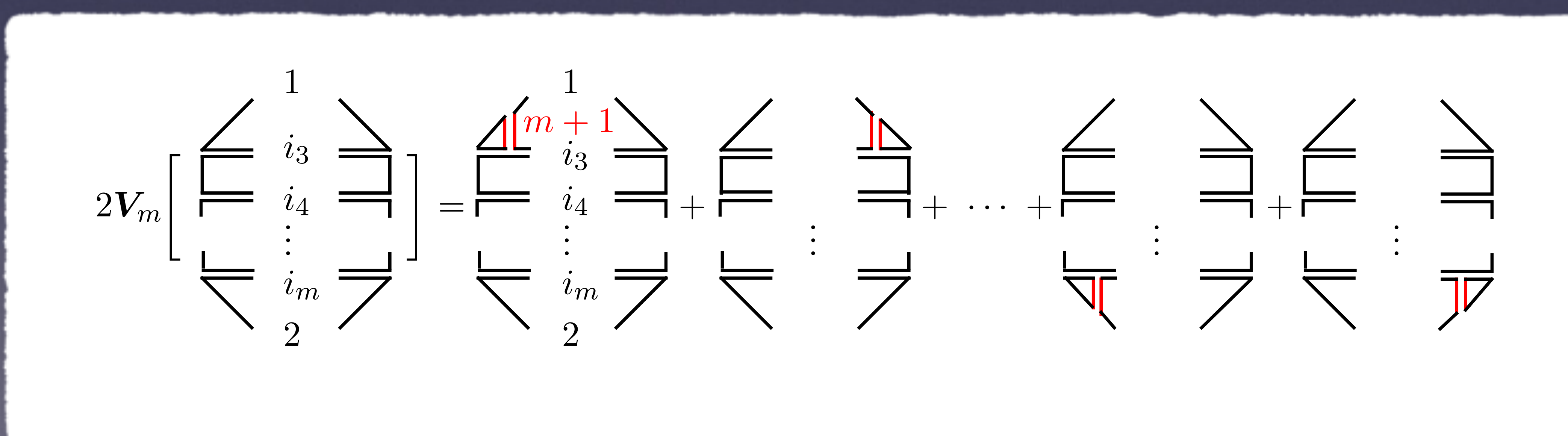
$$\boldsymbol{R}_m = \boldsymbol{\Gamma}^{(1)}_{m,m+1} = 4 \sum_{(ij)} \boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,R} \, W^k_{ij} \, \Theta^{n\bar{n}}_{\text{in}}(k)$$

Virtual has the same form as the real-emission contribution, because the principal-value part of the propagator of the emission does not contribute.

LL resummation

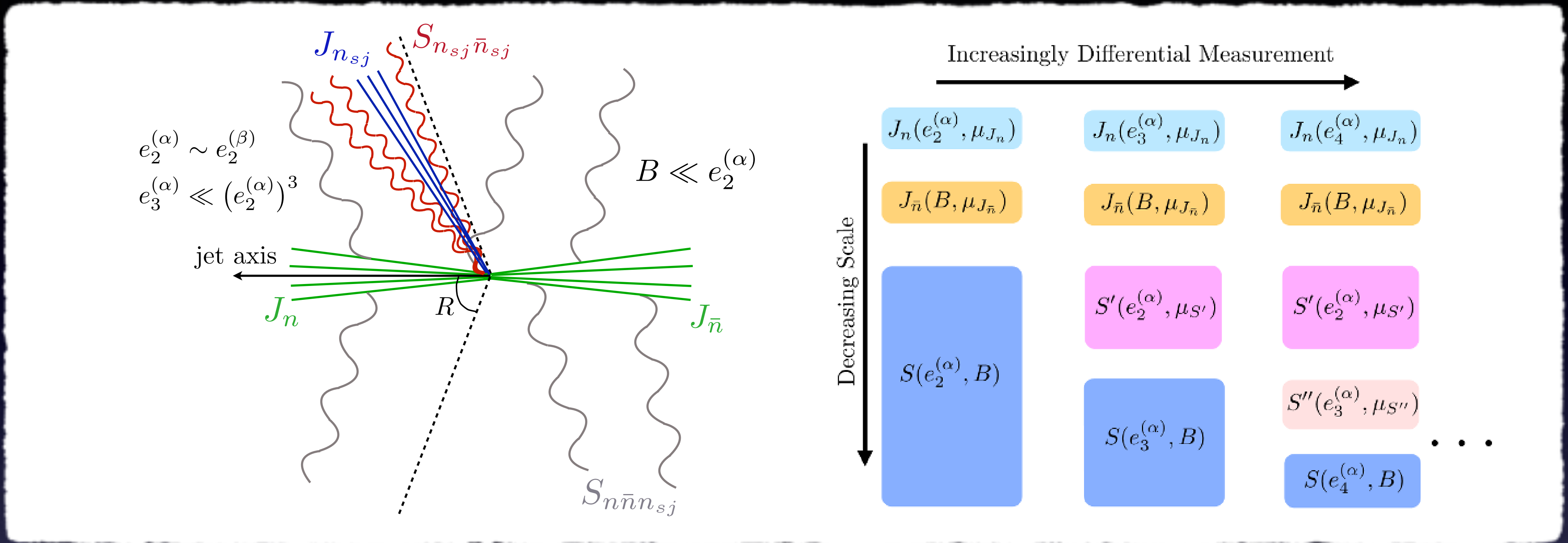
In the large N_C limit the color structure becomes trivial





Comparison with LMN approach

(Larkoski, Moult and Neill, 1501.04596)

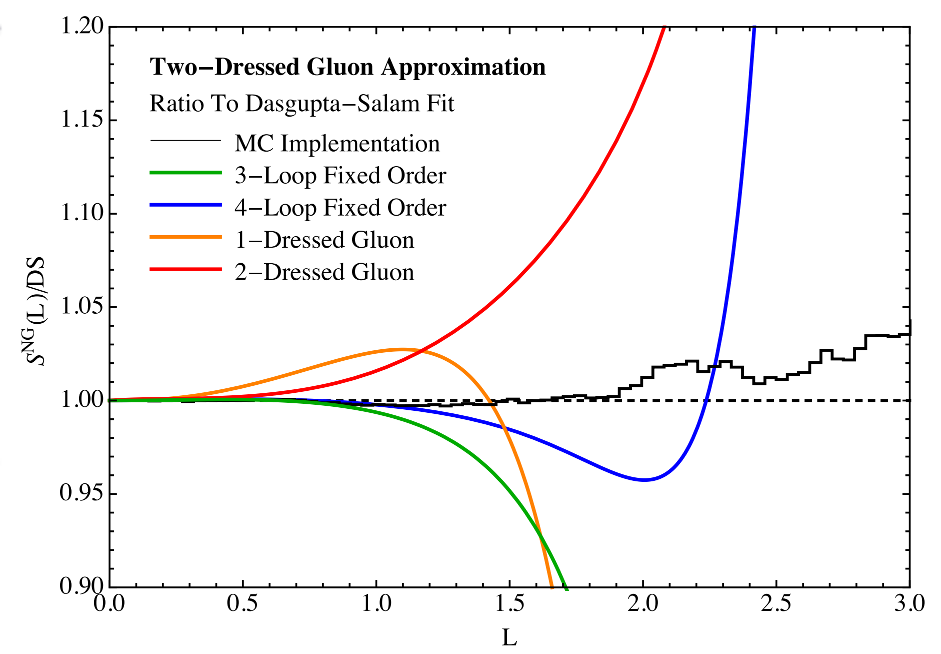


- LMN perform differential measurements to isolate regions where soft subjets give rise to NGLs. Resummation of the GLs associated with subjet observables resums part of the NGLs.
- We derive factorization theorem directly for NG observables: Resummation of NGLs with RG. Soft Wilson lines along energetic particles instead of soft subjets.
- LMN method involves a tower of effective theories with more and more d.o.f;
- We work with a single theory with only two d.o.f: hard and soft. NGLs get factorized into hard and soft logs.

Dressed-gluon expansion

- Even at LL accuracy, terms with arbitrary many subjets contribute. Not clear by which parameter higher-order terms suppressed.

	L^1	L^2	L^3	L^4
One-Dressed	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{6}$	$-\frac{\pi^4}{720}$
Two-Dressed	0	0	$-\frac{\zeta(3)}{12}$	$\frac{\pi^4}{480}(1 \pm 0.05)$
Sum	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{12}$	$\frac{\pi^4}{1440}(1 \pm 0.2)$
Exact	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{12}$	$\frac{\pi^4}{34560}$



- Our RG resummation method is standard (but the RG is complicated!). Clear which ingredients are needed for a given log accuracy.

Open questions in LMN approach

- The problems with traditional global factorization theorems become visible only at NNLO
  - Have evaluated all ingredients to this accuracy and verified that we reproduce the full QCD result. Would be worthwhile to do the same in their approach
- One expects that a factorization theorem for a jet cross section with additional measurements is at least as complicated as the factorization theorem we obtain: Multi-Wilson-line operators in LMN approach?
- We find that the operators with different multiplicities of energetic particles mix under renormalization. This effect should be present in some form in their approach.

Comparison with approach of Caron-Huot

(1501.03754)

- Caron-Huot defines **colour density matrix:**

$$\sigma[U] = \sum_n \int d\Pi_n \, [A_n^{a_1 \cdots a_n}(\{p_i\})]^* \, U^{a_1 b_1}(\theta_1) \cdots U^{a_n b_n}(\theta_n) \, \left[A_n^{b_1 \cdots b_n}(\{p_i\})\right]$$

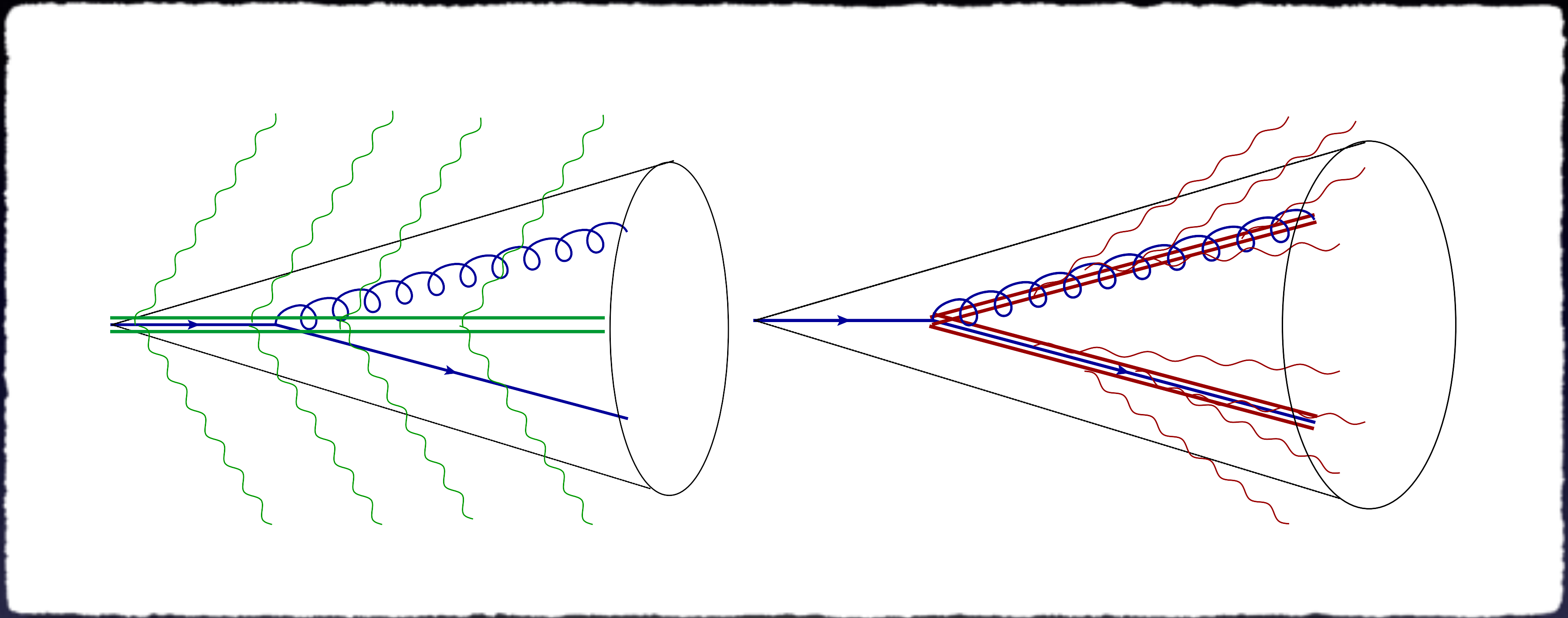
- Here unitary matrices $U(\theta)$ are used to track the contributions from different particle multiplicities.

$$\left[\mu \frac{d}{d\mu} + \beta \frac{d}{d\alpha_s}\right] \sigma^{\text{ren}}[U;\mu] = K(U, \delta/\delta U, \alpha_s(\mu), \epsilon) \sigma^{\text{ren}}[U;\mu]$$

- The one-loop expression for K are in one-to-one correspondence to our anomalous dimensions, and the LL resumed results are the same as ours.

- Beyond LL accuracy, the relation is less immediate. Caron-Huot doesn't distinguish hard and soft partons but multiplies every parton by a matrix $U(\theta)$, and also doesn't include the Wilson line structure which is a important feature of our formula.

Coft factorization



For cone-jet processes with narrow cones, small angle soft radiation became relevant

- collinear and soft ("coft")
- resolves individual collinear partons: operators with multiple Wilson lines

$$1 - 12t - 36.3399\ t^2 + 1186.55\ t^3 - 4768.05\ t^4$$

Method of region expansion

To isolate the different contributions, one expand the amplitudes as well as the phase space constrains in each momentum region.

- Generic soft mode has O(1) angle: after expansion, it is always outside the jet

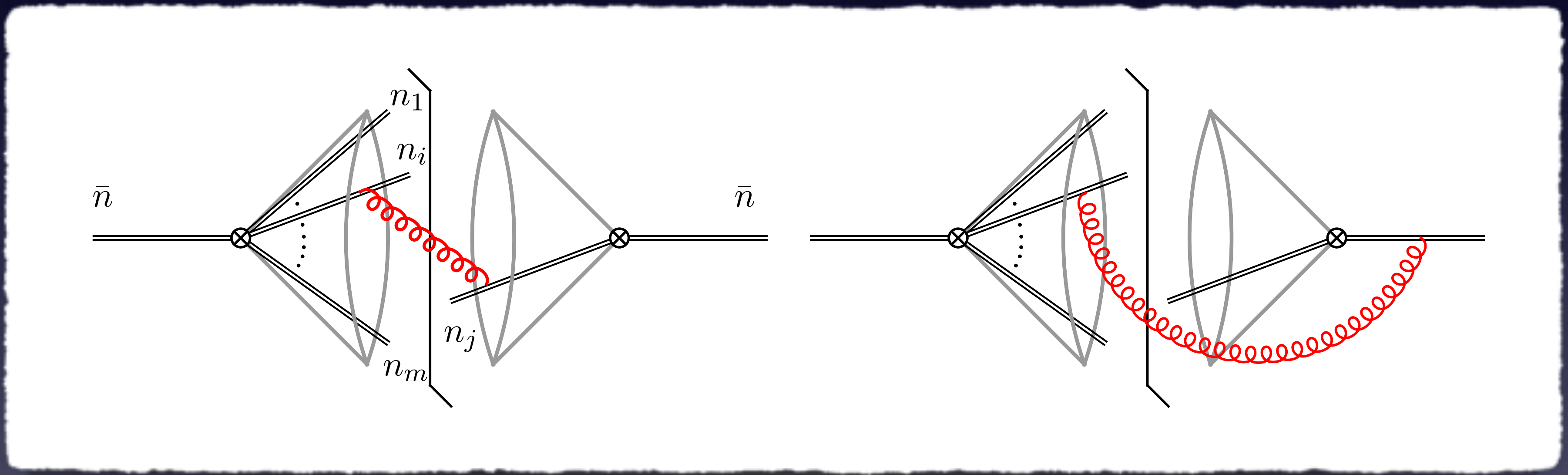
- Collinear mode has large energy. Can never go outside the jet

- Coft mode can be inside or outside, but its contribution to momentum inside the jet is negligible.

Expansion is performed on the integrand level: the full result is obtained after combining the contributions from the different regions.

One-loop renormalization for the narrow-angle jet process

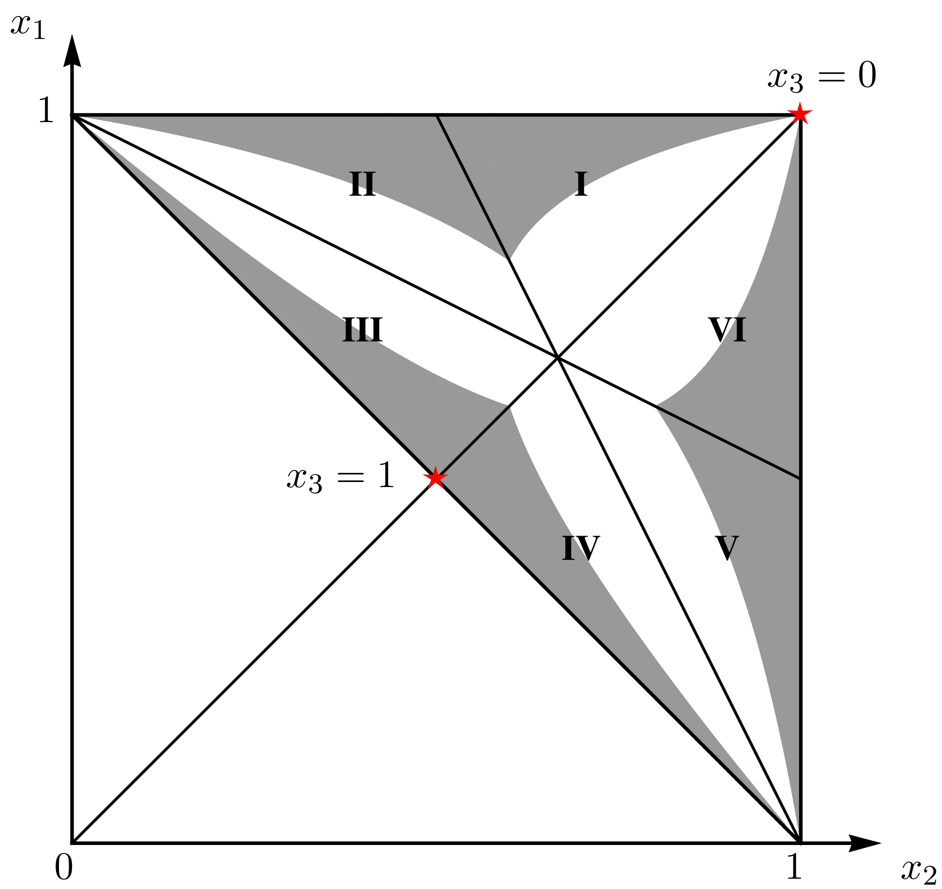
$$\frac{1}{2}\mathcal{H}^{(1)} \cdot \mathbf{1} + \frac{1}{2}\widetilde{\mathcal{S}}^{(1)} \cdot \mathbf{1} + \boldsymbol{z}^{(1)}_{m,m} + \boldsymbol{z}^{(1)}_{m,m+1} + \widetilde{\boldsymbol{\mathcal{U}}}^{(1)}_m = \text{fin.}$$



$$\widetilde{\mathcal{U}}^{(1)}_m(\{\underline{n}\},\epsilon) = -\frac{1}{\epsilon}\sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left[\ln\left(1-\hat{\theta}_i^2\right) + \ln\left(1-\hat{\theta}_j^2\right) - \ln\left(1 - 2\cos\phi_j \hat{\theta}_i \hat{\theta}_j + \hat{\theta}_i^2 \hat{\theta}_j^2\right)\right]$$

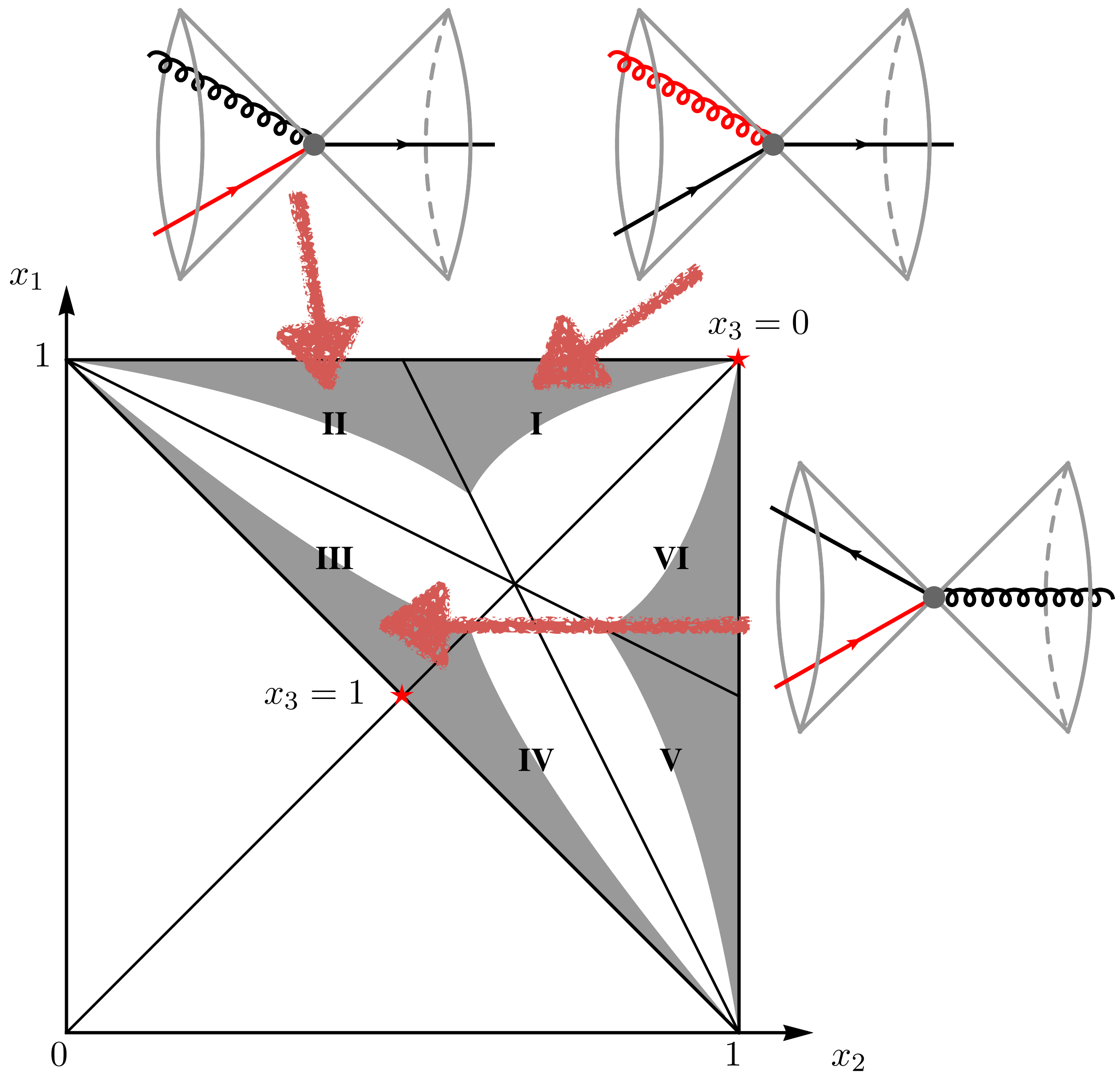
$$-\frac{2}{\epsilon}\sum_{i=1}^{l} \boldsymbol{T}_0 \cdot \boldsymbol{T}_i \ln\left(1-\hat{\theta}_i^2\right) + \boldsymbol{T}_0 \cdot \boldsymbol{T}_0 \left(-\frac{2}{\epsilon^2} + \frac{4\, L_{Q\tau\delta}}{\epsilon}\right)$$

Hard Function $\mathcal{H}_3$



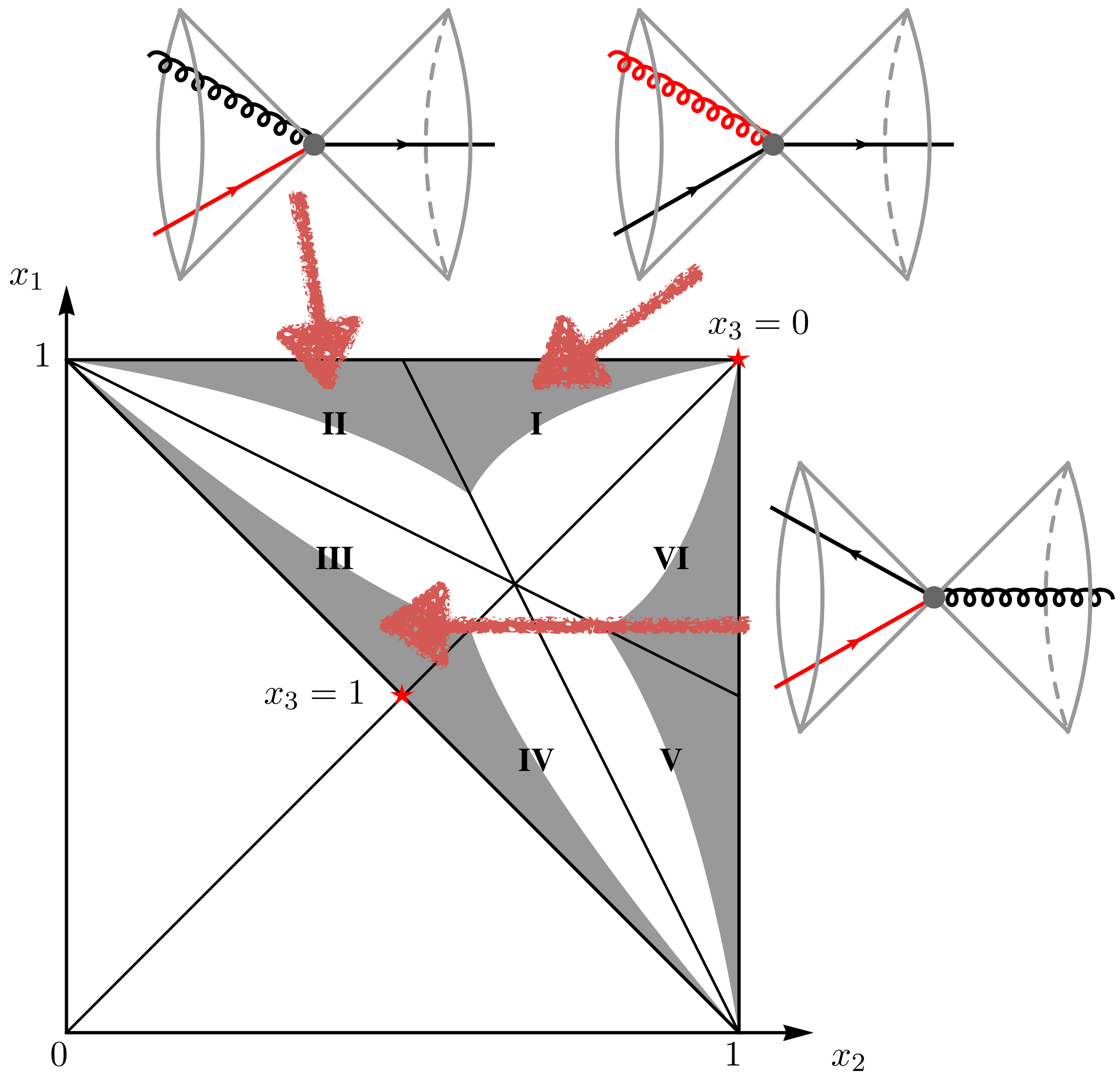
$x_i = 2E_i/Q$

Hard Function $\mathcal{H}_3$



$x_i = 2E_i/Q$

Hard Function $\mathcal{H}_3$



$x_i = 2E_i/Q$

region I:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} u^{-1-2\epsilon}\, v^{-1-\epsilon}\, h_3^{\mathbf{I}}(u, v, \delta, \epsilon)$$

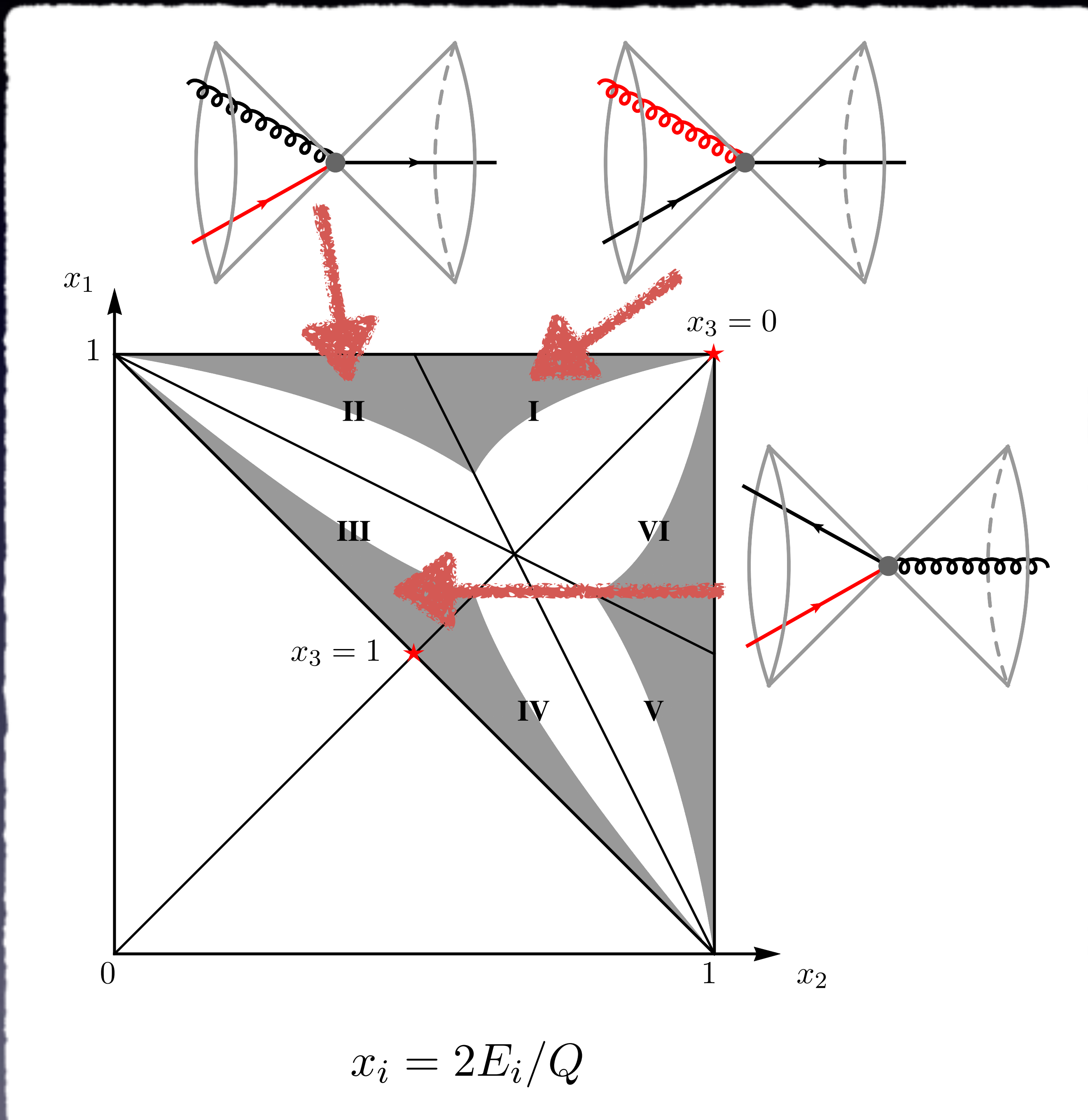
region II:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} v^{-1-\epsilon}\, h_3^{\mathbf{II}}(u, v, \delta, \epsilon)$$

region III:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} h_3^{\mathbf{III}}(u, v, \delta, \epsilon)$$

Hard Function $\mathcal{H}_3$



region I:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} u^{-1-2\epsilon} v^{-1-\epsilon} h_3^{\mathbf{I}}(u, v, \delta, \epsilon)$$

region II:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} v^{-1-\epsilon} h_3^{\mathbf{II}}(u, v, \delta, \epsilon)$$

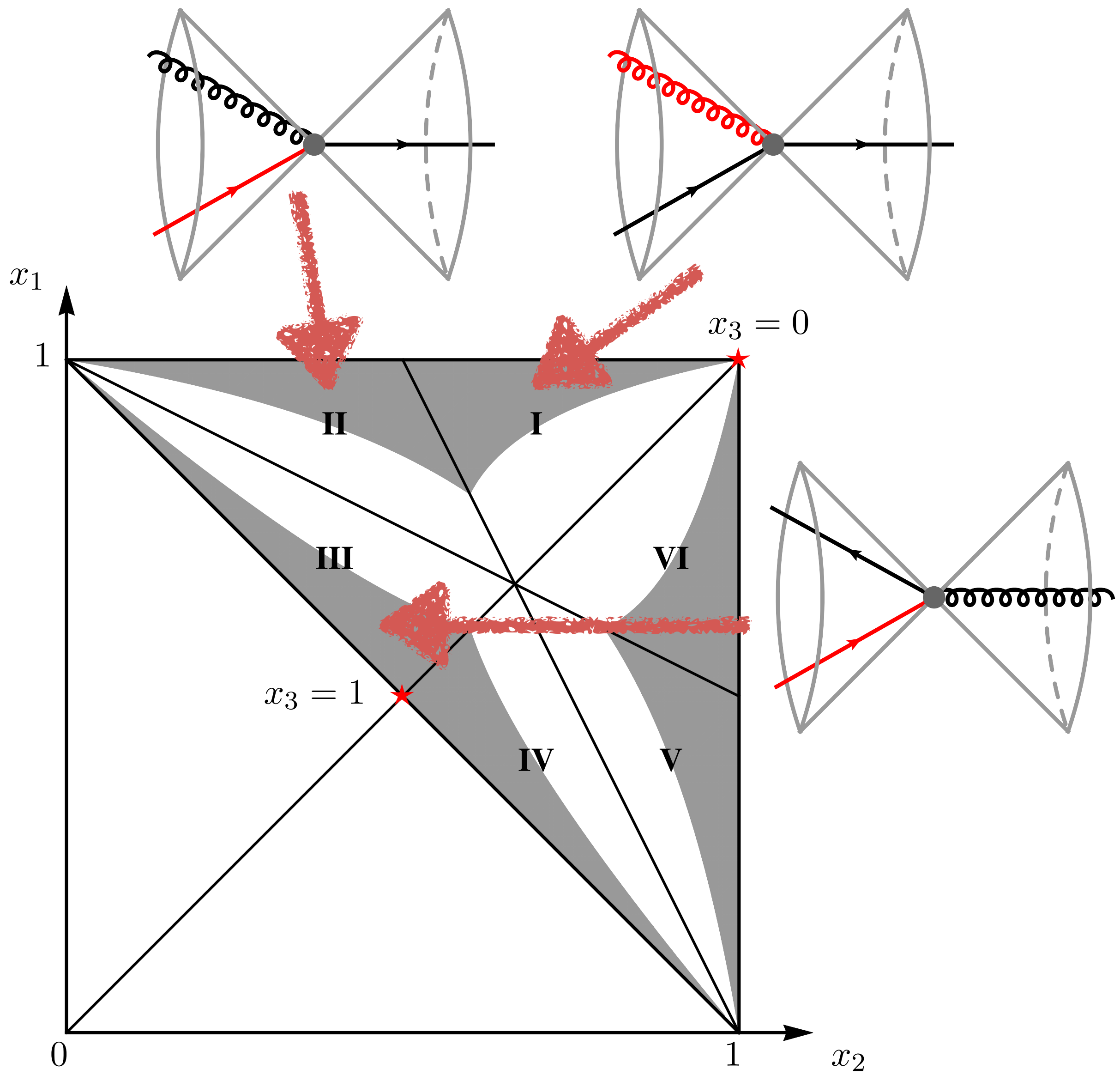
region III:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} h_3^{\mathbf{III}}(u, v, \delta, \epsilon)$$

$$\langle \mathcal{H}_3^{(1)} \otimes \mathbf{1} \rangle$$

NLO

Hard Function $\mathcal{H}_3$



$x_i = 2E_i/Q$

region I:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} u^{-1-2\epsilon}\, v^{-1-\epsilon}\, h_3^{\mathbf{I}}(u, v, \delta, \epsilon)$$

region II:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} v^{-1-\epsilon}\, h_3^{\mathbf{II}}(u, v, \delta, \epsilon)$$

region III:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} h_3^{\mathbf{III}}(u, v, \delta, \epsilon)$$

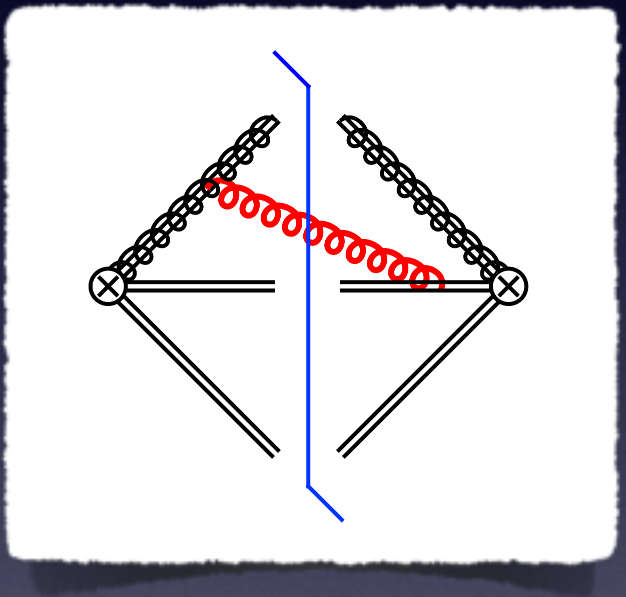
$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \mathbf{1} \rangle$ — NLO

$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \boldsymbol{\mathcal{S}}_3^{(1)} \rangle$ — NNLO

Soft Function $\mathcal{S}_3$

$$\boldsymbol{\mathcal{S}}_3^{(1)}\left(\{\underline{n}\},\epsilon\right) = \frac{4}{\epsilon} S_\epsilon \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_k \cdot n_j} \Theta_{\text{out}}(n_k)$$

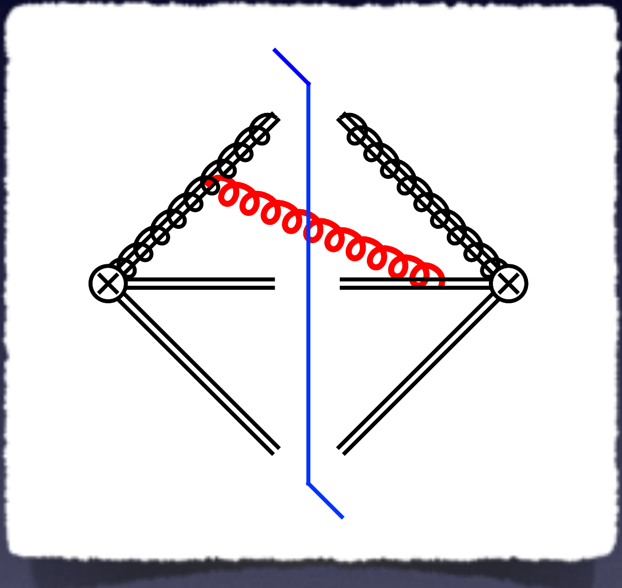
$$(i,j) = (1,2), (1,3), (2,3)$$



Soft Function $\mathcal{S}_3$

$$\boldsymbol{\mathcal{S}}_3^{(1)}(\{\underline{n}\},\epsilon) = \frac{4}{\epsilon} S_\epsilon \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_k \cdot n_j} \Theta_{\text{out}}(n_k)$$

$(i,j) = (1,2), (1,3), (2,3)$



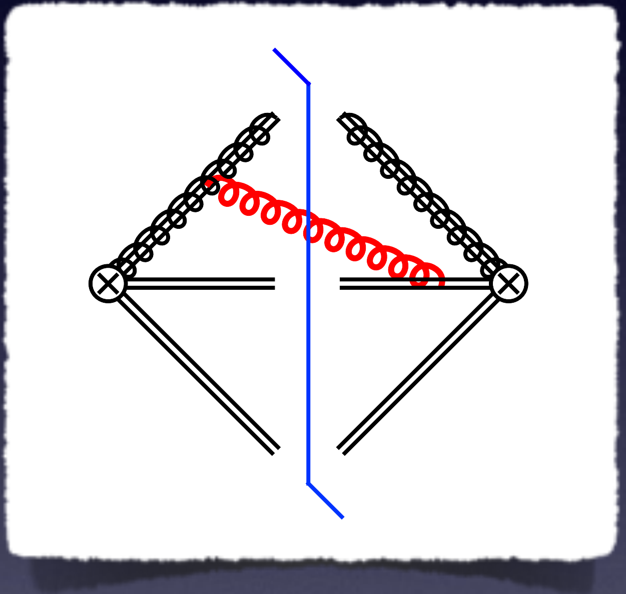
$$\left\langle \boldsymbol{\mathcal{S}}_3^{(1)} \right\rangle = \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \left[C_F \, s_3^F(u,v,\delta,\epsilon) + C_A \, s_3^A(u,v,\delta,\epsilon) \right]$$

$$s_3^A = \frac{1}{\epsilon} A_{\text{I}}^{[-1]}(u,v,\delta) + A_{\text{I}}^{[0]}(u,v,\delta) + \epsilon \, A_{\text{I}}^{[1]}(u,v,\delta)$$

Soft Function $\mathcal{S}_3$

$$\boldsymbol{\mathcal{S}}_3^{(1)}(\{\underline{n}\},\epsilon) = \frac{4}{\epsilon} S_\epsilon \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_k \cdot n_j} \Theta_{\text{out}}(n_k)$$

$$(i,j) = (1,2), (1,3), (2,3)$$



$$\left\langle \boldsymbol{\mathcal{S}}_3^{(1)} \right\rangle = \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \left[C_F \, s_3^F(u,v,\delta,\epsilon) + C_A \, s_3^A(u,v,\delta,\epsilon)\right]$$

$$s_3^A = \frac{1}{\epsilon} A_{\text{I}}^{[-1]}(u,v,\delta) + A_{\text{I}}^{[0]}(u,v,\delta) + \epsilon \, A_{\text{I}}^{[1]}(u,v,\delta)$$

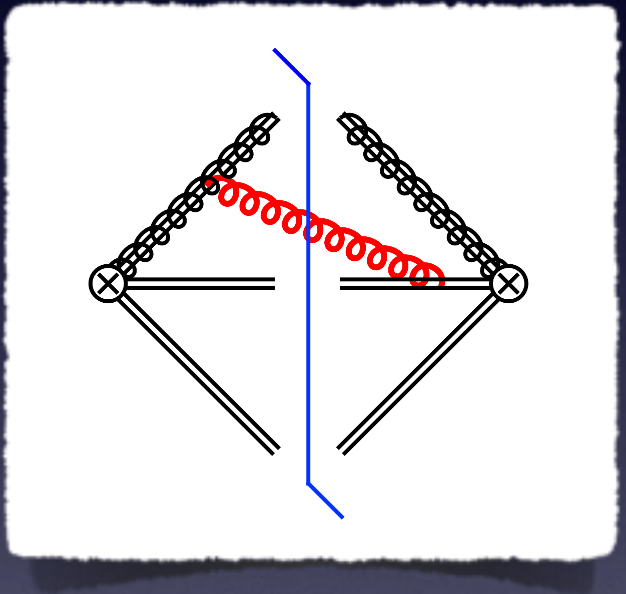
$$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \boldsymbol{\mathcal{S}}_3^{(1)} \rangle = C_\epsilon \left(\frac{\mu}{Q}\right)^{2\epsilon} \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \delta^{-2\epsilon} \left[C_F^2 \, M_F(\delta,\epsilon) + C_F \, C_A \, M_A(\delta,\epsilon)\right]$$

$$M_A(\delta,\epsilon) = \frac{1}{\epsilon^2}\left[-8\,\text{Li}_2(\delta^4) + \frac{4\pi^2}{3}\right] + \frac{2M_A^{[1]}(\delta)}{\epsilon}$$

Soft Function $\mathcal{S}_3$

$$\boldsymbol{\mathcal{S}}_3^{(1)}(\{\underline{n}\},\epsilon) = \frac{4}{\epsilon} S_\epsilon \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_k \cdot n_j} \Theta_{\text{out}}(n_k)$$

$$(i,j) = (1,2), (1,3), (2,3)$$



$$\left\langle \boldsymbol{\mathcal{S}}_3^{(1)} \right\rangle = \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \left[C_F \, s_3^F(u,v,\delta,\epsilon) + C_A \, s_3^A(u,v,\delta,\epsilon) \right]$$

$$s_3^A = \frac{1}{\epsilon} A_{\text{I}}^{[-1]}(u,v,\delta) + A_{\text{I}}^{[0]}(u,v,\delta) + \epsilon \, A_{\text{I}}^{[1]}(u,v,\delta)$$

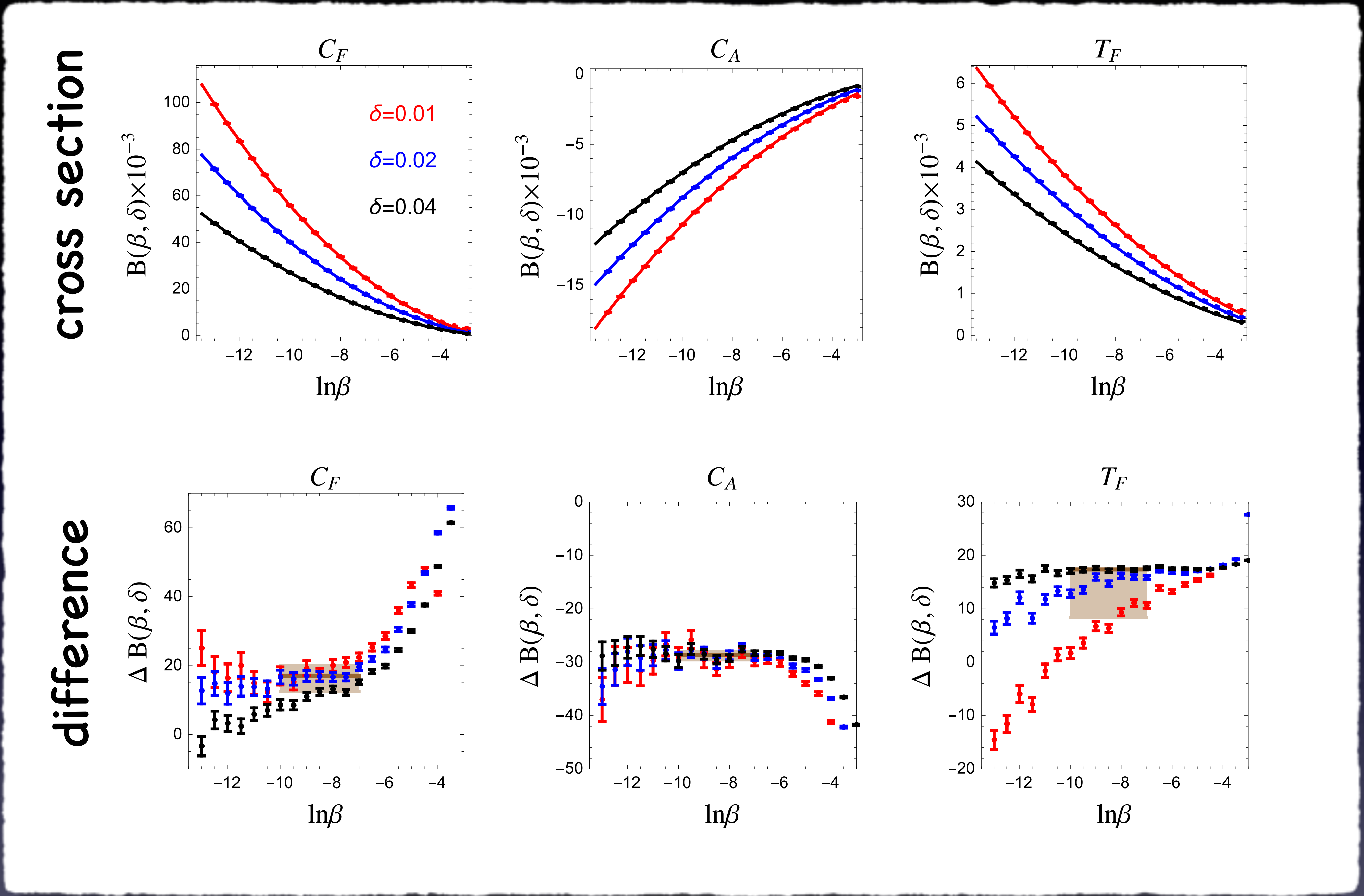
$$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \boldsymbol{\mathcal{S}}_3^{(1)} \rangle = C_\epsilon \left(\frac{\mu}{Q}\right)^{2\epsilon} \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \delta^{-2\epsilon} \left[C_F^2 \, M_F(\delta,\epsilon) + C_F \, C_A \, M_A(\delta,\epsilon) \right]$$

$$M_A(\delta,\epsilon) = \frac{1}{\epsilon^2}\left[-8\,\text{Li}_2(\delta^4) - \frac{4\pi^2}{3} \right] + \frac{2M_A^{[1]}(\delta)}{\epsilon}$$

NGLs !!!

Coft Scale

We emphasize that the coft modes have very low virtuality $p_t^2 = \Lambda_t^2 = (Q\beta\delta)^2$, much lower than the virtuality of the collinear and soft modes. The presence of this low physical scale might have important implications for the relevance of non-perturbative effects. These are suppressed by the ratio $\Lambda_{\rm QCD}/\Lambda_t$, where $\Lambda_{\rm QCD} \sim 0.5\,{\rm GeV}$ is a scale associated with strong QCD dynamics. Non-perturbative corrections to jet processes can thus be much larger than the naive expectation $\Lambda_{\rm QCD}/Q$. For example, for a jet opening angle $\alpha = 10^\circ$ ($\delta \approx 0.09$) and 5% of the collision energy outside the jets ($\beta = 0.1$), one obtains $\Lambda_t \approx 1\,{\rm GeV}$ for $Q = 100\,{\rm GeV}$. It would be interesting to explore phenomenological consequences of this low-scale physics.



Data point from EVENT2, solid lines are our prediction. Difference yields unknown constants

$$c_2^F = 17.1^{+3.0}_{-4.7}\,, \qquad c_2^A = -28.7^{+0.7}_{-1.0}\,, \qquad c_2^f = 17.3^{+0.3}_{-9.0}\,.$$

Note: EVENT2 suffers from numerical instability in n_f channel.