F-Theory GUTs and Discrete Symmetry

Supersymmetry: from M-Theory to the LHC

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What is F-theory?

F-Theory and Symmetries

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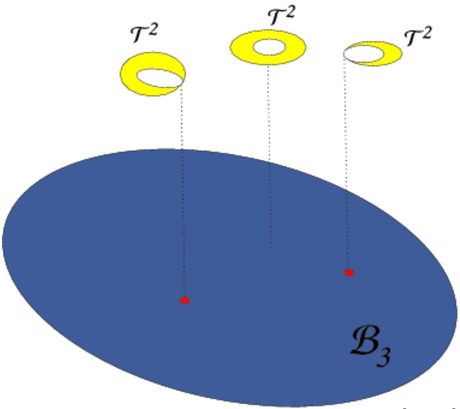
The maximum symmetry enhancement is E_8, which acts as a parent symmetry for any GUT group . . .

$$E_8 \supset E_6 \times SU(3)_\perp$$
$$E_8 \supset SO(10) \times SU(4)_\perp$$
$$E_8 \supset SU(5) \times SU(5)_\perp$$

F-Theory and Symmetries



arXiv:1203.6277 [hep-th]

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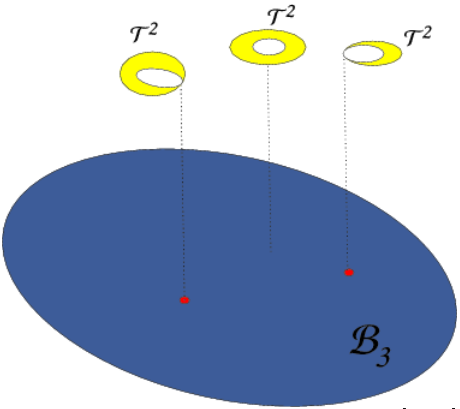

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$$E_8 \rightarrow SU(5)_{GUT} \times SU(5)_{\perp}$$

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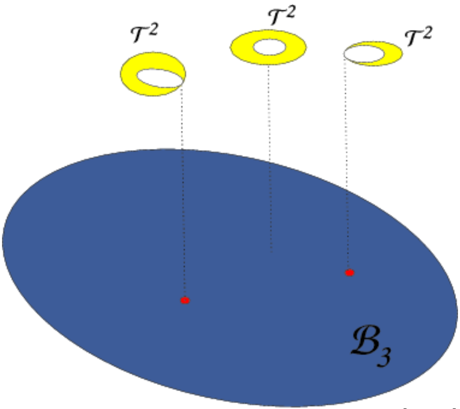

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Depending on how the weights of the **perpendicular group** identify under **"monodromy action"** we can have a **family symmetry** structure accompanying our matter.

F-Theory and Symmetries

The Weierstrass equation for elliptically fibred spaces:

$$y^2 = x^3 + f(z)x + g(z)$$

This can be written in the so-called Tate form:

$$y^2 + \alpha_1 xy + \alpha_3 y = x^3 + \alpha_2 x^2 + \alpha_4 x + \alpha_6$$

The spectral cover equation for SU(5) is a cleaner, more instructive form of this equation:

$$\mathcal{C}_5 : \; b_5 + b_4 s + b_3 s^2 + b_2 s^3 + b_1 s^4 + b_0 s^5$$

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The roots of this equation can be identified with the weights of the fundamental representation of the perpendicular group, which are paired with the antisymmetric representation of the GUT SU(5) - the 10s.

F-Theory - Spectral Cover Equation

The 10s of an $SU(5)$ singularity are described by the Spectral cover equation:

$$\mathcal{C}_5 : \ b_5 + b_4 s + b_3 s^2 + b_2 s^3 + b_1 s^4 + b_0 s^5 = b_0 \prod_{i=1}^{5} (s + t_i)$$

The roots of the spectral cover equation are identified as the weights of the 5 of $SU(5)_\perp$, which in turn specifies the defining equation of the 10 representation of the GUT group:

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Similarly, we have a way to determine our five-curves of the GUT group:

$$\sum_{n=1}^{10} c_n s^{10-n} = b_0 \prod_{i<j} (s - t_i - t_j)$$

$$R = b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 \propto \prod_{i<j} (t_i + t_j) \, .$$

F-Theory and Symmetries

$$\mathcal{C}_5 = b_5 + b_4 s + b_3 s^2 + b_2 s^3 + b_1 s^4 + b_0 s^5 = \sum_k b_k s^{5-k}$$

In general, the spectral cover equation can be factorised. Depending on how the roots are related, there may be monodromy actions relating the roots. For example, if $\mathcal{C}_5$ factorises:

$$\mathcal{C}_5 \to (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s)(a_6 + a_7 s)(a_8 + a_9 s) = 0$$

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Assuming the quadratic part cannot be factorised in the same field as the original b_k coefficients, the two roots can be shown to be:

$$s_\pm = \frac{-a_2 \pm \sqrt{w}}{2a_3}$$
$$w = e^{i\theta}|w|$$
$$\sqrt{w} = e^{i\theta/2}\sqrt{|w|}$$

Under $\theta \to \theta + 2\pi$, the roots interchange: they are related by the action.

A Model with a D_4 Monodromy

The $\mathcal{C}_4$ Spectral Cover

The most interesting classes of Family symmetry groups are S_4 and a number of its subgroups. This corresponds to a splitting of the spectral cover:

$$\mathcal{C}_5 \to \mathcal{C}_4 \times \mathcal{C}_1$$
$$\left(a_1 + a_2 s + a_3 s^2 + a_4 s^3 + a_5 s^4\right) \times \left(a_6 + a_7 s\right)$$

The quartic part encompasses a monodromy action on four weights of $SU(5)_\perp$. The most general action on four elements is S_4

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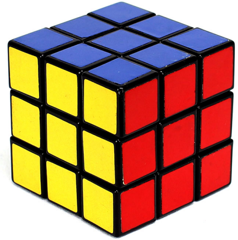

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D_4 from F-theory

The equation defining the properties of the matter curves for the 10s of SU(5) is the s^0 term of the spectral cover equation, while the equation for the 5s arises due to consistency conditions:

$$b_5 = a_1 a_6$$

$$R = \left(a_2^2 a_7 + a_2 a_3 a_6 \mp a_0 a_1 a_6^2\right)\left(a_3 a_6^2 + (a_2 a_6 + a_1 a_7) a_7\right)$$

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Curve	Equation	Homology	N	$SU(5)_\perp$ weight
10_a	a_1	$\eta - 5c_1 - \chi$	$-N$	t_1
10_b	a_6	χ	$+N$	t_5
5_c	$a_2^2 a_7 + a_2 a_3 a_6 \mp a_0 a_1 a_6^2$	$2\eta - 7c_1 - \chi$	$-N$	$2t_1$
5_d	$a_3 a_6^2 + (a_2 a_6 + a_1 a_7) a_7$	$\eta - 3c_1 + \chi$	$+N$	$t_1 + t_5$

D_4 from F-theory

The spectral cover equation has an additional symetry property that we can also exploit:

$$\begin{aligned}
\sigma : s &\rightarrow s e^{i\phi} \\
b_k &\rightarrow b_k e^{i(\chi + (k-6)\phi)} \\
\sum_k b_k s^{5-k} &\rightarrow e^{i(\chi - \phi)} \sum_k b_k s^{5-k}
\end{aligned}$$

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$$\sum_k b_k s^{5-k} \to e^{i(\chi-\phi)} \sum_k b_k s^{5-k}$$

When we factorise the spectral cover in any way, the resulting parts must be consistent with this symmetry. Consider:

$$C_5 \to C_4 \times C_1$$

$$a_n \to e^{i(3-n)\phi} a_n$$

For a given choice of $\phi = \frac{2\pi}{N}$, the coefficients of the split spectral cover will transform differently. This can be used to give a matter parity in non-*ad hoc* way.

D_4 from F-theory

Consider $N = 2$:

$$a_n \to e^{i(3-n)\pi} a_n$$

This will give a parity that alternates:

$$a_1, a_3, a_5, a_7 \to -$$
$$a_2, a_4, a_6 \to +$$

While the defining equation of the 10s of the GUT group is:

$$b_5 = a_1 a_6$$

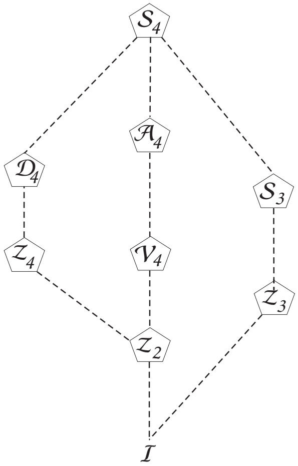
So the curves naturally have different parities.

a_n	$N = 2$	$N = 3$	$N = 4$	$N = 5$
a_1	$-$	α^2	β^2	γ^2
a_2	$+$	α	β	γ
a_3	$-$	1	1	1
a_4	$+$	α^2	β^3	γ^4
a_5	$-$	α	β^2	γ^3
a_6	$+$	1	β	γ^2
a_7	$-$	α^2	1	γ

We are of course free to experiment with other choices for N.

In each case, we also have some freedom to use the phases we discarded for this example. This will shift $a_{1,\ldots,5}$ or $a_{6,7}$ depending on phase choice.

Klein Groups and Geometric Parity



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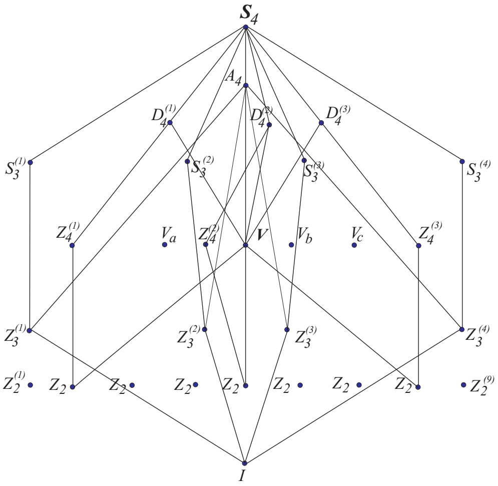

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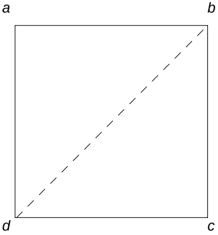
D_4 from F-theory

The generators of D_4 satisfy three relations:

- $A^4 = I$
- $B^2 = I$
- $BAB = A^{-1}$ or $ABA = B$

Geometrically speaking, these correspond to the symmetries of a square: a rotation of $\frac{\pi}{4}$ about the centre point (A) and a flip (B).

A quadruplet is not an irreducible representation of D_4



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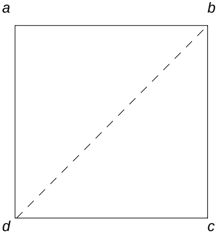

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} t_1 + t_2 + t_3 + t_4 \\ t_1 - t_2 + t_3 - t_4 \\ \sqrt{2}(t_1 - t_3) \\ \sqrt{2}(t_2 - t_4) \end{pmatrix}$$

$$\rightarrow 1_{++} + 1_{+-} + 2$$

Must assume some form of matter curve multifurcation to reconcile.

D_4 from F-theory

Low Energy Spectrum	D_4 rep	$U(1)_{t_5}$	Z_2
Q_3, u_3^c, e_3^c	1_{+-}	0	$-$
u_2^c	1_{++}	1	$+$
u_1^c	1_{++}	0	$+$
$Q_{1,2}, e_{1,2}^c$	2	0	$-$
L_i, d_i^c	1_{+-}	0	$-$
ν_3^c	1_{+-}	0	$-$
$\nu_{1,2}^c$	2	0	$-$
H_u	1_{++}	0	$+$
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- It is possible to construct a model with low energy MSSM content
- The geometric parity gives some R-parity violating operators at low energies
- The only RPV operator type is $10 \cdot \overline{5} \cdot \overline{5} \rightarrow u^c d^c \tilde{d}^c$ - no proton decay, but Neutron-antineutron oscillations.

D_4 from F-theory

There are three trilinear R-Parity violating couplings:

$$10 \cdot \overline{5} \cdot \overline{5} \to QL\tilde{d}^c + u^c d^c \tilde{d}^c + LL\tilde{e}^c$$

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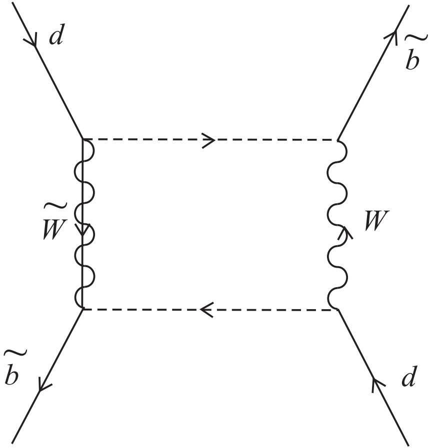
- Atomic parity violation (QLD)
- neutron-antineutron oscillations (UDD)

D_4 from F-theory

Neutron-antineutron oscillations are a seldom considered for BSM physics.

According to Goity and Sher, the dominant process is a boxgraph with W boson and gaugino exchange.

Using this a bound can be set on the coupling λ_{dbu}, coupling to the third generation. This contribution should be largest due to factors of m_b^2/m_W^2 in the decay rate.



D_4 from F-theory

The decay rate for the box process is [hep-ph/9412208]:

$$\Gamma = -\frac{3g^4\lambda^2_{dbu}M^2_{\tilde{b}_{LR}}m_{\tilde{w}}}{8\pi^2 M^4_{\tilde{b}_L}M^4_{\tilde{b}_R}}|\psi(0)|^2\sum_{j,j'}^{u,c,t}\xi_{jj'}J(M^2_{\tilde{w}},M^2_W,M^2_{u_j},M^2_{\tilde{u}_{j'}})$$

$$J(m_1,m_2,m_3,m_4) = \sum_{i=1}^{4}\frac{m_i^4\ln(m_i^2)}{\prod_{k\neq i}(m_i^2-m_k^2)}$$

The experimental bounds on the oscillation time are: $\tau = 1/\Gamma \gtrsim 10^8$

Using the data, along with other known inputs, it is possible to calculate the limits on the coupling. We take $M_{\tilde{b}_L} = M_{\tilde{b}_R} = 500GeV$, scanning over the parameter space of the stop mass.

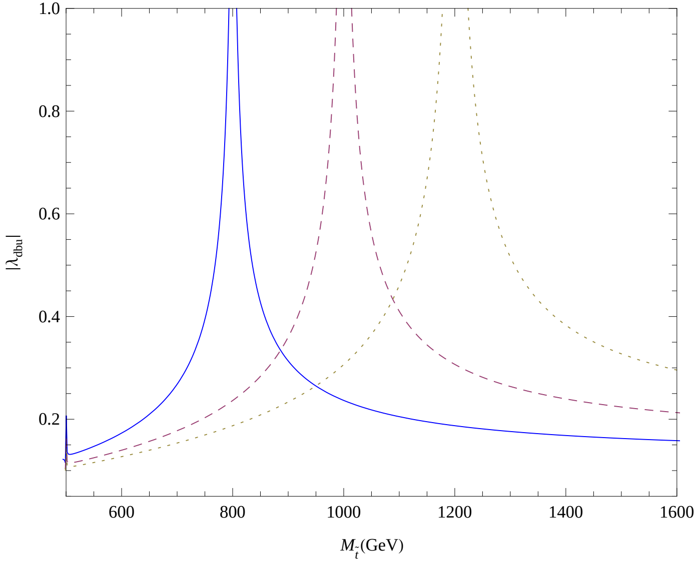


Figure: Bounds on λ_{dbu} using the latest experimental limits. Blue: $M_{\tilde{u}} = M_{\tilde{c}} = 800 GeV$, Dashed: $M_{\tilde{u}} = M_{\tilde{c}} = 1000 GeV$, Dotted: $M_{\tilde{u}} = M_{\tilde{c}} = 1200 GeV$.

D_4 from F-theory

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- In F-theory it is possible to calculate Yukawa couplings directly, by calculating the integral of overlapping wavefunctions
- Taking into account mixing effects this particular coupling is estimated to be of the order $\lambda_{dbu} \leq 10^{-1}$ - which is compatible with the experimental value.

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- The geometric parity implemented in this model gives rise to unexpected R-parity violating effects, which may provide testable predictions of new physics.

F-Theory and Symmetries

- **Diphoton excess from E_6 in F-theory GUTs**
  Offers explanation for the 750 GeV bump using an E_6 inspired model from earlier work [arXiv:1601.00640]-*Karozas, King, Leontaris, AKM*

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Summary

- The variety of symmetry tools available in F-theory give rise to many unique and interesting classes of model
- Geometric parity assignments are a promising way to generate Matter parity without adding it by hand
- It is possible to make models that replicate the MSSM with no RPV...
- and models with interesting signatures - for example neutron-antineutron oscillations without proton decay

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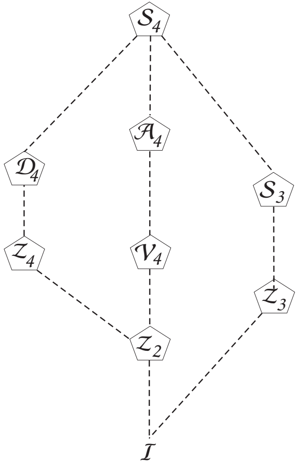

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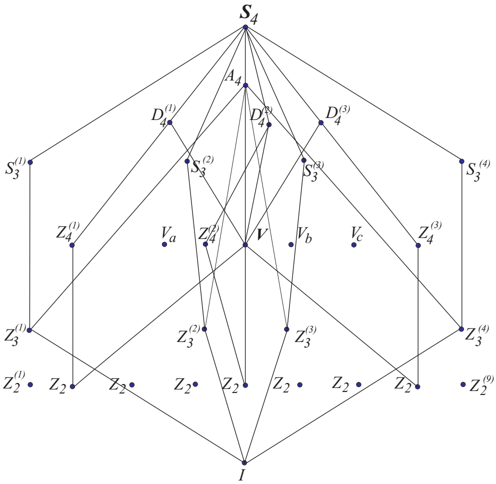

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Klein Groups and Geometric Parity

	S_4 cycles	Trans. A_4	Trans. V_4
4-cycles	(1234), (1243), (1324), (1342), (1423), (1432)	No	No
3-cycles	(123), (124), (132), (134), (142), (143), (234), (243)	Yes	No
2+2-cycles	(12)(34), (13)(24), (14)(23)	Yes	Yes
2-cycles	(12), (13), (14), (23), (24), (34)	No	No
1-cycles	e	Yes	Yes

Klein Groups and Geometric Parity

	S_4 cycles	Trans. A_4	Trans. V_4
4-cycles	(1234), (1243), (1324), (1342), (1423), (1432)	No	No
3-cycles	(123), (124), (132), (134), (142), (143), (234), (243)	Yes	No
2+2-cycles	(12)(34), (13)(24), (14)(23)	Yes	Yes
2-cycles	(12), (13), (14), (23), (24), (34)	No	No
1-cycles	e	Yes	Yes

- Transitive Klein group: $\{(1), (12)(34), (13)(24), (14)(23)\}$

$$\mathcal{C}_5 = \mathcal{C}_4 \times \mathcal{C}_1$$

Klein Groups and Geometric Parity

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2+2-cycles	(12)(34), (13)(24), (14)(23)	Yes	Yes
2-cycles	(12), (13), (14), (23), (24), (34)	No	No
1-cycles	e	Yes	Yes

- Transitive Klein group: $\{(1),(12)(34),(13)(24),(14)(23)\}$

$$\mathcal{C}_5 = \mathcal{C}_4 \times \mathcal{C}_1$$

- Non-transitive Klein group: $\{(1),(12),(34),(12)(34)\}$

$$\mathcal{C}_5 = \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_1$$

Klein Groups and Geometric Parity

Spectral cover equation for a non-transitive Klein monodromy:

$$\mathcal{C}_5 \to \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_1 : (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s + a_6 s^2)(a_7 + a_8 s)$$

The defining equations for the SU(5) matter are then:

$$\begin{aligned} P_{10} =& a_1 a_4 a_7 \\ P_5 =& a_5(a_6 a_7 + a_5 a_8)(a_6 a_7^2 + a_8(a_5 a_7 + a_4 a_8))(a_1 - a_5 a_7 c) \\ & (a_1^2 - a_1(a_5 a_7 + 2a_4 a_8)c + a_4(a_6 a_7^2 + a_8(a_5 a_7 + a_4 a_8))c^2) \end{aligned}$$

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Model has three matter curves that are 10s with different t_i charges, and five that are $5/\overline{5}$s. Enough to build a realistic model!

Implement a model with geometric parity based on existing example: Dudas & Palti [arXiv:1005.5728].

Klein Groups and Geometric Parity

Curve	Charge	Parity	Spectrum
10_1	t_1	i	$M_{10_1} Q + (M_{10_1} - N_1) u^c + (M_{10_1} + N_1) e^c$
10_3	t_3	j	$M_{10_3} Q + (M_{10_3} - N_2) u^c + (M_{10_3} + N_2) e^c$
10_5	t_5	k	$M_{10_5} Q + (M_{10_5} + N_1 + N_2) u^c + (M_{10_5} - N_1 - N_2) e^c$
5_1	$-2t_1$	jk	$M_{5_1} \overline{d^c} + (M_{5_1} - N_1) \overline{L}$
5_{13}	$-t_1 - t_3$	$+$	$M_{5_{13}} \overline{d^c} + (M_{5_{13}} + 2N_1) \overline{L}$
5_{15}	$-t_1 - t_5$	i	$M_{5_{15}} \overline{d^c} + (M_{5_{15}} + N_1) \overline{L}$
5_{35}	$-t_3 - t_5$	j	$M_{5_{35}} \overline{d^c} + (M_{5_{35}} - 2N_1 - N_2) \overline{L}$
5_3	$-2t_3$	$-j$	$M_{5_3} \overline{d^c} + (M_{5_3} + N_2) \overline{L}$

Parameter choices to replicate model of D&P:

$$
\begin{aligned}
N_1 &= M_{5_{15}} = M_{5_{35}} = 0 \\
N_2 &= M_{10_3} = M_{5_1} = 1 = -M_{10_5} = -M_{5_3} \\
M_{10_1} &= 3 = -M_{5_{13}}
\end{aligned}
$$

Original model had *ad hoc* parity assignment

Klein Groups and Geometric Parity

Curve	Charge	Spectrum	All possible assignments							
10_1	t_1	$3Q + 3u^c + 3e^c$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
10_3	t_3	$Q + 2e^c$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
10_5	t_5	$-Q - 2e^c$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
5_1	$-2t_1$	$D_u + H_u$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
5_{13}	$-t_1 - t_3$	$-3\overline{d^c} - 3\overline{L}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
5_{15}	$-t_1 - t_5$	0	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
5_{35}	$-t_3 - t_5$	$-\overline{H}_d$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
5_3	$-2t_3$	$-\overline{D}_d$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$

- Yukawa couplings for the matter of the Standard Model?
- Do exotic processes occur at high rates?
- Operators invariant under $SU(5)_\perp$ charges and the Geometric parity?

Klein Groups and Geometric Parity

Tension between exotic masses and Bilinear R-parity violation: The colour triplets D_u/D_d get masses from:

$$D_uD_d\theta_1\theta_1\theta_3,\ D_uD_d\theta_1\theta_1\theta_6,\ D_uD_d\theta_1\theta_2\overline{\theta}_5,\ D_uD_d\theta_1\theta_3\theta_8,$$
$$D_uD_d\theta_1\theta_6\theta_8,\ D_uD_d\theta_2\overline{\theta}_5\theta_8,\ D_uD_d\theta_3\theta_8\theta_8,\ D_uD_d\theta_6\theta_8\theta_8$$

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Bilinear R-Parity Violating terms at lowest order are very dangerous as they facilitate proton decay at fractions of a second ($\tau_p > 10^{32} yrs$). These must not be allowed in the spectrum:

$$H_uL\theta_1,\ H_uL\theta_8,\ H_uL\theta_1\theta_4,\ H_uL\theta_4\theta_8,\ H_uL\overline{\theta}_5\theta_7$$

Klein Groups and Geometric Parity

Tension between exotic masses and Bilinear R-parity violation: The colour triplets D_u/D_d get masses from:

$$D_uD_d\theta_1\theta_1\theta_3,\ D_uD_d\theta_1\theta_1\theta_6,\ D_uD_d\theta_1\theta_2\overline{\theta}_5,\ D_uD_d\theta_1\theta_3\theta_8,$$
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$$H_uL\theta_1,\ H_uL\theta_8,\ H_uL\theta_1\theta_4,\ H_uL\theta_4\theta_8,\ H_uL\overline{\theta}_5\theta_7$$

Clear problem: can't eliminate dangerous BRPV terms while also integrating out the D_u/D_d matter - θ_1, θ_8 or both in all D_uD_d terms.

Look elsewhere...

Klein Groups and Geometric Parity

Curve	Charge	Parity	Spectrum
10_1	t_1	i	$M_{10_1} Q + (M_{10_1} - N_1) u^c + (M_{10_1} + N_1) e^c$
10_3	t_3	j	$M_{10_3} Q + (M_{10_3} - N_2) u^c + (M_{10_3} + N_2) e^c$
10_5	t_5	k	$M_{10_5} Q + (M_{10_5} + N_1 + N_2) u^c + (M_{10_5} - N_1 - N_2) e^c$
5_1	$-2t_1$	jk	$M_{5_1} \overline{d^c} + (M_{5_1} - N_1) \overline{L}$
5_{13}	$-t_1 - t_3$	$+$	$M_{5_{13}} \overline{d^c} + (M_{5_{13}} + 2N_1) \overline{L}$
5_{15}	$-t_1 - t_5$	i	$M_{5_{15}} \overline{d^c} + (M_{5_{15}} + N_1) \overline{L}$
5_{35}	$-t_3 - t_5$	j	$M_{5_{35}} \overline{d^c} + (M_{5_{35}} - 2N_1 - N_2) \overline{L}$
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Try to implement an MSSM type model with no exotics or BRPV:

Klein Groups and Geometric Parity

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5_1	$-2t_1$	jk	$M_{5_1} \overline{d^c} + (M_{5_1} - N_1) \overline{L}$
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5_3	$-2t_3$	$-j$	$M_{5_3} \overline{d^c} + (M_{5_3} + N_2) \overline{L}$

Try to implement an MSSM type model with no exotics or BRPV:

Large parameter space of models to scan, with many possibilities for interesting physics, but difficult to pin down viable models.

One good option is MSSM-like

$$M_{10_1} = -M_{5_{13}} = 2$$
$$N_1 = M_{10_5} = -M_{5_3} = 1$$
$$N_2 = M_{10_3} = M_{5_1} =$$
$$M_{5_{13}} = M_{5_{35}} = 0$$
$$i = -j = k = -$$

Klein Groups and Geometric Parity

Curve	Charge	Matter Parity	Spectrum
10_1	t_1	$-$	$Q_3 + Q_2 + u_3^c + 3e^c$
10_3	t_3	$+$	$-$
10_5	t_5	$-$	$Q_1 + u_2^c + u_1^c$
5_1	$-2t_1$	$-$	$-\overline{L}_1$
5_{13}	$-t_1 - t_3$	$+$	$2H_u$
5_{15}	$-t_1 - t_5$	$-$	$-\overline{d}_2^c - \overline{d}_1^c - \overline{L}_2$
5_{35}	$-t_3 - t_5$	$+$	$-2\overline{H}_d$
5_3	$-2t_3$	$-$	$-\overline{d}_3^c - \overline{L}_3$
$1_{15} = \theta_7$	$t_1 - t_5$	$-$	N_R^a
$1_{51} = \overline{\theta}_7$	$t_5 - t_1$	$-$	N_R^b

Table: Matter content for a model with the standard matter parity arising from a geometric parity assignment.

Klein Groups and Geometric Parity

Each coupling must be invariant under the $SU(5)_\perp$ charges, t_i. Consider the coupling:

$$10_1 \cdot 10_1 \cdot 5_{13}$$

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Need a singlet to balance this:

$$\overline{\theta}_{1,8} : \quad t_3 - t_1$$

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So the overall charge can be canceled out for the operator:

$$10_1 \cdot 10_1 \cdot 5_{13} \cdot (\overline{\theta}_1 + \overline{\theta}_8)$$

Klein Groups and Geometric Parity

This model has Yukawa couplings for all the generations of quarks and leptons. Consider for example the up-type quarks, which have SU(5) couplings of type $10 \cdot 10 \cdot 5$.

$$
\begin{aligned}
10_1 \cdot 10_1 \cdot 5_{13} \cdot (\bar{\theta}_1 + \bar{\theta}_8) &\rightarrow (Q_3 + Q_2) u_3 H_u (\bar{\theta}_1 + \bar{\theta}_8) \\
10_1 \cdot 10_5 \cdot 5_{13} \cdot \theta_5 &\rightarrow ((Q_3 + Q_2)(u_1 + u_2) + Q_1 u_3) H_u \theta_5 \\
10_5 \cdot 10_5 \cdot 5_{13} \cdot \theta_2 \cdot \theta_5 &\rightarrow Q_1 (u_1 + u_2) H_u \theta_2 \theta_5
\end{aligned}
$$

Singlets must be used to cancel the $SU(5)_\perp$ charges, so we have a series of non-renormalisable Yukawas

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\end{aligned}
$$

Singlets must be used to cancel the $SU(5)_\perp$ charges, so we have a series of non-renormalisable Yukawas

$$
M_{u,c,t} \sim v_u \begin{pmatrix} \epsilon\theta_2\theta_5 & \theta_2\theta_5 & \theta_5 \\ \epsilon^2\theta_5 & \epsilon\theta_5 & \epsilon(\overline{\theta}_1 + \overline{\theta}_8) \\ \epsilon\theta_5 & \theta_5 & \overline{\theta}_1 + \overline{\theta}_8 \end{pmatrix}
$$

The mass matrix is rank 3, with suppressions, ϵ due to the so-called rank theorem, helping to give a hierarchy.

Klein Groups and Geometric Parity

The spectrum contains two singlets that do not have vacuum expectation values, which protects the model from dangerous operators. These singlets, $\theta_7 = N_R^a$ and $\overline{\theta}_7 = N_R^b$, also serve as candidates for right-handed neutrinos. For $\theta_7 = N_R^a$:

$$\overline{5}_3 \cdot 5_{13} \cdot \theta_7 \cdot \overline{\theta}_5 \to L_3 N_R^a H_u \overline{\theta}_5$$

$$\overline{5}_{15} \cdot 5_{13} \cdot \theta_7 \cdot (\overline{\theta}_1 + \overline{\theta}_8) \to L_2 N_R^a H_u (\overline{\theta}_1 + \overline{\theta}_8)$$

$$\overline{5}_1 \cdot 5_{13} \cdot \theta_7 \cdot (\overline{\theta}_1 + \overline{\theta}_8) \cdot \theta_2 \to L_1 N_R^a H_u (\overline{\theta}_1 + \overline{\theta}_8) \theta_2$$

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And we also have the operators arising from the N_R^b singlet:

$$\overline{5}_3 \cdot 5_{13} \cdot \overline{\theta}_7 \cdot (\overline{\theta}_1 + \overline{\theta}_8) \cdot \theta_2 \to L_3 N_R^b H_u (\overline{\theta}_1 + \overline{\theta}_8) \theta_2$$
$$\overline{5}_{15} \cdot 5_{13} \cdot \overline{\theta}_7 \cdot \theta_2 \cdot \theta_5 \to L_2 N_R^b H_u \theta_2 \theta_5$$
$$\overline{5}_1 \cdot 5_{13} \cdot \overline{\theta}_7 \cdot \theta_5 \to L_1 N_R^b H_u \theta_5$$

The combination of these operators should reduce the hierarchy and increase mixing for neutrinos.

Klein Groups and Geometric Parity

The right-handed neutrinos also get a Majorana mass:

$$\frac{\langle\theta_2\rangle^2}{\Lambda}\overline{\theta}_7^2 \; + \; \frac{\langle\overline{\theta}_2\rangle^2}{\Lambda}\theta_7^2 \; + \; M\theta_7\overline{\theta}_7$$

This will allow the Seesaw mechanism to be implemented, giving a light effective neutrino mass.

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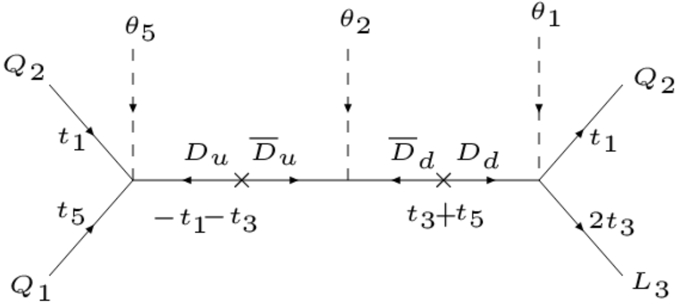
The model has the interesting feature of requiring two copies of the Higgs

$$5_{13} \cdot \overline{5}_{35} \cdot \theta_2 \to M_{ij} H_u^i H_d^j \to M \begin{pmatrix} \epsilon_h^2 & \epsilon_h \\ \epsilon_h & 1 \end{pmatrix} \begin{pmatrix} H_u^1 \\ H_u^2 \end{pmatrix} \begin{pmatrix} H_d^1 & H_d^2 \end{pmatrix}$$

The rank theorem tells us that because they are on the same matter curve, only one gets a mass, while the others must have suppressed mass terms: one Higgs will be light, with another having a mass close to the GUT scale.

Klein Groups and Geometric Parity

There are no parity violating operators in the spectrum, however we should still consider dimension six proton decay.



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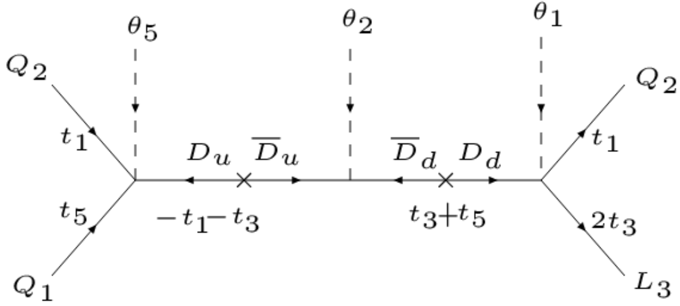

There are no D_u/D_d in the low energy spectrum, however they could appear at the string scale. The process should be highly suppresed due to this, and the non-renormalisability of the internal components of the process.

F-Theory - Monodromy and Discrete Symmetry

The monodromy is best understood by closely examining the quadratic part of the factorised spectral cover:

$$(a_1 + a_2 s + a_3 s^2) = 0$$
$$s_\pm = \frac{-a_2 \pm \sqrt{a_2 - 4a_1 a_3}}{2a_3}$$

we see that since $\sqrt{a_2 - 4a_1 a_3} = e^{i\theta/2}\sqrt{|a_2 - 4a_1 a_3|}$, under $\theta \to \theta + 2\pi$, the two solutions interchange.

Since we do not know anything about the global geometry, in semi-local F-theory we must choose our monodromy group.

D_4 from F-theory

In this case $b_k \propto a_n a_m$. Let us assume:

$$a_n \to e^{i\psi_n} e^{i(3-n)\phi} a_n$$

For any coefficient of C_5, $b_k = a_n a_m$:

$$\begin{aligned} b_k \to & b_k e^{i(\chi+(k-6)\phi)} \\ a_n a_m \to & e^{i(\psi_n+\psi_m)} e^{i(6-n-m)\phi} a_n a_m = e^{i(\psi_n+\psi_m)} e^{-i(6-k)\phi} a_n a_m \end{aligned}$$

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This is trivially compliant if:

$$\chi = \psi_n + \psi_m$$

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This is trivially compliant if:

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It can be shown that the phases for each coefficient are correlated. Consider:

$$\begin{aligned} b_5 =& a_1 a_6 \\ b_4 =& a_2 a_6 + a_1 a_7 \end{aligned}$$

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$$a_n \rightarrow e^{i\psi_n} e^{i(3-n)\phi} a_n$$

For any coefficient of C_5, $b_k = a_n a_m$:

$$\begin{aligned} b_k \rightarrow & b_k e^{i(\chi+(k-6)\phi)} \\ a_n a_m \rightarrow & e^{i(\psi_n+\psi_m)} e^{i(6-n-m)\phi} a_n a_m = e^{i(\psi_n+\psi_m)} e^{-i(6-k)\phi} a_n a_m \end{aligned}$$

This is trivially compliant if:

$$\chi = \psi_n + \psi_m$$

It can be shown that the phases for each coefficient are correlated. Consider:

$$\begin{aligned} b_5 =& a_1 a_6 \\ b_4 =& a_2 a_6 + a_1 a_7 \end{aligned}$$

Given the above it must be true that:

$$\chi = \psi_1 + \psi_6 = \psi_2 + \psi_6 = \psi_1 + \psi_7$$

D_4 from F-theory

This symmetry can be translated into a Z_N type parity. For simplicity, let us set all $\psi_i = 0$:

$$\begin{aligned}
\sigma : & s \rightarrow s e^{i\phi} \\
& b_k \rightarrow b_k e^{i(k-6)\phi} \\
& \sum_k b_k s^{5-k} \rightarrow e^{-i\phi} \sum_k b_k s^{5-k} \\
& a_n \rightarrow e^{i(3-n)\phi} a_n
\end{aligned}$$

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This symmetry can be translated into a Z_N type parity. For simplicity, let us set all $\psi_i = 0$:

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If we then let:

$$\phi = \frac{2\pi}{N}$$

The transformation is then of the Z_N type. C_5 will gain a phase, however, if it is factorised into say $C_5 \to C_4 \times C_1$, the coefficients a_n will transform differently.

D_4 from F-theory

Consider $N = 2$:

$$a_n \to e^{i(3-n)\pi} a_n$$

This will give a parity that alternates:

$$a_1, a_3, a_5, a_7 \to -$$
$$a_2, a_4, a_6 \to +$$

While the defining equation of the 10s of the GUT group is:

$$b_5 = a_1 a_6$$

So the curves naturally have different parities.

a_n	$N = 2$	$N = 3$	$N = 4$	$N = 5$
a_1	$-$	α^2	β^2	γ^2
a_2	$+$	α	β	γ
a_3	$-$	1	1	1
a_4	$+$	α^2	β^3	γ^4
a_5	$-$	α	β^2	γ^3
a_6	$+$	1	β	γ^2
a_7	$-$	α^2	1	γ

We are of course free to experiment with other choices for N.

In each case, we also have some freedom to use the phases we discarded for this example. This will shift $a_{1,\ldots,5}$ or $a_{6,7}$ depending on phase choice.

Name	Charge	All possible assignments							
θ_1	$\pm(t_1 - t_3)$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
θ_2	$\pm(t_1 - t_5)$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
θ_3	0	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
θ_4	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
θ_5	$\pm(t_3 - t_5)$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
θ_6	0	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
θ_7	$\pm(t_1 - t_5)$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$
θ_8	$\pm(t_1 - t_3)$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$

Table: Singlet curves and their perpendicular charges and geometric parity