# Supersymmetry and Geometry of Hyperbolic Monopoles

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Supersymmetry: from M-theory to the LHC
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- Euclidean monopoles $(A, \phi)$ are instantons invariant under time translation

$$\begin{aligned} \phi &\to A_4 \Longrightarrow \\ F &= *F \Longleftrightarrow D_k\Phi = -\frac{1}{2}\varepsilon_{ijk}F^{ij} \end{aligned}$$

where $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}$, $\varepsilon_{1234} = -1$, and the duality operator relative to $\mathbb{R}^4$metric

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- Hyperbolic monopoles $(A, \phi)$ are instantons invariant under axial rotation

$$\begin{aligned} \phi &\rightarrow A_\theta \Longrightarrow \\ F &= *F \Longleftrightarrow D_k\Phi = -\frac{1}{2}\varepsilon_{ijk}F^{ij} \end{aligned}$$

The duality operator relative to $\mathbb{S}^1 \times \mathbb{H}^3$ metric

# Moduli Space of Monopoles

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- The presence of the Bogomol'nyi bound is a result of the fact that monopoles have supersymmetric extension. Topological charges appear in the supersymmetry algebra from which one can deduce the B. B. What's the geometry of Supersymmetric non-equivalent Gauge Hyperbolic Monopoles ?

# Puzzles & Plan:

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1. We need to construct supersymmetric Yang-Mills-Higgs theory on $H^3$.
2. For monopoles on $H^3$, the moduli space doesn't inherit a metric from the gauge theory. The natural $L^2$ metric $\int_{H^3}(|\dot{\phi}|^2 + |\dot{A}|^2)$ "doesn't converge".
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3. New methods for defining the complex structures and the connection.

Plan:

1. Construct supersymmetric Yang-Mills-Higgs (SYMH) theory on $H^3$, and find the unbroken supersymmetry transformations.
2. Study the low energy dynamics & construct supermultiplet of zero modes.
3. Linearize the unbroken supersymmetry, close its algebra on shell, and find the defining geometry equations.

# SYMH theory on $H^3$

- In the journey of constructing this theory we follow these steps:
  1- Start from $N = 1$ SYM theory on $\mathbb{R}^{3,1}$.
  2- Euclideanise to $(1, 1)$ SYM theory on $\mathbb{R}^4$.
  3- Close the supersymmetry algebra off-shell (We want to have auxiliary field).
  4- Reduce the theory along one of the Euclidean dimensions ($x^4$) to obtain a $(1, 1)$ SYMH theory on $\mathbb{R}^3$.
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- After spending your summer on these calculations you get the following supersymmetric Yang Mills Higgs Lagrangian on $H^3$

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2}|D\Phi|^2 - \frac{1}{2}P^2 - i\mathcal{X}^\dagger \not{D}\psi - \mathcal{X}^\dagger [\Phi, \psi] - \frac{i}{2\ell}\mathcal{X}^\dagger\psi$$

- The left susy transformations

$$\delta_L A_i = i\mathcal{X}^\dagger \sigma_i \epsilon_L \quad \delta_L \Phi = \mathcal{X}^\dagger \epsilon_L \quad \delta_L \mathcal{X}^\dagger = 0$$

$$\delta_L \psi = P\epsilon_L + i\left(\frac{1}{2}\varepsilon_{ijk}F^{ij} - D_k\Phi\right)\sigma^k \epsilon_L$$

$$\delta_L P = i\left(\not{D}\mathcal{X}\right)^\dagger \epsilon_L + \left[\Phi, \mathcal{X}^\dagger \epsilon_L\right] - \frac{i}{2\ell}\mathcal{X}^\dagger \epsilon_L$$

- The right susy transformations

$$\delta_R A_i = -i\epsilon_R^\dagger \sigma_i \psi \quad \delta_R \Phi = -\epsilon_R^\dagger \psi \quad \delta_R \psi = 0$$

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# SYMH theory on $H^3$ and The Unbroken Susy

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- Clearly if one $\delta$ preserves BPS configurations the other will not. The BPS configurations are precisely the hyperbolic monopoles. Hence hyperbolic monopoles are supersymmetric "half BPS saturated". For our definition of hyperbolic monopoles, $\delta_R$ is the unbroken supersymmetry.

# Low Energy Dynamics

- The solutions of the Bogomol'nyi equation are static solutions. However, reading the geometry requires having some dynamics, through introducing "time". Once we add time to the configurations, the field eqs. can be interpreted as equation of motion of the monopoles in a some manifold (moduli space). But any motion, however small, increase the energy from its BPS bound, and thus takes us to a different manifold with different geometry.

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- Nevertheless, there is a dynamic scenario where violating the minimum energy bound is avoidable. If we keep the velocity low, and we start the motion tangent to the static min($U$) solutions, then energy conservation prevent the motion from going far away from this manifold. Motion in min($U$) is a geodesic motion." N. Manton 1982"

- The bosonic zero mode is

$$\dot{A}_\mu = i\zeta_R^\dagger \Gamma_\mu \dot{\psi}_L$$

where $\zeta_R$ satisfy

$$\nabla_i \zeta_R = \frac{i}{2\ell}\Gamma_i \Gamma_4 \zeta_R \qquad\qquad \nabla_4 \zeta_R = 0$$

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- Remark:
  - ⋆ Crucial difference (from the flat case) in the Killing spinors give two sets of solutions in $\mathbb{C}^2$ each parametrized by two basis.

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# Linearization of the Unbroken Susy Transformations

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- Linearizing the unbroken susy transformations will give us the supersymmetry between the fermionic & bosonic collective coordinates.

# Linearization of the Unbroken Susy Transformations

- Linearizing the supersymmetry transformations gives the supersymmetry between the coordinates of the moduli space

$$\delta_R \beta^a = \rho_A \mathcal{F}_b^{(A)a} \theta^b,$$

$$\delta_R \theta^a = -\frac{1}{2} \rho_A \mathcal{F}_b^{(A)a} \dot{\beta}^b + i\theta^c \Gamma^a_{bc} \delta_R \beta^b$$

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and $\Gamma^a_{bc}$ is some connection on the moduli space, which properties will be determined from the closure of the supersymmetry algebra.

- Closing the susy algebra on shell in, 1−Dim, requires:

$$\delta_\rho^2 \beta^a = -\frac{1}{2}\rho^A \rho^B \delta_{AB} \dot{\beta}^a \qquad \delta_\rho^2 \theta^a = -\frac{1}{2}\rho^A \rho^B \delta_{AB} \dot{\theta}^a$$

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- The bosonic susy algebra gives the following identities:

$$\begin{aligned}
\mathcal{F}_a^{(A)b} \mathcal{F}_b^{(B)c} + \mathcal{F}_a^{(B)b} \mathcal{F}_b^{(A)c} &= -2\delta^{AB} \delta_a{}^c \\
\Gamma^a_{bc} &= \Gamma^a_{cb} \\
\nabla_a \mathcal{F}_b^{(A)c} &= 0 \\
\mathcal{F}_b^a \partial_a \mathcal{F}_c^d - \mathcal{F}_c^a \partial_a \mathcal{F}_b^d + \mathcal{F}_a^d \partial_b \mathcal{F}_c^a - \mathcal{F}_a^d \partial_c \mathcal{F}_b^a &= 0
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# Identifying the Geometry & Susy Algebra Closure

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\mathcal{F}_b^a \partial_a \mathcal{F}_c^d - \mathcal{F}_c^a \partial_a \mathcal{F}_b^d + \mathcal{F}_a^d \partial_b \mathcal{F}_c^a - \mathcal{F}_a^d \partial_c \mathcal{F}_b^a &= 0
\end{aligned}$$

- Combining the last two equations we can derive the equation for the connection (obata connection)

$$\Gamma^o{}_{ab}{}^c = -\frac{1}{6}\left[2\partial_{(a}\vec{\mathcal{F}}_{b)}{}^d + \vec{\mathcal{F}}_{(a}{}^e \times \partial_e \vec{\mathcal{F}}_{b)}{}^d\right] \cdot \vec{\mathcal{F}}_d{}^c$$

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- However when we defined the bosonic zero modes (coordinate functions of the moduli space) we chose them to be complex. This means that we have found the geometry of the complexified moduli space of hyperbolic monopoles is hypercomplex, hence one would ask what is the geometry of the real moduli space? Or "What is the geometry, that when complexified gives hypercomplex ?"

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- This real geometry is called pluricomplex, introduced in Jan 2012 by Roger Bielawski & Lorenz Schwachhöfer.

- For a pluricomplex geometry we have 3 integrable complex structures $I, J$ & $K$ that don't behave like unit quatorions (they don't anticommute!).
- The pluricomplex structure still determine a decomposition of the complexified tangent space as $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

$$\begin{aligned} IV &= iV \Rightarrow V \in T^{(1,0)}(M) \\ IV &= -iV \Rightarrow V \in T^{(0,1)}(M) \end{aligned}$$

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- Any integral pluricomplex structure has an associated canonical torsion free connection.
- Let $M$ be a pluricomplex manifold and suppose we are looking at the complex thickening $M^{\mathbb{C}}$ of $M$ with complex structure, say, $\widetilde{I}, \widetilde{J}, \widetilde{K}$.
  - $\widetilde{I}, \widetilde{J}, \widetilde{K}$ are integrable and obey the quatorionic algebra
  - Have a torsion free connection (obatta connection)
  - On $T\left(M^{\mathbb{C}}\right) \exists$ an action of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong Mat\,(2, \mathbb{C})$. In other words, the complex structure act linearly on the vectors in $T\left(M^{\mathbb{C}}\right)$ (zero modes).

# The Limiting Case

- In our approach it's very transparent to see how the geometry of the hyperbolic monopole will converge to the geometry of flat monopoles as $\ell \to \infty$
  - The left and right Killing spinor equations will agree

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  - The obata connection gives the Levi Civita connection by which we can show that the Ricci tensor vanishes. In other words the limiting case has produced the hyperKäler geometry.