# Quantum gravity using SUSY as a formal device


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# Introduction

- Einstein's General Relativity (GR) is a perturbatively non-renormalizable field theory.
- The field is the spacetime metric.
- The asymptotic safety conjecture suggests that there may exist a non-perturbative, non-trivial ultraviolet fixed point for gravity.
- Renormalization Group (RG) flow can be seen intuitively as describing physics at different scales of length by changing the resolution.
- SUSY is not just a theory of BSM particle physics, it can also be used as a more general mathematical device.
- Existing examples are manifestly gauge invariant regularization in QFT and Parisi-Sourlas supersymmetry in statistical field theory.
- My work develops a manifestly diffeomorphism invariant Exact RG, the regularization of which would also use this formal kind of SUSY.

# Kadanoff blocking

In the Ising model, Kadanoff blocking is the process of grouping microscopic spins together to form macroscopic “blocked” spins via a majority rule.

The continuous version integrates out the high-energy modes of a field to give a renormalized field, used in a renormalized action.

The blocking functional, *b*, is defined via the effective Boltzmann factor:

$$e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; \delta\left[\varphi - b\left[\varphi_0\right]\right] e^{-S_{\text{bare}}[\varphi_0]}.$$

There are an infinite number of possible Kadanoff blockings, but a simple linear example is

$$b[\varphi_0](x) = \int_y B(x-y)\varphi_0(y),$$

where the kernel, *B*, contains an infrared cutoff function.

The partition function must be invariant under change of cutoff scale, *Λ*, this will be ensured by construction, i.e.

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; e^{-S_{\text{bare}}[\varphi_0]}.$$

Kadanoff blocking demands a suitable notion of locality that requires us to work exclusively in Euclidean signature.

# Polchinski Flow Equation

Differentiate the effective Boltzmann factor w.r.t. “RG time”:

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = -\int_x \frac{\delta}{\delta \varphi(x)} \int \mathcal{D}\varphi_0 \; \delta\left[\varphi - b\left[\varphi_0\right]\right] \Lambda \frac{\partial b[\varphi_0](x)}{\partial \Lambda} e^{-S_{\text{bare}}[\varphi_0]}.$$

Write the “rate of change of blocking functional” as

$$\Psi(x) = \frac{1}{2} \int_y \dot{\Delta}(x,y) \frac{\delta \Sigma}{\delta \varphi(y)},$$ where $\Sigma = S - 2\hat{S}$, and $\Delta = c(p^2/\Lambda^2)p^{-2}$.

The RG flow of the effective Boltzmann factor is then

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = \int_x \frac{\delta}{\delta \varphi(x)} \left( \Psi(x) e^{-S[\varphi]} \right).$$

The Polchinski flow equation is then written elegantly as

$$\dot{S} = \frac{1}{2} \frac{\delta S}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{1}{2} \frac{\delta}{\delta \varphi} \cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi}.$$

where $f \cdot W \cdot g := \int_x f(x) W\left(-\frac{\partial^2}{\Lambda^2}\right) g(x).$

# Generalization to Yang-Mills theories

A pure gauge theory is constructed in a manifestly gauge invariant way by building the action from covariant derivatives and field-strength tensors:

$$D_\mu := \partial_\mu - iA_\mu, \qquad F_{\mu\nu} := i[D_\mu, D_\nu].$$

To preserve manifest gauge invariance, wavefunction renormalization must be avoided. This is achieved by writing the coupling, *g*, as an overall scaling factor:

$$S[A](g) = \frac{1}{4g^2} tr \int_x F_{\mu\nu} c^{-1}\left(-\frac{D^2}{\Lambda^2}\right) F_{\mu\nu} + \cdots$$

The effective action is then written as a loopwise expansion that is also an expansion in powers of *g:*

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots$$

The *β*-functions also have a similar expansion: $\beta := \Lambda \partial_\Lambda g = \beta_1 g^3 + \beta_2 g^5 + \cdots$

The generalization of the Polchinski flow equation to gauge theories uses a suitably covariantized kernel, the easiest way to do this is to replace the partial derivatives with covariant derivatives:

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta A_\mu} \cdot \{\dot{\Delta}\} \cdot \frac{\delta \Sigma}{\delta A_\mu} - \frac{1}{2}\frac{\delta}{\delta A_\mu} \cdot \{\dot{\Delta}\} \cdot \frac{\delta \Sigma}{\delta A_\mu}.$$

# A refresher on GR

The Riemann tensor, representing spacetime curvature, is written in our sign convention as

$$R^a{}_{bcd} = 2\partial_{[c}\Gamma^a{}_{b]d} + 2\Gamma^a{}_{[c|f}\Gamma^f{}_{|b]d}$$

where $$A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$$

We use the Levi-Civita connection (torsion-free metric connection)

$$\Gamma^a{}_{bc} = \frac{g^{ad}}{2}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$$

We have the Ricci tensor in our sign convention as $$R_{ab} = R^c{}_{acb}$$

Then the Ricci scalar by $$R = g^{ab}R_{ab}$$

Thus the Einstein field equation is

$$R_{ab} - \frac{g_{ab}}{2}R + \Lambda g_{ab} = 8\pi G T_{ab}$$

# Diffeomorphism invariance

Consider a general coordinate transformation

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

We need our theories to be diffeomorphism invariant. This is a surprisingly tough constraint on what we can use.

For some covariant derivative, *D*, metrics transform as

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x) = g_{\mu\nu}(x) + 2 g_{\lambda(\alpha} D_{\beta)} \xi^{\lambda} + \xi \cdot D g_{\alpha\beta} \,.$$

So metric perturbations transform as

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + 2 g_{\lambda(\alpha} D_{\beta)} \xi^{\lambda} + \xi \cdot D g_{\alpha\beta} \,.$$

An advantage of manifest diffeomorphism invariance is that we automatically know that our results are not artefacts of some chosen coordinate system.

# Generalization to gravity

The generalization of the Polchinski flow equation to gravity is

$$\dot{S} = \int_x \frac{\delta S}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)} - \int_x \frac{\delta}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)},$$

The kernel, which transforms as a two-argument generalization of a tensor, is

$$K_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\sqrt{g}} \delta(x-y) \left( g_{\mu(\rho} g_{\sigma)\nu} + j g_{\mu\nu} g_{\rho\sigma} \right) \dot{\Delta}.$$

The de Witt supermetric parameter, *j*, determines the balance of modes propagating in the flow equation. For the "kinetic term" to be a regularized Einstein-Hilbert form, *j* = -1/2.

The effective action then goes like $S = \frac{1}{16\pi G} \int_x \sqrt{g} \left( -R + R_{\mu\nu} \frac{d}{\Lambda^2} R^{\mu\nu} - \frac{1}{2\Lambda^2} R \frac{d}{\Lambda^2} R + \cdots \right),$

where *d* is a function of covariant derivatives that is related to the inverse cutoff.

Writing the action as a loopwise expansion: $S = \frac{1}{\tilde{\kappa}} S_0 + S_1 + \tilde{\kappa} S_2 + \tilde{\kappa}^2 S_3 \cdots$

The *β*-function also expands as $\beta := \Lambda \partial_\Lambda \tilde{\kappa} = \beta_1 + \beta_2 \tilde{\kappa} + \beta_3 \tilde{\kappa}^2 + \cdots$

For this action, we have $\dot{\Delta} = -\frac{2}{\Lambda^2} c' \left( -\frac{\nabla^2}{\Lambda^2} \right),$

which is related to the "effective propagator", Δ, in the fixed-background description.

# Fixed-background description

If we fix a Euclidean background metric, we can define the graviton field as the perturbation to that background:

$$h_{\mu\nu}(x) := g_{\mu\nu}(x) - \delta_{\mu\nu}.$$

The position representation is related to a momentum representation via a Fourier transform:

$$h_{\mu\nu}(x) = \int đp\, \mathrm{e}^{-ip\cdot x} h_{\mu\nu}(p), \quad \text{where} \quad đp := \frac{d^D p}{(2\pi)^D}.$$

The action is defined as a series expansion in the perturbation:

$$\begin{aligned} S \quad = \quad & \int đp\, \delta(p) \mathcal{S}^{\mu\nu}(p) h_{\mu\nu}(p) + \frac{1}{2} \int đp\, đq\, \delta(p+q) \mathcal{S}^{\mu\nu\rho\sigma}(p,q) h_{\mu\nu}(p) h_{\rho\sigma}(q) \\ & + \frac{1}{3!} \int đp\, đq\, đr\, \delta(p+q+r) \mathcal{S}^{\mu\nu\rho\sigma\alpha\beta}(p,q,r) h_{\mu\nu}(p) h_{\rho\sigma}(q) h_{\alpha\beta}(r) + \cdots \end{aligned}$$

In this picture, we are able to define an "effective propagator", $\Delta := \frac{1}{p^2} c\left(\frac{p^2}{\Lambda^2}\right).$

Gravity does not have a unique transverse 2-point function because there are two linearly independent transverse projectors. If we choose the linear combination with Einstein-Hilbert structure, the 2-point function is simply that transverse projector times Δ. This then solves the flow equation at the 2-point level.

# Parisi-Sourlas formalism (SUSY in condensed matter)

Consider a D-dimensional system of spins with a random external field, h. We can write the free energy, using "natural" units of $k_B T$=1, as

$$F[h] = ln \int \mathcal{D}\phi \; exp \left\{ - \int d^D x [\mathcal{L}(x) + h(x)\phi(x)] \right\}$$

where the Lagrangian density is

$$\mathcal{L}(x) = -\frac{1}{2}\phi(x)\partial^2\phi(x) + V(\phi(x))$$

The averaged 2-point function goes like

$$\langle \phi(x)\phi(0) \rangle_R \sim \int \mathcal{D}h \; \phi_h(x)\phi_h(0) exp \left( -\frac{1}{2} \int d^D y h^2(y) \right)$$

where $\phi_h$ is the solution to the field equation

$$-\partial^2\phi + V'(\phi) + h = 0$$

# Equivalence to SUSY

A procedure similar to Fadeev-Popov ghosts rewrites the 2-point function to

$$\langle\phi(x)\phi(0)\rangle \sim \int \mathcal{D}\phi\mathcal{D}\omega\mathcal{D}\psi\mathcal{D}\bar{\psi}\ exp\left[-\int d^D y \mathcal{L}_R(y)\right]\phi(x)\phi(0)$$

where ψ is an anti-commuting (ghost) scalar field and the averaged Lagrangian is

$$\mathcal{L}_R = -\frac{1}{2}\omega^2 + \omega[-\partial^2\phi + V'(\phi)] + \bar{\psi}[-\partial^2 + V''(\phi)]\psi$$

which is supersymmetric under

$$\delta\phi = -\bar{a}\epsilon_\mu x_\mu \psi, \qquad \delta\omega = 2\bar{a}\epsilon_\mu\partial_\mu\psi,$$
$$\delta\psi = 0, \qquad \delta\bar{\psi} = \bar{a}(\epsilon_\mu x_\mu \omega + 2\epsilon_\mu\partial_\mu\phi),$$

where $\bar{a}$ is an infinitesimal, anti-commuting number and $\varepsilon_\mu$ is an arbitrary vector.

# Dimensional reduction

The action, written in a manifestly supersymmetric form, is

$$S = \int d^D x d\bar{\theta} d\theta \left( -\frac{1}{2}\Phi \left[ \partial^2 + \frac{\partial^2}{\partial\bar{\theta}\partial\theta} \right] \Phi + V(\Phi) \right)$$

where θ is a Grassmann number and Φ is the superfield

$$\Phi(x,\theta) = \phi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \theta\bar{\theta}\omega(x)$$

This system is equivalent to a non-supersymmetric system with dimension D-2 via the relation

$$\int d^{D-2}x f(x^2) = \int d^D x d\theta d\bar{\theta} f(x^2 + \bar{\theta}\theta)$$

Green's functions at all orders of perturbation theory for the non-SUSY D-2 system are the same as for the SUSY D-dimensional system. Thus a random external field effectively reduces the dimension by 2.

# SUSY in gauge and gravity ERG

The manifestly gauge invariant ERG for *SU(N)* requires additional regularization at 1-loop level using covariant Pauli-Villars fields. The elegant way to do this is with *SU(N|N)* regularization.

The field is promoted to a supermatrix of bosonic components, *A*, and fermionic components, *B*:

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} + \mathcal{A}^0_\mu \mathcal{I}, \quad \text{and} \quad D_\mu = \partial_\mu - i\mathcal{A}_\mu .$$

The action is built in a similar way:

$$S = \frac{1}{4g^2} str \int \mathcal{F}_{\mu\nu} c^{-1}\left(-\frac{D^2}{\Lambda^2}\right)\mathcal{F}_{\mu\nu} + \cdots$$

$$\text{where} \quad str \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = tr_1 X^{11} - tr_2 X^{22} .$$

This supersymmetry is spontaneously broken by a super-Higgs mechaism with a mass at order *Λ*, so that the physical *SU(N)* can be recovered at low energy.

A similar procedure is required for the 1-loop manifestly diffeomorphism invariant ERG for gravity.

# Summary

- Supersymmetry is broader than just a property of BSM physics.
- SUSY appears in statistical field theory in the Parisi-Sourlas formalism for spin-glasses, where it is related to dimensional reduction by 2.
- A related form of SUSY appears as a regularization mechanism in the manifestly gauge invariant ERG and is expected to be useful in the diffeomorphism invariant case also.