Analysis of Peak Power Efficiency and Droop from a Nonuniform Transmission Line*

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***Abstract*—A mathematical expression of the output voltage from a nonuniform transmission line (NTL) with an arbitrary input pulse was deduced. Due to this mathematical expression, two transmission characteristics of NTLs with linear, exponential and Gaussian impedance profiles were further clarified. The first one is that the peak power efficiencies of NTLs with a half-sine input voltage are quantified as functions of Ψ (the ratio of the output impedance to the input impedance of the NTLs) and Γ (the ratio of the pulse width to the one-way transit time of the NTLs). The second one is that the top of an initially rectangle input voltage pulse falls at the terminal of the NTL and that the ratio of the droop to the top of the output voltage is also quantified as a function of Ψ and Γ.**

Keywords—nonuniform transmission line, analytical method, peak power efficiency, droop

I. Introduction

In the past decade, a number of architectures have been proposed for the design of future pulsed power Z-pinch drivers [1-3]. In these architectures nonuniform transmission lines (NTLs) were used as water-insulated radial-transmission-line impedance transformers to combine the outputs of several-hundred terawatt-level pulse generators to produce a petawatt-level pulse. In order to know the transmission characteristics of NTLs with different impedance profiles, studies were made with analytical method [4, 5], numerical simulation [6-9] and experiment [10]. Analytical method is the best way to investigate the transmission characteristics of NTLs because it is presented as mathematical expressions which offer a clear view into how variables affect the result [5]. However, the characteristic impedance varies along the NTL, which brings much difficulty to investigate the transmission characteristics using analytical method, especially in time domain. Thus most researchers dealt with NTLs in frequency domain (steady-state) except for limited cases in which the transient behavior of NTLs in time domain was studied. Hsue and his colleague investigated the step response of a cascaded multiple-section line and gave a mathematical expression of the output voltage at the load end [4]. They focused on the first arriving wave due to the internal transmission-reflection components being a rather complicated function to analyze. Two years ago, we deduced a mathematical expression of the output voltage from NTLs in time domain and clarified the high-pass and pulse-compression characteristics of NTLs using theoretical analysis of the analytical expressions [5]. However, that mathematical expression was only for an input voltage of half-sine shape. Moreover, other transmission characteristics of NTLs also need to be investigated. For example, the peak power efficiencies of NTLs with linear, exponential and Gaussian impedance profiles with a half-sine input voltage are quantified as functions of Ψ (the ratio of the output impedance to the input impedance of the NTLs) and Γ (the ratio of the pulse width to the one-way transit time of the NTLs) [7], and the top of an initially rectangle input voltage pulse falls at the output terminal of the NTL [11].

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In this paper, we extended the mathematical expression of the output voltage proposed two years ago [5] and got a similar one which is for NTLs with an arbitrary input voltage pulse. Then the two transmission characteristics mentioned above were investigated by the analytical analysis of the new mathematical expression.

II. Output Voltage from an NTL

A. Mathematical Expression of the Output Voltage

In [5], we considered the lossless NTL as a cascaded multiple section line where all the sections have the same length, as shown in Fig. 1. In this paper, we will use the same model. If the length of each section is short enough, the *i*th section can be assumed uniform and characterized by its characteristic impedance Z_i(i=0,1,2,…,m). So the one-way transit time of the NTL is given by

$$T_{\text{line}} = (m+1)\cdot\Delta t \tag{1}$$

where Δt is the one-way transit time of each section.

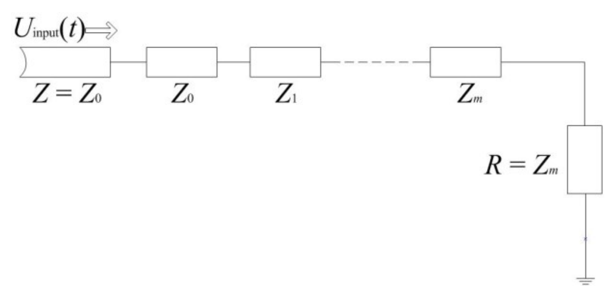

Fig. 1. Cascaded multiple-section line with impedance-matching terminals.

Since we are interested in the transmission characteristics of the NTL itself, the impedance should be matched at the both terminals of the NTL. A semi-infinite uniform transmission line with an characteristic impedance of Z_0 is inserted in between the pulse generator and the input terminal of the NTL. And a resistor $R=Z_m$ is used as the load connecting at the output terminal of the NTL.

The only difference from the model in [5] is that the input voltage wave is not restricted to a half-sine shape. It can be an arbitrary pulse such as a rectangle wave. It is given by

$$U_{\text{input}}(t)=\begin{cases}0, & t<0\\ f(t), & 0\le t\le t_{\text{end}}\\ 0, & t>t_{\text{end}}\end{cases} \tag{2}$$

where t_{end} is the width of the input voltage wave and $f(t)$ is an arbitrary function.

Using the same method as in [5], we can obtain the mathematical expression of the output voltage

$$U_{\text{output}}(t+T_{\text{line}})=\rho_0\cdot U_{\text{input}}(t)+\sum_{i=1}^{l}\rho_0\rho_i\cdot U_{\text{input}}(t-2i\cdot\Delta t) \tag{3}$$

where l is the largest integer that is not larger than $t/(2\Delta t)$ or $(m-1)$. ρ_0 and $\rho_i(i=1,2,\ldots,l)$ are the coefficients determined only by the characteristic impedances of the line sections and are given by

$$\rho_0=\prod_{j=1}^{m}\frac{2Z_j}{Z_{j-1}+Z_j}>0 \tag{4}$$

$$\rho_1=\sum_{j=1}^{m-1}\left(\frac{Z_{j+1}-Z_j}{Z_{j+1}+Z_j}\cdot\frac{Z_{j-1}-Z_j}{Z_{j-1}+Z_j}\right)<0 \tag{5}$$

$$\rho_i=\sum_{j=1}^{m-i}\left[\frac{Z_{j+i}-Z_{j+i-1}}{Z_{j+i}+Z_{j+i-1}}\cdot\frac{Z_{j-1}-Z_j}{Z_{j-1}+Z_j}\cdot\prod_{k=j}^{j+i-2}\frac{4Z_kZ_{k+1}}{(Z_k+Z_{k+1})^2}\right]<0,\ i=2,3,\cdots \tag{6}$$

B. Variables in the Mathematical Expression

Now we will look into all variables in (3) to see what makes a difference to $U_{\text{output}}(t+T_{\text{line}})$.

1) $U_{\text{input}}(t-2i\cdot\Delta t)$ is determined by $U_{\text{input}}(t)$. $U_{\text{input}}(t)$ and $U_{\text{input}}(t-2i\cdot\Delta t)$ represent that the input pulse has an influence to the output voltage. It is easy to understand this.

2) We will focus on l. Due to (1), we can obtain

$$\frac{t}{2\Delta t}=\frac{t}{2\times\frac{T_{\text{line}}}{m+1}}=\frac{(m+1)t}{2T_{\text{line}}} \tag{7}$$

To ensure that (3) is correct, the length of each line section should be short enough, which means $\Delta t\to 0$ and $m\to+\infty$. Thus m is not a variable for $U_{\text{output}}(t+T_{\text{line}})$. In this case

$$\lim_{m\to+\infty}\frac{\frac{t}{2\Delta t}}{m-1}=\lim_{m\to+\infty}\frac{(m+1)}{2(m-1)}\cdot\frac{t}{T_{\text{line}}}=\frac{t}{2T_{\text{line}}} \tag{8}$$

Thus when $t\le 2T_{\text{line}}$, l is the largest integer that is not larger than $t/(2\Delta t)$, which equals to $(m+1)t/(2T_{\text{line}})$. When $t>2T_{\text{line}}$, l is the largest integer that is not larger than $(m-1)$. So for any selected time t, l is determined only by T_{line}.

3) From (4)-(6), we can see that if all $Z_i(i=0,1,2,\ldots,m)$ are multiplied by the same coefficient, $\rho_i(i=0,1,2,\ldots,l)$ remain the same. Thus $U_{\text{output}}(t+T_{\text{line}})$ remains the same. Particularly, for NTLs with linear, exponential and Gaussian [11] impedance profiles, when Ψ remains the same, $Z_i(i=0,1,2,\ldots,m)$ are all proportional to the input impedance Z_0. Thus $\rho_i(i=0,1,2,\ldots,l)$ are determined only by Ψ for these NTLs.

In conclusion, for any selected time t, the output voltage $U_{\text{output}}(t+T_{\text{line}})$ is determined by $U_{\text{input}}(t)$, T_{line}, and $\rho_i(i=0,1,2,\ldots,l)$. For NTLs with linear, exponential and Gaussian impedance profiles, $\rho_i(i=0,1,2,\ldots,l)$ are determined only by Ψ.

III. Peak Power Efficiency of an NTL

Hu and his colleagues found that the peak power efficiencies of NTLs with linear, exponential and Gaussian impedance profiles with a half-sine input voltage pulse are quantified as functions of $\Psi(\Psi=Z_m/Z_0)$ and $\Gamma(\Gamma=t_{\text{end}}/T_{\text{line}})$ using a 1-D circuit model [7]. In this section, we will clarify this transmission characteristic using analytical method.

The peak power efficiency of an NTL is defined by

$$\eta=\left[\frac{(U_{\text{output}})_{\max}}{(U_{\text{input}})_{\max}}\right]^2\cdot\frac{1}{\Psi} \tag{9}$$

where $(U_{\text{output}})_{\max}$ is the amplitude of $U_{\text{output}}(t+T_{\text{line}})$ and $(U_{\text{input}})_{\max}$ is the amplitude of $U_{\text{input}}(t)$. As the NTL is a linear system, $U_{\text{output}}(t+T_{\text{line}})$is proportional to $(U_{\text{input}})_{\max}$. So $(U_{\text{input}})_{\max}$ should be assumed to remain the same. We suppose that there are two NTLs systems named System A and System B. We need to clarify when the two systems have the same Γ and Ψ, they have the same $(U_{\text{output}})_{\max}$.

1) We assume that Γ, Ψ, t_{end} and T_{line} of the two systems are the same. Z_m and Z_0 are different. A half-sine input voltage pulse can be determined totally by $(U_{\text{input}})_{\max}$ and t_{end}, so it is also the same in the two systems. Due to Section II *B. 2)*, we can know that for any selected time t, l is the same. Due to Section II *B. 3)*, we can know $\rho_i(i=0,1,2,\ldots,l)$ are the same. As all variables in (3) are the same, $U_{\text{output}}(t+T_{\text{line}})$ is the same. Thus $(U_{\text{output}})_{\max}$ is the same.

2) We assume that Γ, Ψ, Z_m and Z_0 of the two systems are the same. t_{end} and T_{line} are different. We assume that the input voltage waves of the two systems are respectively

$$U_{\text{inputA}}(t)=\begin{cases}0, & t<0\\ \sin\omega_{\text{A}}t, & 0\le t\le t_{\text{endA}}\\ 0, & t>t_{\text{endA}}\end{cases} \tag{10}$$

$$U_{\text{inputB}}(t)=\begin{cases}0, & t<0\\ \sin\omega_{\text{B}}t, & 0\le t\le t_{\text{endB}}\\ 0, & t>t_{\text{endB}}\end{cases} \tag{11}$$

It is obvious that

$$\omega_{\text{A}}t_{\text{endA}}=\omega_{\text{B}}t_{\text{endB}}=\pi \tag{12}$$

$$\frac{t_{\text{endA}}}{T_{\text{lineA}}}=\Gamma=\frac{t_{\text{endB}}}{T_{\text{lineB}}} \tag{13}$$

Thus

$$\omega_{\text{A}}T_{\text{lineA}}=\omega_{\text{B}}T_{\text{lineB}} \tag{14}$$

Due to the pulse-compression characteristic of NTLs in [5], the maxima of $U_{\text{outputA}}(t+T_{\text{lineA}})$ and $U_{\text{outputB}}(t+T_{\text{lineB}})$ occur at $t_{m\text{A}}\in\left(0,\pi/(2\omega_{\text{A}})\right)$ and $t_{m\text{B}}\in\left(0,\pi/(2\omega_{\text{B}})\right)$, respectively. For any a time instant $t_{\text{A}}\in\left(0,\pi/(2\omega_{\text{A}})\right)$, there must exist a time instant $t_{\text{B}}\in\left(0,\pi/(2\omega_{\text{B}})\right)$ that satisfies the following relation

$$\omega_{\text{A}}t_{\text{A}}=\omega_{\text{B}}t_{\text{B}} \tag{15}$$

Due to (3), we obtain

$$U_{\text{outputA}}(t_{\text{A}}+T_{\text{lineA}})=\rho_0\cdot\sin\omega_{\text{A}}t_{\text{A}}+\sum_{i=1}^{l_A}\rho_0\rho_i\cdot\sin\omega_{\text{A}}\left(t_{\text{A}}-2i\cdot\Delta t_{\text{A}}\right) \tag{16}$$

$$U_{\text{outputB}}(t_{\text{B}}+T_{\text{lineB}})=\rho_0\cdot\sin\omega_{\text{B}}t_{\text{B}}+\sum_{i=1}^{l_B}\rho_0\rho_i\cdot\sin\omega_{\text{B}}\left(t_{\text{B}}-2i\cdot\Delta t_{\text{B}}\right) \tag{17}$$

Where l_{A} is the largest integer that is not larger than $t_{\text{A}}/(2\Delta t_{\text{A}})$ or $(m-1)$, and l_{B} is the largest integer that is not larger than $t_{\text{B}}/(2\Delta t_{\text{B}})$ or $(m-1)$.

Due to (1) and (14), we obtain

$$\omega_{\text{A}}\Delta t_{\text{A}}=\frac{\omega_{\text{A}}T_{\text{lineA}}}{m+1}=\frac{\omega_{\text{B}}T_{\text{lineB}}}{m+1}=\omega_{\text{B}}\Delta t_{\text{B}} \tag{18}$$

Due to (15) and (18), we obtain

$$\frac{t_{\text{A}}}{\Delta t_{\text{A}}}=\frac{\omega_{\text{B}}t_{\text{B}}/\omega_{\text{A}}}{\omega_{\text{B}}\Delta t_{\text{B}}/\omega_{\text{A}}}=\frac{t_{\text{B}}}{\Delta t_{\text{B}}} \tag{19}$$

Thus

$$l_{\text{A}}=l_{\text{B}} \tag{20}$$

Upon substitution of (15), (18) and (20) into (16) and (17), we obtain

$$U_{\text{outputA}}(t_{\text{A}}+T_{\text{lineA}})=U_{\text{outputB}}(t_{\text{B}}+T_{\text{lineB}}) \tag{21}$$

Equation (21) means that for any a time instant $t_{\text{A}}\in\left(0,\pi/(2\omega_{\text{A}})\right)$, there exist a time instant $t_{\text{B}}\in\left(0,\pi/(2\omega_{\text{B}})\right)$ which makes

$$U_{\text{outputA}}(t_{\text{A}}+T_{\text{lineA}})=U_{\text{outputB}}(t_{\text{B}}+T_{\text{lineB}})\le\left(U_{\text{outputB}}\right)_{\max} \tag{22}$$

Thus

$$\left(U_{\text{outputA}}\right)_{\max}\le\left(U_{\text{outputB}}\right)_{\max} \tag{23}$$

Using the same method, we can obtain

$$\left(U_{\text{outputB}}\right)_{\max}\le\left(U_{\text{outputA}}\right)_{\max} \tag{24}$$

Due to (23) and (24), we obtain

$$\left(U_{\text{outputA}}\right)_{\max}=\left(U_{\text{outputB}}\right)_{\max} \tag{25}$$

Thus System A and System B have the same $(U_{\text{output}})_{\max}$.

3) We assume that Γ and Ψ are the same. Z_m, Z_0, t_{end} and T_{line} are different. We suppose a new system named System C. System C's t_{end} and T_{line} are the same with those of System A, while its Z_m and Z_0 are the same with those of System B. Obviously, System C's Γ and Ψ are the same with those of System A and B. Then we can know that $(U_{\text{output}})_{\max}$ of System C is the same with that of System A based on *1)*, and that it is the same with that of System B based on *2)*. So $(U_{\text{output}})_{\max}$ of System A is the same with that of System B.

We can conclude from *1)*, *2)* and *3)* that η of NTLs with linear, exponential and Gaussian impedance profiles with a half-sine input voltage pulse are quantified as functions of Ψ and Γ.

IV. Droop of the Output Voltage Wave from an NTL

The output voltage wave of an NTL with a rectangle input voltage pulse is not a rectangle wave [11]. The top falls as shown in Figure 2. $U_{\text{output}}(t+T_{\text{line}})$ is lower than $U_{\text{output}}(T_{\text{line}})$, which is called the droop. In this section, we will clarify that the ratio of the droop to the top $\left[U_{\text{output}}(T_{\text{line}})-U_{\text{output}}(t_{\text{end}}+T_{\text{line}})\right]/U_{\text{output}}(T_{\text{line}})$ is quantified only as a function of Ψ and Γ for NTLs with linear, exponential and Gaussian impedance profiles.

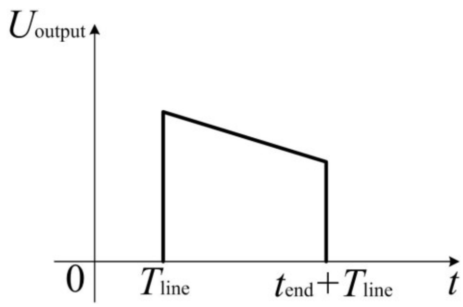

Fig. 2. Droop of the output voltage wave from an NTL with a rectangle input voltage pulse.

Due to (2) and (3), we can obtain

$$U_{\text{output}}(T_{\text{line}}) = \rho_0 \cdot f(t) + \sum_{i=1}^{l_0} \rho_0 \rho_i \cdot f(0 - 2i \cdot \Delta t) \tag{26}$$

$$U_{\text{output}}(t_{\text{end}} + T_{\text{line}}) = \rho_0 \cdot f(t_{\text{end}}) + \sum_{i=1}^{l_{\text{end}}} \rho_0 \rho_i \cdot f(t_{\text{end}} - 2i \cdot \Delta t) \tag{27}$$

where l_0 is the largest integer that is not larger than $0/(2\Delta t)$ or $(m\text{-}1)$, so $l_0=0$. l_{end} is the largest integer that is not larger than $t_{\text{end}}/(2\Delta t)$ or $(m\text{-}1)$. For a rectangle input voltage pulse, $f(t)$ is a constant in (2). So we can define $f(t)=U$. Thus (26) and (27) are

$$U_{\text{output}}(T_{\text{line}}) = \rho_0 U \tag{28}$$

$$U_{\text{output}}(t_{\text{end}} + T_{\text{line}}) = \rho_0 U + \sum_{i=1}^{l_{\text{end}}} \rho_0 \rho_i U \tag{29}$$

We define $h(m)$ is the ratio of the droop to the top when the number of line sections is $(m+1)$. So we just need to calculate

$$\begin{aligned} \lim_{m\to+\infty} h(m) &= \lim_{m\to+\infty} \frac{\left[U_{\text{output}}(T_{\text{line}}) - U_{\text{output}}(t_{\text{end}} + T_{\text{line}})\right]}{U_{\text{output}}(T_{\text{line}})} \\ &= \lim_{m\to+\infty} \frac{\rho_0 U - \left(\rho_0 U + \sum_{i=1}^{l_{\text{end}}} \rho_0 \rho_i U\right)}{\rho_0 U} = -\lim_{m\to+\infty} \sum_{i=1}^{l_{\text{end}}} \rho_i \end{aligned} \tag{30}$$

From Section II, we can know that $\rho_i(i=1,2,\ldots,l_{\text{end}})$ are determined only by Ψ for NTLs with linear, exponential and Gaussian impedance profiles.

l_{end} is the largest integer that is not larger than $t_{\text{end}}/(2\Delta t)$ or $(m\text{-}1)$. Due to (7), we obtain

$$\frac{t_{\text{end}}}{2\Delta t} = \frac{(m+1)t_{\text{end}}}{2T_{\text{line}}} = \frac{(m+1)\Gamma}{2} \tag{31}$$

$$\lim_{m\to+\infty} \frac{\frac{t_{\text{end}}}{2\Delta t}}{m-1} = \lim_{m\to+\infty} \frac{(m+1)\Gamma}{2(m-1)} = \frac{\Gamma}{2} \tag{32}$$

Thus when $\Gamma \leq 2$, l_{end} is the largest integer that is not larger than $t_{\text{end}}/(2\Delta t)$, which equals to $(m+1)\Gamma/2$. When $\Gamma>2$, l_{end} is the largest integer that is not larger than $(m\text{-}1)$. So l_{end} is only determined by Γ and m. Thus from (30) we can know that $\lim_{m\to+\infty} h(m)$ is only qualified as a function of Ψ and Γ, which is right the conclusion at the beginning of this section.

V. Conclusions

The output voltage from an NTL with an arbitrary input voltage pulse can be calculated by a mathematical expression in time domain. The peak power efficiency and droop of the output voltage from the NTL can also be investigated using this mathematical expression.

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