Complex Langevin in Lattice QCD: dynamic stabilisation and the phase diagram

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SCIENCE WITHOUT BORDERS

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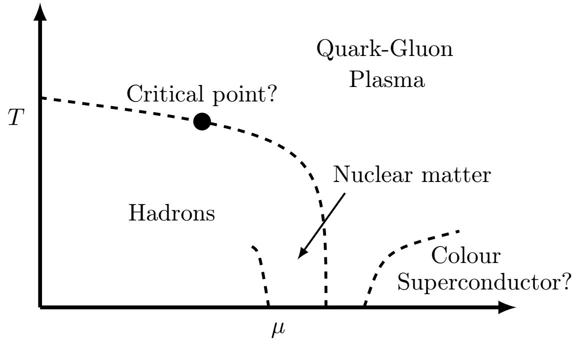
Goal: QCD phase diagram



- Still largely unknown
- Full thermodynamical study of QCD (phase transitions, etc)
- Applications in cosmology (e.g. neutron stars, early universe)
- Possible guide for heavy-ion collision experiments

CLE: Motivation

- *Sign problem*: Inclusion of a chemical potential to an Euclidean path integral makes the action complex
- Average values for observables then rely on precise cancellations of oscillating quantities
- In QCD this is manifested in the fermion determinant

$$[\det M(U, \mu)]^* = \det M(U, -\mu^*)$$

  which is, for real chemical potential μ, complex
- Results from Hybrid Monte-Carlo simulations become unreliable if $\mu/T \gtrsim 1$, where the sign problem is severe

CLE: Stochastic quantization I

In the continuum [Damgaard and Hüffel, Physics Reports]

- Add fictitious time dimension θ to gauge fields
- Evolve them according to a Langevin equation

$$\frac{\partial A_\mu^a(x,\theta)}{\partial\theta} = -\frac{\delta S}{\delta A_\mu^a(x,\theta)} + \eta_\mu^a(x,\theta)\,,$$

where S is the QCD action and $\eta_\mu^a(x)$ are white noise fields satisfying

$$\langle\eta_\mu^a(x)\rangle = 0\,, \quad \langle\eta_\mu^a(x)\eta_\nu^b(y)\rangle = 2\delta^{ab}\delta_{\mu\nu}\delta(x-y)\,,$$

- Quantum expectation values are computed as averages over the Langevin time θ after the system reaches equilibrium at $\theta = T_{therm}$

$$\langle O\rangle = \lim_{T\to\infty}\frac{1}{T-T_{therm}}\int_{T_{therm}}^{T} O(\theta)d\theta$$

CLE: Stochastic quantization II

On the lattice

[Damgaard and Hüffel, Physics Reports]

- Evolve gauge links according to the Langevin equations

$$U_{x\mu}(\theta+\varepsilon) = \exp\left[X_{x\mu}\right] U_{x\mu}(\theta)\,,$$

where

$$X_{x\mu} = i\lambda^a(\varepsilon D^a_{x\mu} S\left[U(\theta)\right] + \sqrt{\varepsilon}\,\eta^a_{x\mu}(\theta))\,,$$

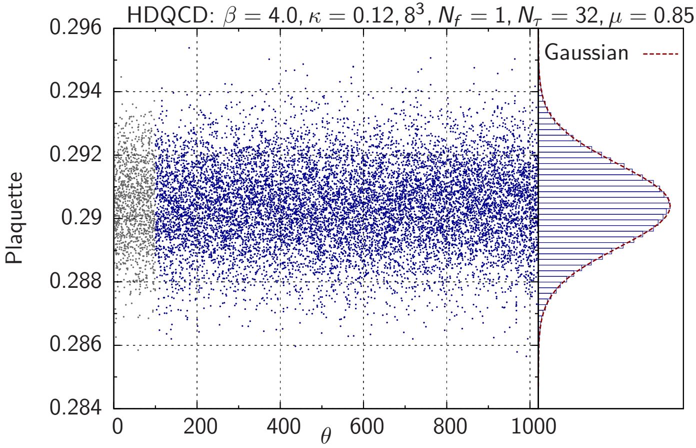
λ^a are the Gell-Mann matrices, ε is the stepsize, $\eta^a_{x\mu}$ are white noise fields satisfying

$$\langle \eta^a_{x\mu} \rangle = 0\,, \quad \langle \eta^a_{x\mu} \eta^b_{y\nu} \rangle = 2\delta^{ab}\delta_{xy}\delta_{\mu\nu}\,,$$

S is the QCD action and $D^a_{x\mu}$ is defined as

$$D^a_{x\mu} f(U) = \frac{\partial}{\partial\alpha} f(e^{i\alpha\lambda^a} U_{x\mu})\Big|_{\alpha=0}$$

CLE: Stochastic quantization II



Typical evolution of observable as function of the Langevin time.

CLE: Complexification I

Complexification

[Aarts, Stamatescu, hep-lat/0807.1597]

- Allow gauge fields to be complex, i.e., $\mathbb{R} \ni A_\mu^a(x) \to A_\mu^a(x) \in \mathbb{C}$
- On the lattice this means SU$(3) \ni U_{x\mu} \to U_{x\mu} \in$ SL$(3, \mathbb{C})$
- Use $U_{x\mu}^{-1}$ instead of $U_{x\mu}^\dagger$ as
  - it keeps the action holomorphic;
  - they coincide on SU(3) but on SL$(3, \mathbb{C})$ it is U^{-1} that represents the backwards-pointing link.
- That means the plaquette is now

$$U_{x,\mu\nu} = U_{x\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^{-1} U_{x,\nu}^{-1}\,,$$

and the Wilson action reads

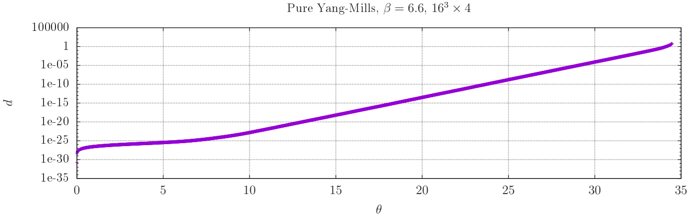
$$S[U] = -\frac{\beta}{3} \sum_x \sum_{\mu<\nu} \mathbf{Tr} \left[\frac{1}{2} \left(U_{x,\mu\nu} + U_{x,\mu\nu}^{-1} \right) - \mathbb{1} \right]$$

CLE: Complexification II

- The $\mathrm{SL}(3,\mathbb{C})$ group is a non-compact manifold, which means the gauge links can get arbitrarily far from $\mathrm{SU}(3)$
- During simulations monitor the distance from the unitary manifold with

$$d = \frac{1}{\Omega}\mathbf{Tr}\sum_{x,\mu}\left[U_{x\mu}U_{x\mu}^{\dagger} - \mathbb{1}\right] \geq 0$$

- d can increase because of the fermion determinant but also from numerical imprecisions – therefore it is necessary to keep it under control without violating the original $\mathrm{SU}(3)$ gauge invariance



CLE: Complexification - Gauge cooling

Gauge cooling

[Seiler, Sexty, Stamatescu, hep-lat/1211.3709]

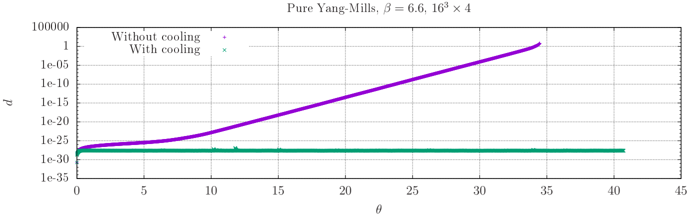
- Use gauge transformations to keep the system as close as possible to SU(3), i.e., minimise the imaginary part of $A^a_\mu(x)$

$$U_{x\mu} \to \Lambda_x U_{x\mu} \Lambda^{-1}_{x+\mu}\,, \quad \Lambda_x = \exp\left[-\varepsilon\alpha\lambda^a f^a_x\right]$$

where

$$f^a_x = 2\mathbf{Tr}\left[\lambda^a\left(U_{x\mu}U^\dagger_{x\mu} - U^\dagger_{x-\mu,\mu}U_{x-\mu,\mu}\right)\right]$$

- The parameter α and the number of cooling steps are chosen adaptively based on the distance d.



Heavy-dense QCD

Heavy-dense approximation

[Aarts, Stamatescu, hep-lat/0807.1597]

- Heavy quarks $\rightarrow$ spatial part of fermion determinant does not contribute, but temporal part is exact ($\kappa \rightarrow 0$):

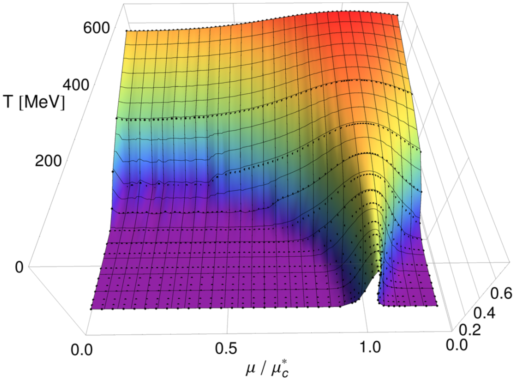
$$\det M(U,\mu) = \prod_{\vec{x}} \left\{ \det \left[1 + (2\kappa e^{\mu})^{N_\tau} \mathcal{P}_{\vec{x}} \right]^2 \times \det \left[1 + \left(2\kappa e^{-\mu}\right)^{N_\tau} \mathcal{P}_{\vec{x}}^{-1} \right]^2 \right\}$$

- Polyakov loop

$$\mathcal{P}_{\vec{x}} = \prod_{\tau} U_4(\vec{x},\tau)$$

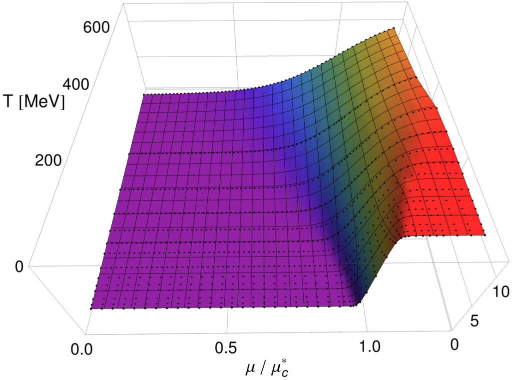
- Exhibits the sign problem: $[\det M(U,\mu)]^* = \det M(U,-\mu^*)$
- $\mu = \mu_c^* \equiv -\ln(2\kappa)$ marks the transition at zero temperature ($N_\tau \rightarrow \infty$)

Polyakov loop in HDQCD at $V = 10^3$, $\beta = 5.8$ and $\kappa = 0.04$



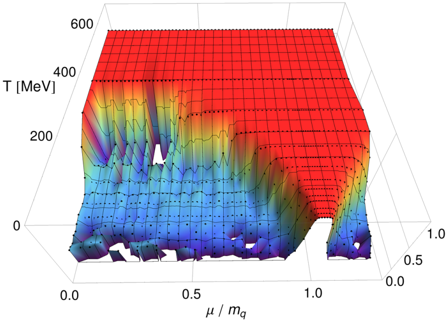
Average Polyakov loop as function of temperature and chemical potential. Regions of confinement ($\langle \mathcal{P} \rangle = 0$) and deconfinement are visible.

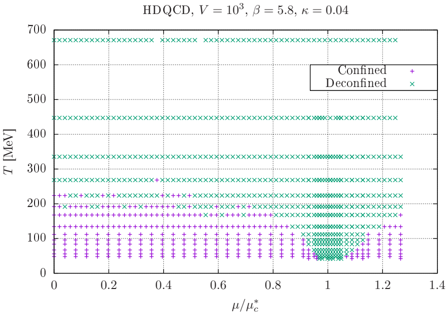
Quark density in HDQCD at $V = 10^3$, $\beta = 5.8$ and $\kappa = 0.04$



Average quark density as function of temperature and chemical potential. Transition to higher densities smoothens with temperature.

Binder cumulant for the Polyakov loop





Binder cumulant: $B = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$

Confinement: $B = 0$

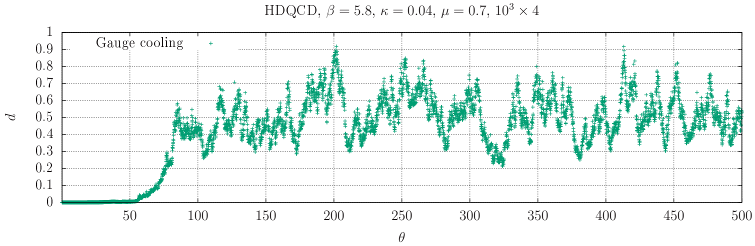
Deconfinement: $B = 2/3$

Dynamic Stabilisation

- *Dynamic stabilisation*: Addition of new term to the drift to keep the system as close as possible to SU(3), i.e., minimise the imaginary part of $A_\mu^a(x)$

$$M_x^a = i b_x^a \left(\sum_c b_x^c b_x^c \right)^2 \, , \quad b_x^a = \mathrm{Tr} \left[\lambda^a \sum_\nu U_{x\nu} U_{x\nu}^\dagger \right] .$$

- M_x^a is irrelevant compared to the drift coming from the action in the continuum limit and, to leading order, depends only on the imaginary part of the gauge fields.



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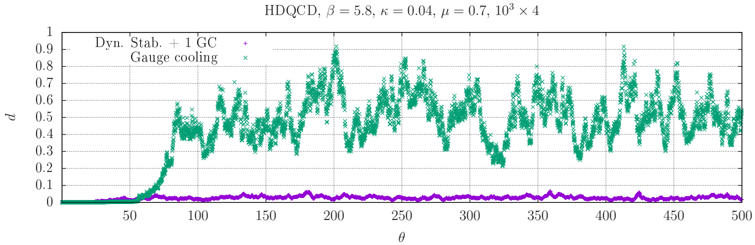

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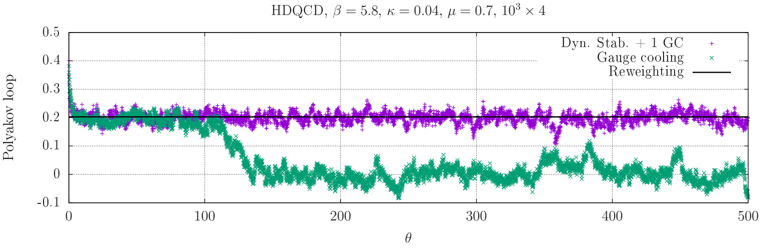
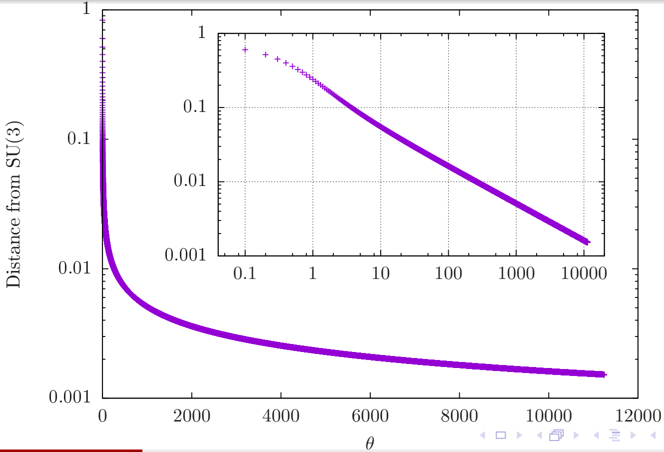


Dynamic stabilisation - Example I

- The new drift, with α being a control coefficient, reads

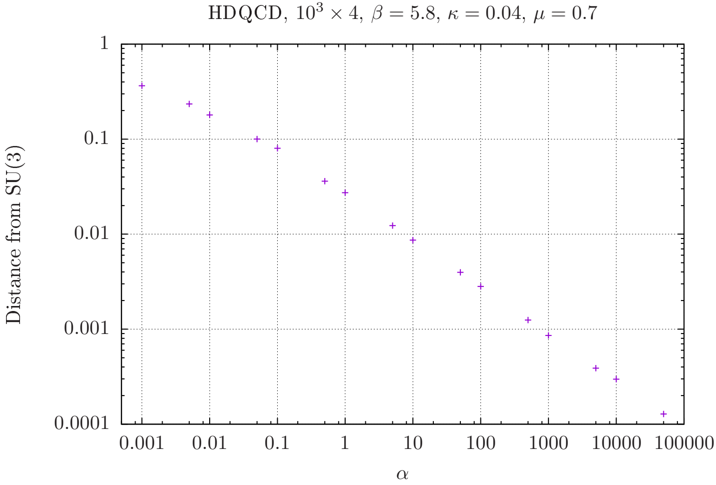
$$X_{x\nu} = i\lambda^a \left(\epsilon D^a_{x,\nu} S + \epsilon \alpha M^a_x + \sqrt{\epsilon}\, \eta^a_{x,\nu} \right) .$$

- Test run with $S = 0$ to see how DS acts of the distance from SU(3)



Dynamic stabilisation - Example II

- α is seen to control how far from the unitary manifold the system is



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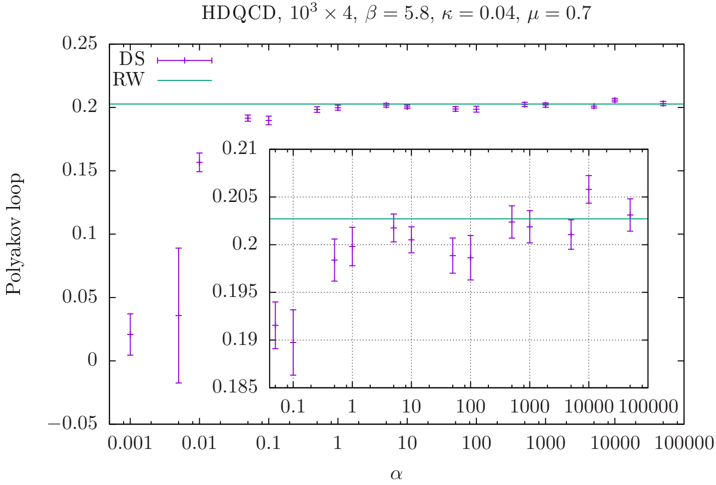

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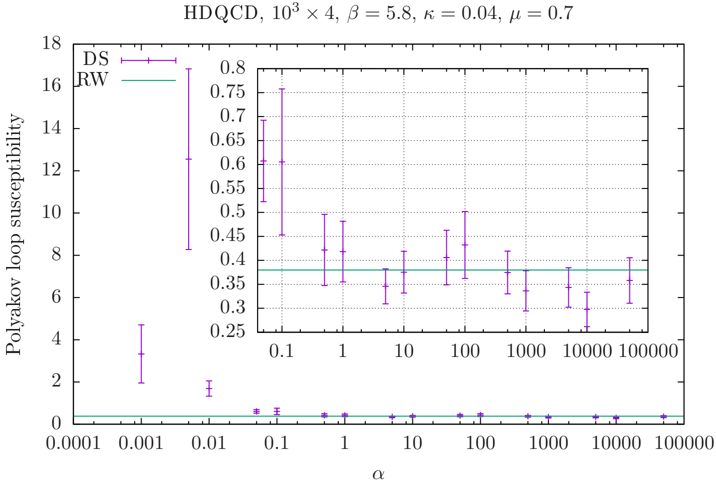


Full QCD with Staggered Quarks

CLE for QCD with Staggered Quarks

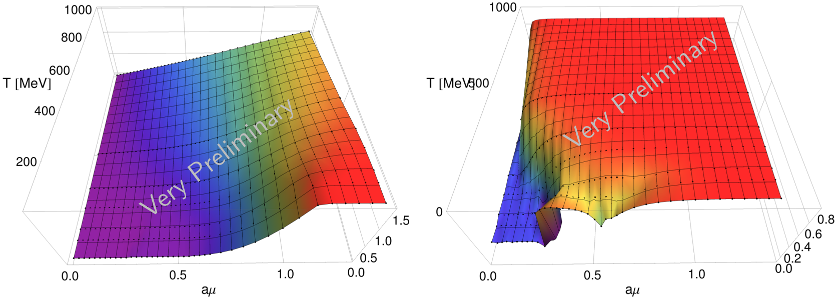
[Sexty, hep-lat/1307.7748]

- Derivative of the fermion determinant is added to the drift

$$X_{x\mu} \to X_{x\mu} + i\frac{N_F}{4}\lambda^a \mathrm{Tr}\left[M^{-1}(U,\mu)D^a_{x\mu}M(U,\mu)\right]$$

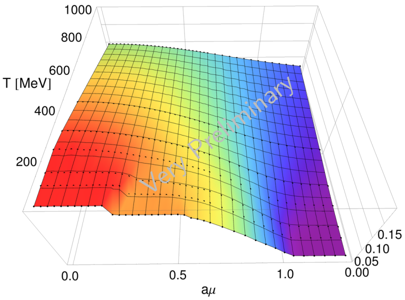
- The inversion is carried out using the conjugate gradient algorithm
- Trace is evaluated employing the bilinear noise scheme

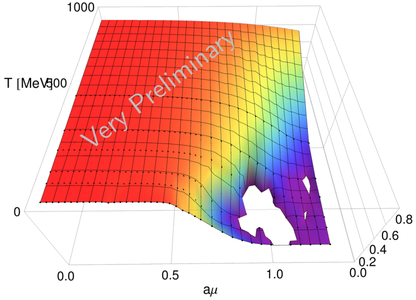
Quark density at $V = 8^3$, $\beta = 6.4$ and $m = 0.05$



Average quark density as function of temperature and chemical potential.
Drop in Binder cumulant under investigation.

Chiral condensate at $V = 8^3$, $\beta = 6.4$ and $m = 0.05$





Average chiral condensate as function of temperature and chemical potential. Deconfined region very evident.


ONE DOES NOT SIMPLY
INVERT THE FERMION MATRIX
imgflip.com

Summary and Outlook

Summary

- Complex Langevin Equation allows simulations of theories that exhibit the sign problem
- CLE + Gauge Cooling gives results consistent with expectations for heavy quarks
- CLE + Dynamic Stabilisation improves control over the distance from $\mathrm{SU}(3)$
- Initial analyses of the QCD phase diagram with heavy and dynamical quarks

Outlook

- Improve issues related to the inversion of the fermion matrix
- Map the vicinity of the phase boundary using dynamical quarks