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Light quark mass differences
in the $\pi - \eta - \eta'$ system.

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Relevance of the $\pi_0 - \eta$ and $\pi_0 - \eta'$ mixing angles $\epsilon$ and $\epsilon'$

- In the determination of light quark mass ratio $\frac{m_u}{m_d}$
- In the decays of $\eta \to 3\pi$ and $\eta' \to 3\pi$
- In the determination of the ratio $\frac{\epsilon}{\epsilon'}|_{CP}$ related to $K^0 \to \pi^0\pi^0$
- Studies of spontaneous breaking of strong $CP$ symmetry.
- Deeper insight in studies of restoration of chiral and $U_A(1)$ symmetries in the QCD phase diagram at finite $T, \mu, B$. 
## Isospin breaking observables

<table>
<thead>
<tr>
<th>Effect</th>
<th>$m_u - m_d$</th>
<th>EM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{\pi^+} - M_{\pi^0} = 4.6$ MeV</td>
<td>subleading</td>
<td>√</td>
</tr>
<tr>
<td>$M_{K^+} - M_{K^0} = -3.99$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$M_{D^+} - M_{D^0} = 4.78$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$\eta \rightarrow 3\pi$</td>
<td>√</td>
<td>subleading</td>
</tr>
<tr>
<td>$M_{B^+} - M_{B^0} = -0.33$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$M_{n} - M_{p} = 1.29$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$M_{\Sigma^-} - M_{\Sigma^0} = 4.8$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$M_{\Sigma^+} - M_{\Sigma^0} = -3.27$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$M_{\Xi^-} - M_{\Xi^0} = 6.48$ MeV</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>$\rho - \omega$ mixing</td>
<td>√</td>
<td>subleading</td>
</tr>
<tr>
<td>$\pi^0 - \eta$, $\pi^0 - \eta'$ mixings</td>
<td>√</td>
<td>subleading</td>
</tr>
<tr>
<td>$F_{\pi^+}/F_{\pi^0}$</td>
<td>subleading</td>
<td>√</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow \gamma\gamma$</td>
<td>√</td>
<td>subleading</td>
</tr>
</tbody>
</table>
Current quark mass values

PDG:2014 At renormalization scale \( \mu = 2 \text{GeV} \).

Mass Ratios: RG invariant.

Lattice data: Isospin symmetric limit \( m_u = m_d = \bar{m} \)

\[ \bar{m} = 3.4 \pm 0.25 \quad m_s = 93.5 \pm 2.5 \text{MeV} \quad \frac{m_s}{\bar{m}} = 27.5 \pm 0.3 \]

Estimations of \( m_u \) and \( m_d \):

Combined Lattice data + isospin breaking effects from ChPT + phenomenology: MILC collaboration (2009), BMW Collaboration (2011), Flavinet Lattice Averaging Group (2011),

\[ m_u = 2.15(15) \text{MeV} \quad m_d = 4.70(20) \text{MeV} \quad \frac{m_u}{m_d} = 0.46(5) \]

Our ratio: \( m_u / m_d \sim 0.46 \) (NLO in \( N_c \))

Moderate correction \(~ 20\% \) to
LO ChPT \( m_u / m_d = 0.56 \) Weinberg:1977
**MAIN ASSUMPTIONS**

1. Scales of nonperturbative QCD:

\[ \Lambda_{QCD} \sim \Lambda_{\text{conf}} < \Lambda_{\chi_{SB}} \approx 4\pi f_{\pi}. \]

In the regime \( \Lambda_{\text{conf}} < \Lambda < \Lambda_{\chi_{SB}} \) the induced effective interaction between quarks is of the form

\[ L = \bar{q} \left( i\gamma^\mu \partial_\mu - m \right) q + \frac{G}{\Lambda^2} \left( \bar{q} \Gamma q \right)^2 + \frac{K}{\Lambda^5} \left( \bar{q} \Gamma q \right)^3 + \ldots, \]

where \( \Lambda \) is determined by the chiral symmetry-breaking scale. If instantons are responsible for multi-quark interactions, then

\[ \Lambda \sim \rho^{-1} \approx \left( 0.33 \text{ fm} \right)^{-1} \]
2. Chiral symmetry restrictions

The color quark fields possess definite transformation properties with respect to the chiral flavor $U(3)_R \otimes U(3)_L$ global symmetry of the QCD Lagrangian with massless quarks.

To study scalar and pseudoscalar modes it is convenient to introduce the $U(3)$ Lie-algebra valued field:

$$\Sigma = (s_a - ip_a) \frac{\lambda^a}{2}, \quad s_a = \bar{q} \lambda_a q, \quad p_a = \bar{q} \lambda_a i \gamma_5 q.$$ 

and the external source $\chi$, which generate the explicit symmetry breaking effects - future mass terms and mass dependent interactions, with the transformation properties:

$$\Sigma' = V_R \Sigma V_L^+, \quad \chi' = V_R \chi V_L^+.$$
MULTI-QUARK INTERACTIONS WITHOUT DERIVATIVES

\[ L_i \propto \frac{g_i}{\Lambda^{\gamma}} \chi^\alpha \Sigma^\beta \]

a). Dimensional arguments:


Therefore,

\[ \alpha + 3\beta - \gamma = 4 \]
b). The regime of dynamical chiral symmetry breaking: effective potential is

\[ V_{\text{eff}}(\sigma) = \]

\[ \sim \Lambda^3 \quad \sim \frac{\Lambda^4}{\Lambda^2} \quad \sim \frac{\Lambda^6}{\Lambda^5} \quad \sim \frac{\Lambda^8}{\Lambda^8} \quad \sim \frac{\Lambda^{2\beta}}{\Lambda^\gamma} \]

i.e. the leading contributions to the effective potential give the vertices with

\[ 2\beta - \gamma \geq 0 \]
Combining both restrictions we come to the conclusion that only vertices with
\[ \alpha + \beta \leq 4 \]
must be taken into account at leading order.

1). \( \alpha = 0, \beta = 1,2,3,4 \) these are 4, 6 and 8-quark interactions:

where
\[ L_{\text{int}} = L_{4q} + L_{6q} + L_{8q}, \]

\[ L_{4q} = \frac{\bar{G}}{\Lambda^2} Tr(\Sigma^+ \Sigma), \quad L_{6q} = \frac{\bar{K}}{\Lambda^5} \left( \det \Sigma + \det \Sigma^+ \right), \]

\[ L^{(1)}_{8q} = \frac{\bar{g}_1}{\Lambda^8} \left[ Tr \left( \Sigma^+ \Sigma \right) \right]^2, \quad L^{(2)}_{8q} = \frac{\bar{g}_2}{\Lambda^8} Tr \left( \Sigma^+ \Sigma \Sigma^+ \Sigma \right). \]
2). There are only six classes of vertices depending on external sources $\chi$, they are:

$$\alpha = 1, \beta = 1, 2, 3; \quad \alpha = 2, \beta = 1, 2; \quad \alpha = 3, \beta = 1.$$  

This group contains 11 terms:

$$L_\chi = \sum_{i=0}^{10} L_i,$$

$$L_0 = -Tr\left(\Sigma^+ \chi + \chi^+ \Sigma\right),$$

$$L_1 = -\frac{K_1}{\Lambda} e_{ijk} e_{mnl} \Sigma_{im} \chi_{jn} \chi_{kl} + h.c.,$$

$$L_2 = \frac{K_2}{\Lambda^3} e_{ijk} e_{mnl} \chi_{im} \Sigma_{jn} \Sigma_{kl} + h.c.,$$

$$L_3 = \frac{g_3}{\Lambda^6} Tr\left(\Sigma^+ \Sigma \Sigma^+ \chi\right) + h.c.,$$
\[ L_4 = \frac{g_4}{\Lambda^6} Tr(\Sigma^+\Sigma)Tr(\Sigma^+\chi) + h.c., \]
\[ L_5 = \frac{g_5}{\Lambda^4} Tr(\Sigma^+\chi\Sigma^+\chi) + h.c., \]
\[ L_6 = \frac{g_6}{\Lambda^4} Tr(\Sigma \Sigma^+\chi\chi^+ + \Sigma^+\Sigma\chi^+\chi), \]
\[ L_7 = \frac{g_7}{\Lambda^4} (Tr\Sigma^+\chi + h.c.)^2, \]
\[ L_8 = \frac{g_8}{\Lambda^4} (Tr\Sigma^+\chi - h.c.)^2, \]
\[ L_9 = -\frac{g_9}{\Lambda^2} Tr(\Sigma^+\chi\chi^+\chi) + h.c., \]
\[ L_{10} = -\frac{g_{10}}{\Lambda^2} Tr(\chi^+\chi)Tr(\chi^+\Sigma) + h.c. \]
Explicit chiral symmetry breaking interactions

Put $\chi = \frac{\mu}{2}$ with $\mu = \text{diag}(\mu_u, \mu_d, \mu_s)$

\[
\frac{\bar{K}^2}{\Lambda^3} \quad \frac{\bar{g}^6 \ldots 8}{\Lambda^4} \quad \frac{\bar{g}^{3,4}}{\Lambda^6}
\]
Current quark mass term

\[ \Lambda^2 N_c \times (1 + \frac{\bar{K}_1}{\Lambda N_c} + \frac{\bar{g}_{9,10}}{\Lambda^2 N_c}) \]

\[ \mu \xrightarrow{\approx} \hat{m} \mathcal{O}(\Lambda^2 \times N_c) \]

Kaplan Manohar ambiguity allows to set \( \bar{K}_1 = \bar{g}_9 = \bar{g}_{10} = 0 \)

\( \mu \rightarrow \hat{m} \)
\( \mathcal{N}_c \) assignments:

\[ \Sigma \sim \mathcal{N}_c; \quad \Lambda \sim \mathcal{N}_c^0 \sim 1; \quad \chi \sim \mathcal{N}_c^0 \sim 1 \]

- Then we get exactly that the diagrams which survive as \( \Lambda \to \infty \) also survive as \( \mathcal{N}_c \to \infty \) and comply with the usual requirements:

- Leading quark contribution to the vacuum energy from 4q interactions known to be of order \( \mathcal{N}_c \to \mathbb{G} \sim \frac{1}{\mathcal{N}_c} \)

- \( U_A(1) \) anomaly contribution (\( i\zeta \frac{1}{2} t \) Hooft interaction) is suppressed by one power of \( \frac{1}{\mathcal{N}_c} \to \kappa \sim \frac{1}{\mathcal{N}_c^3} \).

- Zweig’s rule violating effects are always of order \( \frac{1}{\mathcal{N}_c} \) with respect to leading contribution: e.g. \( g_1 \sim \frac{1}{\mathcal{N}_c^4} \).
• We have $L_{4q}$ and $L_0$ of $O(N_c)$ and all other terms in the Lagrangian of $O(N_c^0)$.

• Non OZI-violating Lagrangian pieces scaling as $O(N_c^0)$ represent NLO contributions with one internal quark loop in $N_c$ counting. The coupling encodes the admixture of four quark component $\bar{q}q\bar{q}q$ to the leading $\bar{q}q$ at $N_c \to \infty$.

• Diagrams tracing Zweig’s rule violation:
  $\kappa, \kappa_1, \kappa_2, g_1, g_4, g_7, g_8, g_{10}$

• Diagrams with admixture of 4 quark and 2 quark states:
  $g_2, g_3, g_5, g_6, g_9$.

• Phenomenology of terms $L_0, G, \kappa, g_1, g_2$ have been mostly studied until now. The role of the remaining 10 terms should be carefully addressed to be consistent with the generic $\frac{1}{N_c}$ expansion of QCD.
BOSONIZATION

Let us introduce in the vacuum functional

\[ Z = \int dq \, d\bar{q} \exp\left( i \int d^4x L \right) \]

the functional unity (Alkofer, Reinhardt, 1988)

\[ 1 = \int \prod_a ds_a \, dp_a \, \delta(s_a - \bar{q} \lambda_a q) \delta(p_a - \bar{q} i \gamma_5 \lambda_a q) \]

\[ = \int \prod_a ds_a \, dp_a \, d\sigma_a \, d\phi_a \exp\left\{ i \int d^4x \left[ \sigma_a \left( s_a - \bar{q} \lambda_a q \right) + \phi_a \left( p_a - \bar{q} i \gamma_5 \lambda_a q \right) \right] \right\} \]

thus obtaining

\[ Z = \int \prod_a d\sigma_a \, d\phi_a \, dq \, d\bar{q} \exp\left( i \int d^4x L_{q\bar{q}} \right) \int \prod_a ds_a \, dp_a \exp\left( i \int d^4x L_{aux} \right). \]

Gaussian integral
heat kernel expansion
stationary phase approx.
Here

\[ L_{q\bar{q}} = \bar{q} \left[ i\gamma^\mu \partial_\mu - (\sigma + i\gamma_5 \phi) \right] q \equiv \bar{q} D q \]

\[ L_{aux} = s_a \left( \sigma_a - m_a \right) + p_a \phi_a + L_{int}(s,p) + \sum_{i=2}^{8} L'_i(s,p,\mu) \]

- quartic polynomial in auxiliary fields
- cubic polynomial in auxiliary fields

\[ L_{int}(s,p) = L_{4q} + L_{6q} + L_{8q} \]
INTEGRATION OVER AUXILIARY FIELDS

The stationary phase trajectory are given by the extremum conditions

\[ \frac{\partial L_{aux}}{\partial s_a} = 0, \quad \frac{\partial L_{aux}}{\partial p_a} = 0. \]

which must be fulfilled in the neighborhood of the uniform vacuum state of the theory, i.e. \( \sigma \rightarrow \sigma + M, \quad \langle \sigma \rangle = 0. \) We seek solutions in the form

\[ s_{st}^a = h_a + h_{ab}^{(1)} \sigma_b + h_{abc}^{(1)} \sigma_b \sigma_c + h_{abc}^{(2)} \phi_b \phi_c + \ldots \]

\[ p_{st}^a = h_{ab}^{(2)} \phi_b + h_{abc}^{(3)} \sigma_b \phi_c + \ldots \]

We are led to the result:

\[ L_{aux} = h_a \sigma_a + \frac{1}{2} h_{ab}^{(1)} \sigma_a \sigma_b + \frac{1}{2} h_{ab}^{(2)} \phi_a \phi_b + \ldots \]
THE TOTAL LAGRANGIAN OF THE BOSONIZED THEORY

a). The gap equation

\[ h_i + \frac{N_c}{6\pi^2} M_i \left[ 3I_0 - \left( 3M_i^2 - M^2 \right) I_1 \right] = 0. \]

where

\[ M^2 = M_u^2 + M_d^2 + M_s^2. \]
... solved self-consistently with

\[ M_i - m_i + \frac{\kappa}{4} t_{ijk} h_j h_k + \frac{h_i}{2} (2G + g_1 h^2 + g_4 mh) + \frac{g_2}{2} h_i^3 \]
\[ + \frac{m_i}{4} \left[ 3g_3 h_i^2 + g_4 h^2 + 2(g_5 + g_6) m_i h_i + 4g_7 mh \right] \]
\[ + \kappa_2 t_{ijk} m_j h_k = 0. \] (1)

\( t_{ijk} \) is a totally symmetric quantity, whose nonzero components are \( t_{uds} = 1 \); there is no summation over the open index \( i \) but we sum over the dummy indices, e.g.
\[ h^2 = h_u^2 + h_d^2 + h_s^2, \]
\[ mh = m_u h_u + m_d h_d + m_s h_s. \]
b.) Small perturbations

\[ L_{\text{kin}} + L_{\text{mass}} = \frac{N_c I_1}{16\pi^2} \text{tr}(\partial \phi)^2 + \frac{N_c I_0}{4\pi^2} \phi_a^2 \]

\[ - \frac{N_c I_1}{24\pi^2} \left\{ \left[ \phi_u^2 (2M_u^2 - M_d^2 - M_s^2) + \phi_d^2 (2M_d^2 - M_u^2 - M_s^2) \right] + \phi_s^2 (2M_s^2 - M_u^2 - M_d^2) \right\} + \frac{1}{2} h^{(2)}_{ab} \phi_a \phi_b + \ldots \] (2)

\( h^{(2)}_{ab} \) carries all the dependence on the model couplings.

\( M_i \) only indirectly through gap equations.

The kinetic term requires a redefinition of meson fields

\[ \phi_a = g \phi^R_a \quad g^2 = \frac{4\pi^2}{N_c I_1} = \frac{(M_u + M_d)^2}{2f^2_\pi}. \]

\[ \phi_u = \phi_3 + \frac{\sqrt{2}\phi_0 + \phi_8}{\sqrt{3}} = \phi_3 + \eta_{ns} \quad \phi_d = -\phi_3 + \frac{\sqrt{2}\phi_0 + \phi_8}{\sqrt{3}} = -\phi_3 + \eta_{ns} \quad \phi_s = \sqrt{\frac{2}{3}} \phi_0 + \frac{2\phi_8}{\sqrt{3}} = \sqrt{2} \eta_s. \]
Defining $m_\Delta = \frac{1}{2}(m_d - m_u)$, $m_\Sigma = \frac{1}{2}(m_d + m_u)$, $h_\Delta = \frac{1}{2}(h_d - h_u)$ and $h_\Sigma = \frac{1}{2}(h_d + h_u)$, one has

$$
\sqrt{6}(h_{03}^{(2)})^{-1} = h_\Delta(2g_2 h_\Sigma + \kappa + g_3 m_\Sigma)
\]
\[ + m_\Delta[g_3 h_\Sigma + 2(\kappa_2 - g_8(m_s + 2m_\Sigma))
\] - (g_5 - g_6)m_\Sigma].
\]

(3)

and a quite similar expression for $(h_{38}^{(2)})^{-1}$.

$(h_{03}^{(2)})^{-1} \neq 0$ ONLY if NLO in $N_c$ terms contribute.

(i) ESB couplings $\neq 0 \rightarrow$ Explicit $m_i$ dependence.
(ii) Absence of ESB couplings $\rightarrow$ effects of ESB present in difference of the condensates $h_\Delta \neq 0$ if the conventional QCD mass term $m_u \neq m_d$.
(iii) In case (ii) only the 't Hooft $\sim \kappa$ and and the $8q \sim g_2$ contribute.
For comparison

\[ 3\sqrt{2}(h_{08}^{(2)})^{-1} = \kappa(h_s - h_\Sigma) + g_2(h_\Delta^2 - h_s^2 + h_\Sigma^2) \\
- g_3(h_\Sigma m_\Sigma + m_\Delta h_\Delta - h_s m_s) + 2\kappa_2(m_s - m_\Sigma) \\
+ (g_5 - g_6)(m_\Sigma^2 + m_\Delta^2 - m_s^2) \\
- 2g_8(m_s^2 + m_s m_\Sigma + 2m_\Sigma^2). \]  \( (4) \)
Meson Mass matrix diagonalization

$\pi_0, \eta, \eta'$ are related to the symmetric pseudoscalar meson mass matrix $B_{ij}$ by $S = \mathcal{U} \mathcal{V}$

\[
(\phi_3, \phi_0, \phi_8) S^{-1} S \begin{pmatrix} B_{33} & B_{03} & B_{38} \\ B_{03} & B_{00} & B_{08} \\ B_{38} & B_{08} & B_{88} \end{pmatrix} S^{-1} S \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix},
\]

(5)

\[
\begin{pmatrix} \phi_3 \\ \eta_{ns} \\ \eta_s \end{pmatrix} = \mathcal{V} \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix}
\]

(6)

\[
\mathcal{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix},
\]

(7)
Unitary transformation $\mathcal{U}$ relates strange non-strange basis to the physical states, Kroll:2005

$$
\begin{pmatrix}
\pi^0 \\
\eta \\
\eta'
\end{pmatrix} = \mathcal{U}(\epsilon_1, \epsilon_2, \psi)
\begin{pmatrix}
\phi_3 \\
\eta_{ns} \\
\eta_s
\end{pmatrix},
$$

(8)

$\mathcal{U}$ is linearized in the $\pi^0 - \eta$ and $\pi^0 - \eta'$ mixing angles, because $\phi_3$ couples weakly to the $\eta_{ns}$ and $\eta_s$ states, decoupling in the isospin limit, while the mixing for the $\eta - \eta'$ system is strong.

$$
\mathcal{U} =
\begin{pmatrix}
1 & \epsilon_1 + \epsilon_2 \cos \psi & -\epsilon_2 \sin \psi \\
-\epsilon_2 - \epsilon_1 \cos \psi & \cos \psi & -\sin \psi \\
-\epsilon_1 \sin \psi & \sin \psi & \cos \psi
\end{pmatrix}
$$

(9)

$\epsilon = \epsilon_2 + \epsilon_1 \cos \psi$, $\epsilon' = \epsilon_1 \sin \psi$. 
The diagonalization of the mass matrix can also be done exactly using the explicit analytical expressions for the eigenvalues of a symmetric $3 \times 3$ matrix$^1$

$$
\begin{align*}
\lambda_1 &= \xi - \sqrt{\varsigma} \left( \cos [\varphi] + \sqrt{3} \sin [\varphi] \right) \\
\lambda_2 &= \xi - \sqrt{\varsigma} \left( \cos [\varphi] - \sqrt{3} \sin [\varphi] \right) \\
\lambda_3 &= \xi + 2\sqrt{\varsigma} \cos [\varphi],
\end{align*}
$$

where we use the abbreviations: $\xi = \frac{\text{tr}[M]}{3}$, $\mathcal{M} = M - \xi \mathbb{1}$, $\varsigma = \frac{1}{6} \sum_i \sum_j (\mathcal{M}_{ij})^2$, $\vartheta = \frac{1}{2} \det [\mathcal{M}]$, $\varphi = \frac{1}{6} \text{ArcTan} \left[ \frac{\sqrt{\varsigma^2 - \vartheta}}{\vartheta} \right]$.$^1$

---

$^1$ O. K. Smith, Commun. ACM 4, 168 (1961), ISSN 0001-0782
The eigenvectors can then be obtained by normalizing the vectors given by:

\[ \vec{v}_i = \left( (\vec{M}_1 - \lambda_i \hat{e}_1) \times (\vec{M}_2 - \lambda_i \hat{e}_2) \right)^* , \]

where \( \vec{M}_j \) corresponds to the \( j \) column of \( M \). These are then used to build the diagonalization matrix which using the standard parametrization of the CKM matrix (with the abbreviation \( C_{ij} \equiv \cos[\theta_{ij}], S_{ij} \equiv \sin[\theta_{ij}] \)):

\[ \theta_{12} = -\epsilon \quad \theta_{13} = -\epsilon' \quad \theta_{23} = \psi \]

\[ U = \begin{bmatrix}
C_{12}C_{13} & -S_{12}C_{23} - C_{12}S_{13}S_{23} & S_{12}S_{23} - C_{12}S_{13}C_{23} \\
S_{12}C_{13} & C_{12}C_{23} - S_{12}S_{13}S_{23} & -S_{12}S_{23} - S_{12}S_{13}C_{23} \\
S_{13} & C_{13}S_{23} & C_{13}C_{23}
\end{bmatrix} \]
Weak decay constants

Model’s axial-vector current

\[ A^a_\mu = \frac{1}{4} \text{tr} \left[ \{ \sigma^R + Mg^{-1}, \partial_\mu \phi^R \} - \{ \partial_\mu \sigma^R, \phi^R \} \right] \lambda_a \] + \ldots \ (10)

\[ \langle 0 | A^a_\mu (0) | \phi^b_R \rangle = if^{ab} p_\mu. \]

\[ \langle 0 | A^{1+i2}_\mu (0) | \pi (p) \rangle = i \sqrt{2} f_\pi p_\mu \]

\[ \langle 0 | A^{4+i5}_\mu (0) | K (p) \rangle = i \sqrt{2} f_K p_\mu \]

\[ f_\pi = \frac{M_u + M_d}{2g} \quad f_K = \frac{M_u + M_s}{2g}. \]

\( \sigma^R, \phi^R \): normalized fields

\( M = M_a \lambda_a \): constituent quark mass matrix
Table: The pseudoscalar masses and weak decay constants (MeV) in the isospin limit used as input (marked with *) for different sets of the model. Sets a, b contain ESB interactions and allow for a fit of the scalar masses and their strong decays as well, \( m_\sigma = 550 \) MeV, \( m_\kappa = 850 \) MeV, \( m_{a_0} = m_{f_0} = 980 \) MeV Osipov:2013; set (c) does not. Set (a) corresponds to \( \theta_P = -12^\circ \), b) to \( \theta_P = -15^\circ \).

<table>
<thead>
<tr>
<th>Sets</th>
<th>( m_\pi )</th>
<th>( m_K )</th>
<th>( m_\eta )</th>
<th>( m_{\eta'} )</th>
<th>( f_\pi )</th>
<th>( f_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a,b</td>
<td>138*</td>
<td>494*</td>
<td>547*</td>
<td>958*</td>
<td>92*</td>
<td>113*</td>
</tr>
<tr>
<td>c</td>
<td>138*</td>
<td>494*</td>
<td>475*</td>
<td>958*</td>
<td>92*</td>
<td>115.7</td>
</tr>
</tbody>
</table>

\( m_K < m_\eta \): only with ESB interactions.
Table: The mixing angles in the $\eta - \eta'$ system in isospin limit, and related weak decay constants Osipov:2006, Osipov:2015 and in comparison with different approaches.

<table>
<thead>
<tr>
<th>Sets</th>
<th>$\theta_P^\circ$</th>
<th>$\theta_0^\circ$</th>
<th>$\theta_8^\circ$</th>
<th>$f_0^2 / f_\pi^2$</th>
<th>$f_8^2 / f_\pi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>-12*</td>
<td>-1.42</td>
<td>-21.37</td>
<td>1.172</td>
<td>1.318</td>
</tr>
<tr>
<td>$b$</td>
<td>-15*</td>
<td>-4.42</td>
<td>-24.37</td>
<td>1.172</td>
<td>1.322</td>
</tr>
<tr>
<td>$c$</td>
<td>-14.5</td>
<td>-2.82</td>
<td>-24.78</td>
<td>1.197</td>
<td>1.365</td>
</tr>
<tr>
<td>Goity:2002 CHPT</td>
<td>-10.5</td>
<td>-1.5</td>
<td>-20.0</td>
<td>1.24</td>
<td>1.31</td>
</tr>
<tr>
<td>Kaiser:2000 CHPT</td>
<td>-</td>
<td>-4.</td>
<td>-20.5</td>
<td>1.10</td>
<td>1.28</td>
</tr>
<tr>
<td>de Fazio:2000 sum rules</td>
<td>-</td>
<td>-15.6</td>
<td>-10.8</td>
<td>1.39</td>
<td>1.39</td>
</tr>
</tbody>
</table>

$\theta_0 = \psi - \arctan(\sqrt{2} \frac{M_s}{M_u})$

$\theta_8 = \psi - \arctan(\sqrt{2} \frac{M_u}{M_s})$

$\psi = \theta_P + \arctan\sqrt{2}$

$f_0^2 = \frac{2f_K^2 + f_\pi^2}{3} + \frac{f_\pi^2}{6} \left( \frac{M_s}{M_u} - 1 \right)^2$

$f_8^2 = \frac{4f_K^2 - f_\pi^2}{3} + \frac{(M_s - M_u)^2}{3g^2}$
Table: Empirical input used in the fits with isospin breaking, sets A and B with ESB interactions, set C without. Primes indicate which masses of the pion and kaon multiplets have been used for the fit, the other being output. Masses in units of MeV, angle $\psi$ in degrees.

<table>
<thead>
<tr>
<th>Sets</th>
<th>$m^0_\pi$</th>
<th>$m^{\pm}_\pi$</th>
<th>$m_\eta$</th>
<th>$m'_\eta$</th>
<th>$m^0_K$</th>
<th>$m^{\pm}_K$</th>
<th>$f_\pi$</th>
<th>$f_K$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A,B</td>
<td>136'</td>
<td>136.6</td>
<td>547</td>
<td>958</td>
<td>500</td>
<td>494'</td>
<td>92</td>
<td>113</td>
<td>39.7</td>
</tr>
<tr>
<td>C</td>
<td>136'</td>
<td>137.0</td>
<td>477</td>
<td>958</td>
<td>501</td>
<td>497'</td>
<td>92</td>
<td>116</td>
<td>39.7</td>
</tr>
</tbody>
</table>

ESB interactions do not account for empirical $m^0_\pi - m^{+}_\pi \approx 4.6$MeV. EM interactions must be taken into account.

ESB largely determine fraction of empirical $m^{+}_K - m^{0}_K \sim -4$MeV. Overestimate by $\sim 2$MeV in sets A, B.
Table: \( \frac{m_u}{m_d} = 0.46 \), current and constituent quark masses \( m_u, m_d, m_s \), \( M_u, M_d, M_s \) in MeV, \( \pi^0 - \eta, \pi^0 - \eta' \) mixing angles \( \epsilon \) and \( \epsilon' \).

<table>
<thead>
<tr>
<th>( m_u )</th>
<th>( m_d )</th>
<th>( m_s )</th>
<th>( M_u )</th>
<th>( M_d )</th>
<th>( M_s )</th>
<th>( \epsilon )</th>
<th>( \epsilon' )</th>
<th>( \frac{\epsilon}{\epsilon'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.179</td>
<td>4.760</td>
<td>95*</td>
<td>372</td>
<td>375</td>
<td>544</td>
<td>0.014*</td>
<td>0.0037*</td>
<td>3.78</td>
</tr>
<tr>
<td>2.166</td>
<td>4.733</td>
<td>95*</td>
<td>372</td>
<td>375</td>
<td>544</td>
<td>0.017*</td>
<td>0.0045*</td>
<td>3.95</td>
</tr>
<tr>
<td>3.774</td>
<td>8.246</td>
<td>194</td>
<td>373</td>
<td>380</td>
<td>573</td>
<td>0.022</td>
<td>0.0025</td>
<td>8.78</td>
</tr>
</tbody>
</table>

Table: \( \epsilon \) and \( \epsilon' \) values in the literature.

<table>
<thead>
<tr>
<th>Source</th>
<th>( \epsilon )</th>
<th>( \epsilon' )</th>
<th>( \frac{\epsilon}{\epsilon'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feldmann:1999 phen.</td>
<td>0.014</td>
<td>0.0037</td>
<td>3.78</td>
</tr>
<tr>
<td>Kroll:2005 phen.</td>
<td>0.017 ± 0.002</td>
<td>0.004 ± 0.001</td>
<td>4.25 ± 1.1</td>
</tr>
<tr>
<td>Goity:2002 ChPt NLO</td>
<td>0.014 ÷ 0.016</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Coon:1986 phen.</td>
<td>0.021</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>BES:2004 Exp.</td>
<td>0.030 ± 0.002</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Tippens:2001 Exp.</td>
<td>0.026 ± 0.007</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Conclusions

- The explicit symmetry breaking interactions of the generalized NJL Lagrangian considered are crucial to obtain the phenomenological quoted value for the ratio $\frac{\epsilon}{\epsilon'}$.

- We obtain values for the $\epsilon$ mixing angle which lie within the results discussed in the literature. Unfortunately the value for $\epsilon'$ is much less discussed.

- We obtain $\epsilon$ and $\epsilon'$ reasonably close to the ones indicated in Feldmann:1999, Kroll:2005 for current quark mass values in agreement with the presently quoted average values.

- The sets with explicit breaking interactions are also the ones which yield the best fits to other empirical data within the model variants.