.

eQCD 2016

6-11 March 2016, Costa da Caparica

Light quark mass differences in the $\pi - \eta - \eta'$ system.

A. A. Osipov, B. Hiller, A. H. Blin, J. Moreira

U. Coimbra, Portugal

Osipov, Hiller, Blin, *1501.06146 [hep-ph]*, *Phys.Rev. D88 (2013) 5, 054032*, *EPJA 49(2013)14*

.

Relevance of the $\pi_0 - \eta$ and $\pi_0 - \eta'$ mixing angles ϵ and ϵ'

- *In the determination of light quark mass ratio $\frac{m_u}{m_d}$*
- *In the decays of $\eta \to 3\pi$ and $\eta' \to 3\pi$*
- *In the determination of the ratio $\frac{\epsilon}{\epsilon'}|_{CP}$ related to $K^0 \to \pi^0\pi^0$*
- *Studies of spontaneous breaking of strong CP symmetry.*
- *Deeper insight in studies of restoration of chiral and $U_A(1)$ symmetries in the QCD phase diagram at finite T, μ, B .*

Isospin breaking observables

Effect	$m_u - m_d$	EM
$M_{\pi^+} - M_{\pi^0} = 4.6$ MeV	subleading	$\surd$
$M_{K^+} - M_{K^0} = -3.99$ MeV	$\surd$	$\surd$
$M_{D^+} - M_{D^0} = 4.78$ MeV	$\surd$	$\surd$
$\eta \to 3\pi$	$\surd$	subleading
$M_{B^+} - M_{B^0} = -0.33$ MeV	$\surd$	$\surd$
$M_n - M_p = 1.29$ MeV	$\surd$	$\surd$
$M_{\Sigma^-} - M_{\Sigma^0} = 4.8$ MeV	$\surd$	$\surd$
$M_{\Sigma^+} - M_{\Sigma^0} = -3.27$ MeV	$\surd$	$\surd$
$M_{\Xi^-} - M_{\Xi^0} = 6.48$ MeV	$\surd$	$\surd$
$\rho - \omega$ mixing	$\surd$	subleading
$\pi^0 - \eta$, $\pi^0 - \eta'$ mixings	$\surd$	subleading
F_{π^+}/F_{π^0}	subleading	$\surd$
$\pi^0 \to \gamma\gamma$	$\surd$	subleading

Current quark mass values

PDG:2014 At renormalization scale $\mu = 2GeV$.
Mass Ratios: RG invariant.

Lattice data: Isospin symmetric limit $m_u = m_d = \bar{m}$

$\bar{m} = 3.4 \pm 0.25 \qquad m_s = 93.5 \pm 2.5 MeV \qquad \frac{m_s}{\bar{m}} = 27.5 \pm 0.3$

Estimations of m_u *and* m_d*:*

Combined Lattice data + isospin breaking effects from ChPT + phenomenology: MILC collaboration (2009), BMW Collaboration (2011), Flavinet Lattice Averaging Group (2011),

$m_u = 2.15(15) MeV \qquad m_d = 4.70(20) MeV \qquad \frac{m_u}{m_d} = 0.46(5)$

Our ratio: $m_u/m_d \sim 0.46$ *(NLO in* N_c *)*

Moderate correction $\sim 20\%$ *to*
LO ChPT $m_u/m_d = 0.56$ *Weinberg:1977*

MAIN ASSUMPTIONS

1. Scales of nonperturbative QCD:

$$\Lambda_{QCD} \sim \Lambda_{conf} < \Lambda_{\chi SB} \approx 4\pi f_{\pi} .$$

In the regime $\Lambda_{conf} < \Lambda < \Lambda_{\chi SB}$ the induced effective interaction between quarks is of the form

$$L = \bar{q}\left(i\gamma^{\mu}\partial_{\mu} - m\right)q + \frac{\overline{G}}{\Lambda^2}\left(\bar{q}\Gamma q\right)^2 + \frac{\overline{\kappa}}{\Lambda^5}\left(\bar{q}\Gamma q\right)^3 + \ldots,$$

where Λ is determined by the chiral symmetry-breaking scale. If instantons are responsible for multi-quark interactions, than

$$\Lambda \sim \rho^{-1} \approx \left(0.33\ fm\right)^{-1}$$

2. Chiral symmetry restrictions

The color quark fields possess definite transformation properties with respect to the chiral flavor $U(3)_R \otimes U(3)_L$ global symmetry of the QCD Lagrangian with massless quarks

To study scalar and pseudoscalar modes it is convenient to introduce the U(3) Lie-algebra valued field:

$$\Sigma = \left(s_a - ip_a\right)\frac{\lambda_a}{2}, \quad s_a = \bar{q}\lambda_a q, \quad p_a = \bar{q}\lambda_a i\gamma_5 q.$$

and the external source χ, which generate the explicit symmetry breaking effects – future mass terms and mass dependent interactions, with the transformation properties:

$$\Sigma' = V_R \Sigma V_L^+, \quad \chi' = V_R \chi V_L^+.$$

MULTI-QUARK INTERACTIONS WITHOUT DERIVATIVES

$$L_i \propto \frac{\overline{g}_i}{\Lambda^\gamma} \chi^\alpha \Sigma^\beta$$

a). Dimensional arguments:

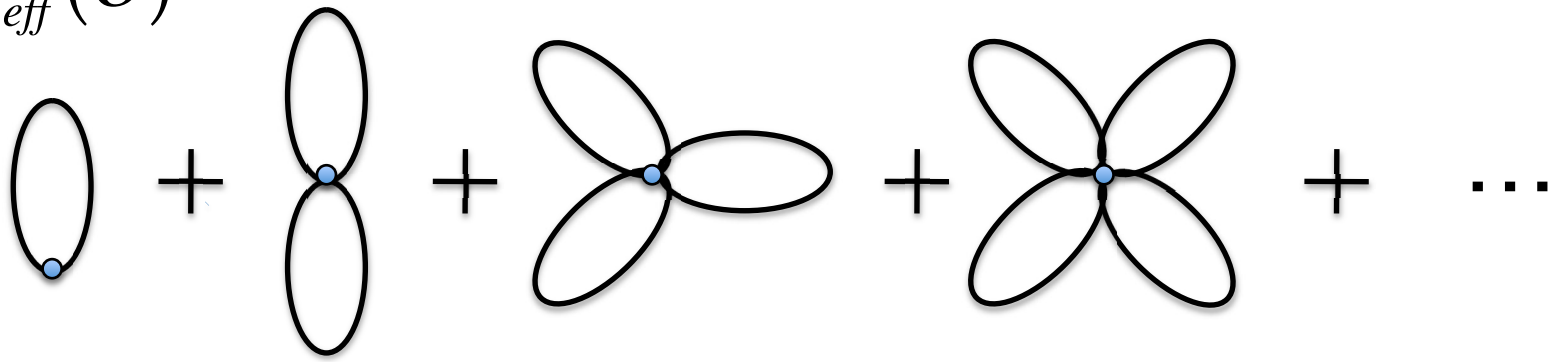
$$[\Lambda] = M, \quad [\chi] = M, \quad [\Sigma] = M^3, \quad [L_i] = M^4.$$

Therefore,

$$\alpha + 3\beta - \gamma = 4$$

b). The regime of dynamical chiral symmetry breaking: effective potential is

$$V_{eff}(\sigma) =$$



$$\sim \Lambda^3 \qquad \sim \frac{\Lambda^4}{\Lambda^2} \qquad \sim \frac{\Lambda^6}{\Lambda^5} \qquad \sim \frac{\Lambda^8}{\Lambda^8} \qquad \sim \frac{\Lambda^{2\beta}}{\Lambda^{\gamma}}$$

i.e. the leading contributions to the effective potential give the vertices with

$$2\beta - \gamma \geq 0$$

Combining both restrictions we come to the conclusion that only vertices with

$$\alpha + \beta \leq 4$$

must be taken into account at leading order.

1). $\alpha = 0, \beta = 1,2,3,4$ these are 4, 6 and 8-quark interactions:

where

$$L_{\text{int}} = L_{4q} + L_{6q} + L_{8q},$$

$$L_{4q} = \frac{\overline{G}}{\Lambda^2} Tr\left(\Sigma^+\Sigma\right), \qquad L_{6q} = \frac{\overline{\kappa}}{\Lambda^5}\left(\det\Sigma + \det\Sigma^+\right),$$

$$L_{8q}^{(1)} = \frac{\overline{g}_1}{\Lambda^8}\left[Tr\left(\Sigma^+\Sigma\right)\right]^2, \quad L_{8q}^{(2)} = \frac{\overline{g}_2}{\Lambda^8} Tr\left(\Sigma^+\Sigma\Sigma^+\Sigma\right).$$

2). There are only six classes of vertices depending on external sources χ, they are:

$$\alpha = 1, \beta = 1,2,3; \quad \alpha = 2, \beta = 1,2; \quad \alpha = 3, \beta = 1.$$

This group contains 11 terms:

$$L_\chi = \sum_{i=0}^{10} L_i ,$$

$$L_0 = -Tr\left(\Sigma^+ \chi + \chi^+ \Sigma\right),$$

$$L_1 = -\frac{\overline{\kappa}_1}{\Lambda} e_{ijk} e_{mnl} \Sigma_{im} \chi_{jn} \chi_{kl} + h.c.,$$

$$L_2 = \frac{\overline{\kappa}_2}{\Lambda^3} e_{ijk} e_{mnl} \chi_{im} \Sigma_{jn} \Sigma_{kl} + h.c.,$$

$$L_3 = \frac{\overline{g}_3}{\Lambda^6} Tr\left(\Sigma^+ \Sigma \Sigma^+ \chi\right) + h.c.,$$

$$L_4 = \frac{\overline{g}_4}{\Lambda^6} Tr\left(\Sigma^+\Sigma\right) Tr\left(\Sigma^+\chi\right) + h.c.,$$

$$L_5 = \frac{\overline{g}_5}{\Lambda^4} Tr\left(\Sigma^+\chi\Sigma^+\chi\right) + h.c.,$$

$$L_6 = \frac{\overline{g}_6}{\Lambda^4} Tr\left(\Sigma\Sigma^+\chi\chi^+ + \Sigma^+\Sigma\chi^+\chi\right),$$

$$L_7 = \frac{\overline{g}_7}{\Lambda^4}\left(Tr\,\Sigma^+\chi + h.c.\right)^2,$$

$$L_8 = \frac{\overline{g}_8}{\Lambda^4}\left(Tr\,\Sigma^+\chi - h.c.\right)^2,$$

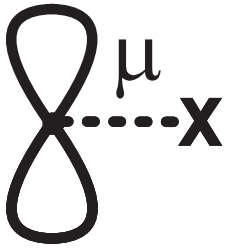
$$L_9 = -\frac{\overline{g}_9}{\Lambda^2} Tr\left(\Sigma^+\chi\chi^+\chi\right) + h.c.,$$

$$L_{10} = -\frac{\overline{g}_{10}}{\Lambda^2} Tr\left(\chi^+\chi\right) Tr\left(\chi^+\Sigma\right) + h.c.$$

Explicit chiral symmetry breaking interactions

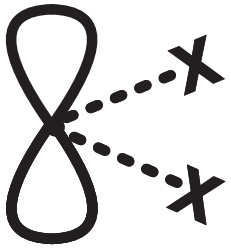
Put $\chi = \frac{\mu}{2}$ *with* $\mu = diag(\mu_u, \mu_d, \mu_s)$

$$\frac{\bar{K}_2}{\Lambda^3}$$



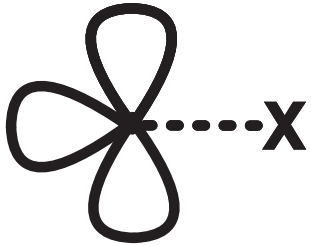
$$\propto \quad \Lambda$$

$$\frac{\bar{g}_{6..8}}{\Lambda^4}$$



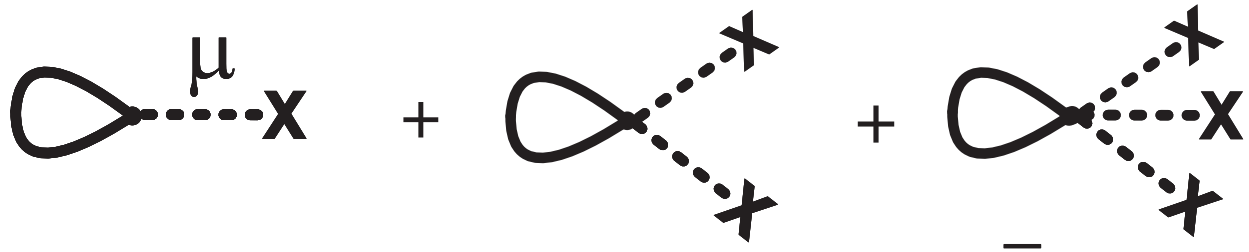
$$\propto \quad \Lambda^0$$

$$\frac{\bar{g}_{3,4}}{\Lambda^6}$$



$$\propto \quad \Lambda^0$$

Current quark mass term



$$\Lambda^2 N_c \, x \left(1 + \frac{\bar{K}_1}{\Lambda N_c} + \frac{\bar{g}_{9,10}}{\Lambda^2 N_c} \right)$$

$$= \quad \propto \; \hat{m} \; \mathcal{O}(\Lambda^2 \, x \, N_c)$$

Kaplan Manohar ambiguity allows to set $\bar{K}_1 = \bar{g}_9 = \bar{g}_{10} = 0$

$\mu \to \hat{m}$

N_c assignments:

$\Sigma \sim N_c$; $\Lambda \sim N_c^0 \sim 1$; $\chi \sim N_c^0 \sim 1$

- *Then we get exactly that the diagrams which survive as $\Lambda \to \infty$ also surive as $N_c \to \infty$ and comply with the usual requirements:*

- *Leading quark contribution to the vacuum energy from 4q interactions known to be of order $N_c \to G \sim \frac{1}{N_c}$*

- *$U_A(1)$ anomaly contribution (i¿$\frac{1}{2}$t Hooft interaction) is suppressed by one power of $\frac{1}{N_c} \to \kappa \sim \frac{1}{N_c^3}$.*

- *Zweig's rule violating effects are always of order $\frac{1}{N_c}$ with respect to leading contribution:e.g. $\to g_1 \sim \frac{1}{N_c^4}$.*

.

- *We have L_{4q} and L_0 of $\mathcal{O}(N_c)$ and all other terms in the Lagrangian of $\mathcal{O}(N_c^0)$.*

- *Non OZI-violating Lagrangian pieces scaling as $\mathcal{O}(N_c^0)$ represent NLO contributions with one internal quark loop in N_c counting. The coupling encodes the admixture of four quark component $\bar{q}q\bar{q}q$ to the leading $\bar{q}q$ at $N_c \to \infty$.*

- *Diagrams tracing Zweig's rule violation:*
$\kappa, \kappa_1, \kappa_2, g_1, g_4, g_7, g_8, g_{10}$

- *Diagrams with admixture of 4 quark and 2 quark states:*
g_2, g_3, g_5, g_6, g_9.

- *Phenomenology of terms L_0, G, κ, g_1, g_2 have been mostly studied until now. The role of the remaining 10 terms should be carefully addressed to be consistent with the generic $\frac{1}{N_c}$ expansion of QCD.*

BOSONIZATION

Let us introduce in the vacuum functional

$$Z = \int dq\, d\bar{q} \exp\left(i \int d^4 x L\right)$$

the functional unity (Alkofer, Reinhardt, 1988)

$$1 = \int \prod_a ds_a dp_a \delta\left(s_a - \bar{q}\lambda_a q\right)\delta\left(p_a - \bar{q} i\gamma_5 \lambda_a q\right)$$

$$= \int \prod_a ds_a dp_a d\sigma_a d\phi_a \exp\left\{i \int d^4 x \left[\sigma_a\left(s_a - \bar{q}\lambda_a q\right) + \phi_a\left(p_a - \bar{q} i\gamma_5 \lambda_a q\right)\right]\right\}$$

thus obtaining

$$Z = \int \prod_a d\sigma_a d\phi_a \underbrace{dq d\bar{q} \exp\left(i \int d^4 x L_{q\bar{q}}\right)}_{\text{Gaussian integral heat kernel expansion}} \underbrace{\int \prod_a ds_a dp_a \exp\left(i \int d^4 x L_{aux}\right)}_{\text{stationary phase approx.}}.$$

Here

$$L_{q\bar{q}} = \bar{q}\left[i\gamma^{\mu}\partial_{\mu} - \left(\sigma + i\gamma_5\phi\right)\right]q \equiv \bar{q}Dq$$

$$L_{aux} = s_a\left(\sigma_a - m_a\right) + p_a\phi_a + L_{\text{int}}(s,p) + \sum_{i=2}^{8} L_i'\left(s,p,\mu\right)$$

quartic polynomial
in auxiliary fields

cubic polynomial
in auxiliary fields

$$L_{\text{int}}\left(s,p\right) = L_{4q} + L_{6q} + L_{8q}$$

INTEGRATION OVER AUXILIARY FIELDS

The stationary phase trajectory are given by the extremum conditions

$$\frac{\partial L_{aux}}{\partial s_a} = 0, \quad \frac{\partial L_{aux}}{\partial p_a} = 0.$$

which must be fulfilled in the neighborhood of the uniform vacuum state of the theory, i.e. $\sigma \to \sigma + M$, $\langle \sigma \rangle = 0$. We seek solutions in the form

$$s_a^{st} = h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + \ldots$$

$$p_a^{st} = h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\sigma_b\phi_c + \ldots$$

We are led to the result:

$$L_{aux} = h_a\sigma_a + \frac{1}{2}h_{ab}^{(1)}\sigma_a\sigma_b + \frac{1}{2}h_{ab}^{(2)}\phi_a\phi_b + \ldots$$

THE TOTAL LAGRANGIAN OF THE BOSONIZED THEORY

a). The gap equation

$$h_i + \frac{N_c}{6\pi^2} M_i \left[3I_0 - \left(3M_i^2 - M^2 \right) I_1 \right] = 0.$$

where

$$M^2 = M_u^2 + M_d^2 + M_s^2 .$$

... solved self-consostently with

$$
\begin{aligned}
& M_i - m_i + \frac{\kappa}{4} t_{ijk} h_j h_k + \frac{h_i}{2}\left(2G + g_1 h^2 + g_4 mh\right) + \frac{g_2}{2} h_i^3 \\
& + \frac{m_i}{4}\left[3 g_3 h_i^2 + g_4 h^2 + 2(g_5 + g_6) m_i h_i + 4 g_7 mh\right] \\
& + \kappa_2 t_{ijk} m_j h_k = 0. \qquad (1)
\end{aligned}
$$

t_{ijk} is a totally symmetric quantity, whose nonzero components are $t_{uds} = 1$; there is no summation over the open index i but we sum over the dummy indices, e.g.
$h^2 = h_u^2 + h_d^2 + h_s^2, mh = m_u h_u + m_d h_d + m_s h_s$.

.

b.) Small perturbations

$$
\begin{aligned}
L_{kin} + L_{mass} = & \frac{N_c I_1}{16\pi^2} tr(\partial\phi)^2 + \frac{N_c I_0}{4\pi^2}\phi_a^2 \\
- & \frac{N_c I_1}{24\pi^2}\{[\phi_u^2(2M_u^2 - M_d^2 - M_s^2) + \phi_d^2(2M_d^2 - M_u^2 - M_s^2) \\
+ & \phi_s^2(2M_s^2 - M_u^2 - M_d^2)]\} + \frac{1}{2} h_{ab}^{(2)}\phi_a\phi_b + \ldots \qquad (2)
\end{aligned}
$$

$h_{ab}^{(2)}$ *carries all the dependence on the model couplings.*

M_i *only indirectly through gap equations.*

The kinetic term requires a redefinition of meson fields

$\phi_a = g\phi_a^R \qquad g^2 = \frac{4\pi^2}{N_c I_1} = \frac{(M_u+M_d)^2}{2f_\pi^2}.$

$\phi_u = \phi_3 + \frac{\sqrt{2}\phi_0+\phi_8}{\sqrt{3}} = \phi_3 + \eta_{ns} \qquad \phi_d = -\phi_3 + \frac{\sqrt{2}\phi_0+\phi_8}{\sqrt{3}} = -\phi_3 + \eta_{ns} \qquad \phi_s = \sqrt{\frac{2}{3}}\phi_0 - \frac{2\phi_8}{\sqrt{3}} = \sqrt{2}\eta_s.$

Defining $m_\Delta = \frac{1}{2}(m_d - m_u)$, $m_\Sigma = \frac{1}{2}(m_d + m_u)$, $h_\Delta = \frac{1}{2}(h_d - h_u)$ *and* $h_\Sigma = \frac{1}{2}(h_d + h_u)$, *one has*

$$\sqrt{6}(h_{03}^{(2)})^{(-1)} = h_\Delta(2g_2 h_\Sigma + \kappa + g_3 m_\Sigma) + m_\Delta[g_3 h_\Sigma + 2(\kappa_2 - g_8(m_s + 2m_\Sigma) - (g_5 - g_6)m_\Sigma]. \quad (3)$$

and a quite similar expression for $(h_{38}^{(2)})^{(-1)}$.

$(h_{03}^{(2)})^{(-1)} \neq 0$ *ONLY if NLO in* N_c *terms contribute.*

(i) ESB couplings $\neq 0 \rightarrow$ *Explicit* m_i *dependence.*
(ii) Absence of ESB couplings $\rightarrow$ *effects of ESB present in difference of the condensates* $h_\Delta \neq 0$ *if the conventional QCD mass term* $m_u \neq m_d$.
(iii) In case (ii) only the 't Hooft $\sim \kappa$ *and and the 8q* $\sim g_2$ *contribute.*

.

For comparison

$$
\begin{aligned}
&3\sqrt{2}(h_{08}^{(2)})^{(-1)} = \kappa(h_s - h_\Sigma) + g_2(h_\Delta^2 - h_s^2 + h_\Sigma^2) \\
&-g_3(h_\Sigma m_\Sigma + m_\Delta h_\Delta - h_s m_s) + 2\kappa_2(m_s - m_\Sigma) \\
&+(g_5 - g_6)(m_\Sigma^2 + m_\Delta^2 - m_s^2) \\
&-2g_8(m_s^2 + m_s m_\Sigma + 2m_\Sigma^2). \qquad (4)
\end{aligned}
$$

Meson Mass matrix diagonalization

π_0, η, η' *are related to the symmetric pseudoscalar meson mass matrix* B_{ij} *by* $\mathcal{S} = \mathcal{U}\mathcal{V}$

$$(\phi_3, \phi_0, \phi_8)\, S^{-1} S \begin{pmatrix} B_{33} & B_{03} & B_{38} \\ B_{03} & B_{00} & B_{08} \\ B_{38} & B_{08} & B_{88} \end{pmatrix} S^{-1} S \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \phi_3 \\ \eta_{ns} \\ \eta_s \end{pmatrix} = \mathcal{V} \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix} \tag{6}$$

$$\mathcal{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}, \tag{7}$$

Unitary transformation $\mathcal{U}$ relates strange non-strange basis to the physical states , Kroll:2005

$$\begin{pmatrix} \pi^0 \\ \eta \\ \eta' \end{pmatrix} = \mathcal{U}(\epsilon_1, \epsilon_2, \psi) \begin{pmatrix} \phi_3 \\ \eta_{ns} \\ \eta_s \end{pmatrix}, \tag{8}$$

$\mathcal{U}$ is linearized in the $\pi^0 - \eta$ and $\pi^0 - \eta'$ mixing angles, because ϕ_3 couples weakly to the η_{ns} and η_s states, decoupling in the isospin limit, while the mixing for the $\eta - \eta'$ system is strong.

$$\mathcal{U} = \begin{pmatrix} 1 & \epsilon_1 + \epsilon_2 \cos\psi & -\epsilon_2 \sin\psi \\ -\epsilon_2 - \epsilon_1 \cos\psi & \cos\psi & -\sin\psi \\ -\epsilon_1 \sin\psi & \sin\psi & \cos\psi \end{pmatrix} \tag{9}$$

$\epsilon = \epsilon_2 + \epsilon_1 \cos\psi, \quad \epsilon' = \epsilon_1 \sin\psi.$

The diagonalization of the mass matrix can also be done exactly using the explicit analytical expressions for the eigenvalues of a symmetric 3×3 matrix [1]

$$\begin{aligned}
\lambda_1 =& \xi - \sqrt{\varsigma}\left(\cos[\varphi] + \sqrt{3}\sin[\varphi]\right) \\
\lambda_2 =& \xi - \sqrt{\varsigma}\left(\cos[\varphi] - \sqrt{3}\sin[\varphi]\right) \\
\lambda_3 =& \xi + 2\sqrt{\varsigma}\cos[\varphi],
\end{aligned}$$

where we use the abbreviations: $\xi = \frac{\mathrm{tr}[M]}{3}, \quad \mathcal{M} = M - \xi \mathbf{I}, \varsigma = \frac{1}{6}\sum_i \sum_j (\mathcal{M}_{ij})^2, \quad \vartheta = \frac{1}{2}\det[\mathcal{M}], \quad \varphi = \frac{1}{6}\mathrm{ArcTan}\left[\frac{\sqrt{\varsigma^3 - \vartheta}}{\vartheta}\right]$

[1] O. K. Smith, Commun. ACM 4, 168 (1961), ISSN 0001-0782
Joachim Kopp , Int.J.Mod.Phys.C19:523-548,2008

The eigenvectors can then be obtained by normalizing the vectors given by:

$$\vec{v_i} = \left(\left(\overrightarrow{M_1} - \lambda_i \hat{e}_1\right) \times \left(\overrightarrow{M_2} - \lambda_i \hat{e}_2\right)\right)^*,$$

where $\overrightarrow{M_j}$ *corresponds to the* j *column of* M*.*
These are then used to build the diagonalization matrix which using the standard parametrization of the CKM matrix(with the abbreviation $C_{ij} \equiv Cos\,[\theta_{ij}]$*,* $S_{ij} \equiv Sin\,[\theta_{ij}]$*):*

$$\theta_{12} = -\epsilon \qquad \theta_{13} = -\epsilon' \qquad \theta_{23} = \psi$$

$$U = \begin{bmatrix} C_{12}C_{13} & -S_{12}C_{23} - C_{12}S_{13}S_{23} & S_{12}S_{23} - C_{12}S_{13}C_{23} \\ S_{12}C_{13} & C_{12}C_{23} - S_{12}S_{13}S_{23} & -C_{12}S_{23} - S_{12}S_{13}C_{23} \\ S_{13} & C_{13}S_{23} & C_{13}C_{23} \end{bmatrix}$$

.

Weak decay constants

Model's axial-vector current

$$\mathcal{A}^a_\mu = \frac{1}{4} tr[(\{\sigma^R + Mg^{-1}, \partial_\mu \phi^R\} - \{\partial_\mu \sigma^R, \phi^R\})\lambda_a] + \ldots \quad (10)$$

$< 0|\mathcal{A}^a_\mu(0)|\phi^b_R >= if^{ab} p_\mu.$

$< 0|\mathcal{A}^{1+i2}_\mu(0)|\pi(p) >= i\sqrt{2} f_\pi p_\mu$

$< 0|\mathcal{A}^{4+i5}_\mu(0)|K(p) >= i\sqrt{2} f_K p_\mu$

$f_\pi = \frac{M_u + M_d}{2g} \qquad f_K = \frac{M_u + M_s}{2g}.$

σ^R, ϕ^R: *normalized fields*

$M = M_a \lambda_a$: *constituent quark mass matrix*

.

Table: The pseudoscalar masses and weak decay constans (MeV) in the isospin limit used as input (marked with *) for different sets of the model. Sets a, b contain ESB interactions and allow for a fit of the scalar masses and their strong decays as well, $m_\sigma = 550$ MeV, $m_\kappa = 850$ MeV, $m_{a_0} = m_{f_0} = 980$ MeV Osipov:2013; set (c) does not. Set (a) corresponds to $\theta_P = -12^\circ$, b) to $\theta_P = -15^\circ$.

Sets	m_π	m_K	m_η	$m_{\eta'}$	f_π	f_K
a,b	*138**	*494**	*547**	*958**	*92**	*113**
c	*138**	*494**	*475**	*958**	*92**	*115.7*

$m_K < m_\eta$: only with ESB interactions.

Table: The mixing angles in the $\eta - \eta'$ system in isospin limit, and related weak decay constants Osipov:2006, Osipov:2015 and in comparison with different approaches.

Sets	θ_P°	θ_0°	θ_8°	$\frac{f_0}{f_\pi}$	$\frac{f_8}{f_\pi}$
a	-12*	-1.42	-21.37	1.172	1.318
b	-15*	-4.42	-24.37	1.172	1.322
c	-14.5	-2.82	-24.78	1.197	1.365
Feldmann:1998 phen.	-13.3	-6.8	-19.4	1.10	1.19
Feldmann:1998 phen.	-15.4	-9.2	-21.2	1.17	1.26
Goity:2002 CHPT	-10.5	-1.5	-20.0	1.24	1.31
Kaiser:2000 CHPT	-	-4.	-20.5	1.10	1.28
de Fazio:2000 sum rules	-	-15.6	-10.8	1.39	1.39

$$\theta_0 = \psi - arctan(\sqrt{2}\frac{M_s}{M_u}) \qquad \theta_8 = \psi - arctan(\sqrt{2}\frac{M_u}{M_s})$$

$$\psi = \theta_P + arctan\sqrt{2}$$

$$f_0^2 = \frac{2f_K^2+f_\pi^2}{3} + \frac{f_\pi^2}{6}(\frac{M_s}{M_u} - 1)^2 \qquad f_8^2 = \frac{4f_K^2-f_\pi^2}{3} + \frac{(M_s-M_u)^2}{3g^2}$$

.

Table: Empirical input used in the fits with isospin breaking , sets A and B with ESB interactions, set C without. Primes indicate which masses of the pion and kaon multiplets have been used for the fit, the other being output. Masses in units of MeV, angle ψ in degrees.

Sets	m_π^0	$m_\pi^\pm$	m_η	m'_η	m_K^0	$m_K^\pm$	f_π	f_K	ψ
A,B	*136'*	*136.6*	*547*	*958*	*500*	*494'*	*92*	*113*	*39.7*
C	*136'*	*137.0*	*477*	*958*	*501*	*497'*	*92*	*116*	*39.7*

ESB interactions do not account for empirical $m_\pi^0 - m_\pi^+ =\sim 4.6 MeV$ *. EM interactions must be taken into account.*

ESB largely determine fraction of empirical $m_K^+ - m_K^0 \sim -4 MeV$*. Overestimate by* $\sim 2 MeV$ *in sets* A, B*.*

.

Table: $\frac{m_u}{m_d} = 0.46$,current and constituent quark masses m_u, m_d,m_s M_u, M_d, M_s in MeV, $\pi^0 - \eta$, $\pi^0 - \eta'$ mixing angles $\epsilon$and ϵ'.

m_u	m_d	m_s	M_u	M_d	M_s	ϵ	ϵ'	$\frac{\epsilon}{\epsilon'}$
2.179	*4.760*	*95**	*372*	*375*	*544*	*0.014**	*0.0037**	*3.78*
2.166	*4.733*	*95**	*372*	*375*	*544*	*0.017**	*0.0045**	*3.95*
3.774	*8.246*	*194*	*373*	*380*	*573*	*0.022*	*0.0025*	*8.78*

Table: ϵ and ϵ' values in the literature.

	ϵ	ϵ'	$\frac{\epsilon}{\epsilon'}$
Feldmann:1999 phen.	*0.014*	*0.0037*	*3.7*
Kroll:2005 phen.	0.017 ± 0.002	0.004 ± 0.001	4.25 ± 1.1
Goity:2002 ChPt NLO	$0.014 \div 0.016$	-	
Coon:1986 phen.	*0.021*	-	
BES:2004 Exp.	0.030 ± 0.002	-	
Tippens:2001 Exp.	0.026 ± 0.007	-	

.

Conclusions

- *The explicit symmetry breaking interactions of the generalized NJL Lagrangian considered are crucial to obtain the phenomenological quoted value for the ratio* $\frac{\epsilon}{\epsilon'}$.

- *We obtain values for the* ϵ *mixing angle which lie within the results discussed in the literature. Unfortunately the value for* ϵ' *is much less discussed.*

- *We obtain* ϵ *and* ϵ' *reasonably close to the ones indicated in Feldmann:1999, Kroll:2005 for current quark mass values in agreement with the presently quoted average values.*

- *The sets with explicit breaking interactions are also the ones which yield the best fits to other empirical data within the model variants.*