Supersymmetric Quantum Mechanics in One Dimension*

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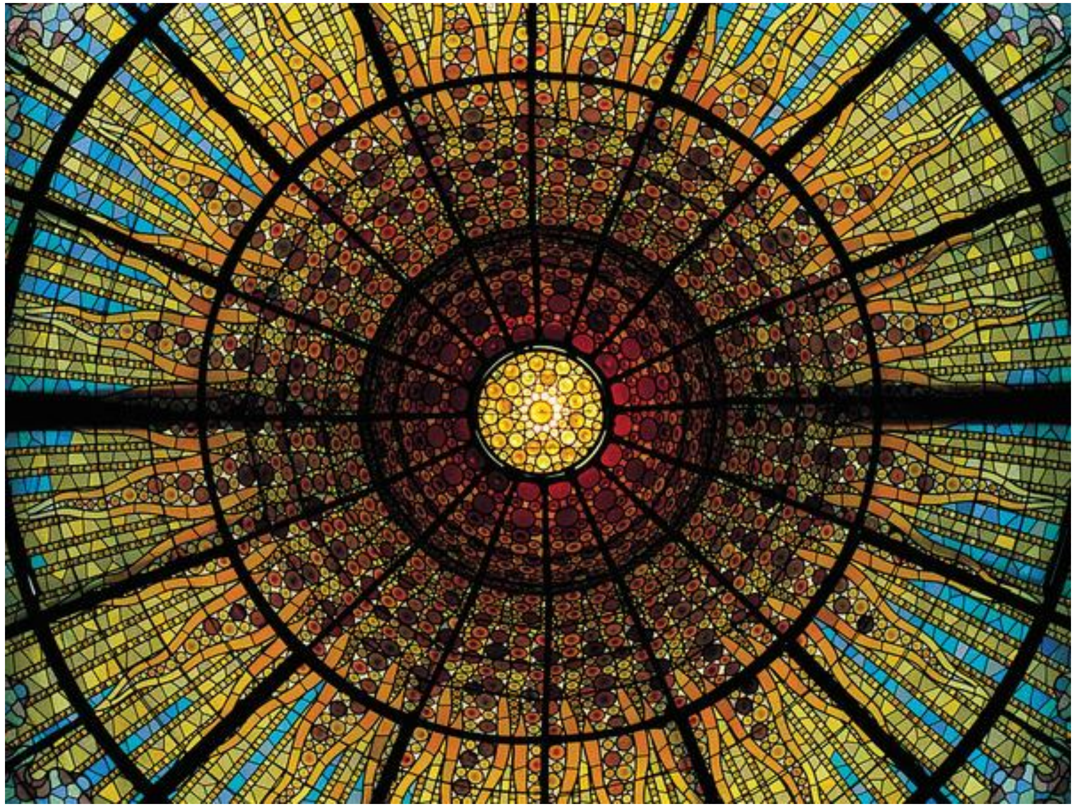
Symmetry

- **What is symmetry?**

A property that remains unchanged under certain transformation.



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Physical Symmetries

Spacetime Symmetry

Depends on space and time

Continuous

Spatial-Time-Spatial temporal symmetries

<u>Examples:</u>
Spatial Translation
Time Translation
Spatial Rotation

Discrete

Symmetry over non continuous change

<u>Examples:</u>
Charge Conjugation
Spatial Parity
Time Reversal

Supersymmetry

Internal Symmetry

Independent of space and time

<u>Example:</u>
Isospin symmetry among proton and neutron is an example

Noether's Theorem

Every Continuous Symmetry yields a conservation law!

Symmetry		Conservation Law
Translation in space	$\leftrightarrow$	Linear Momentum
Translation in time	$\leftrightarrow$	Energy
Rotation	$\leftrightarrow$	Angular Momentum
Gauge transformation	$\leftrightarrow$	charge

Supersymmetry

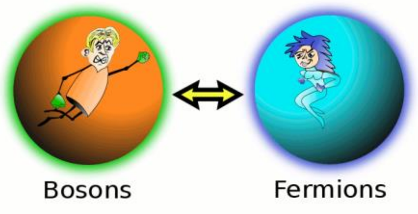
- Two types of particles in the universe:

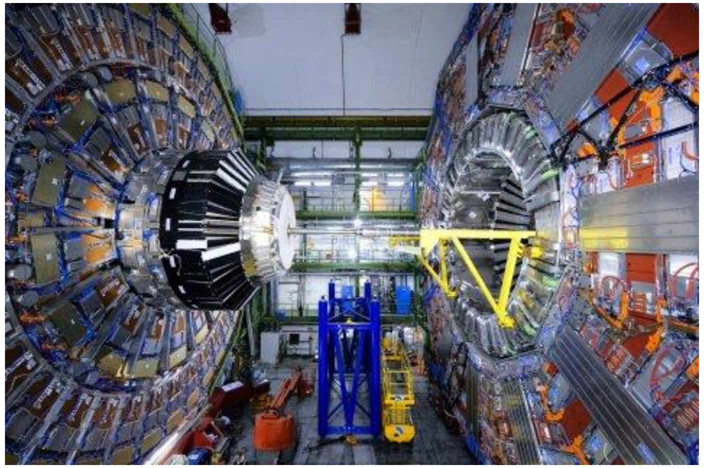
Fermions and Bosons.

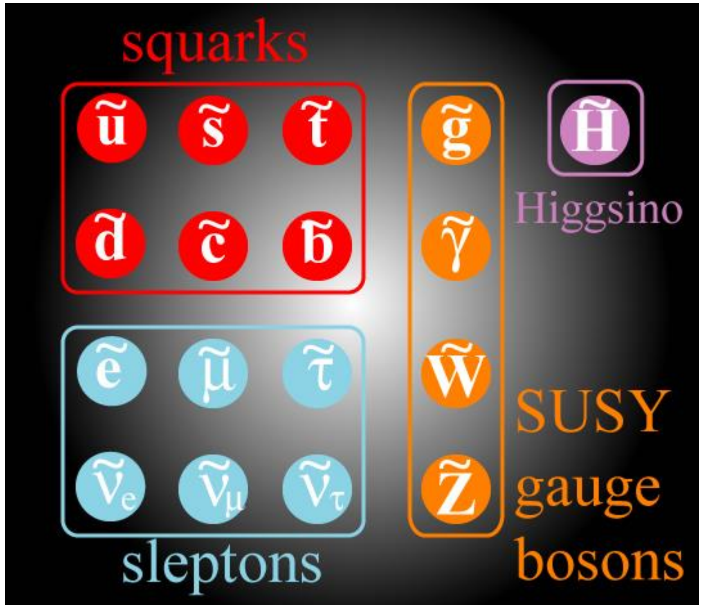
- Fermions: Particles of Matter.
- Bosons: elementary bosons are the force carries.

Supersymmetry (SUSY): a symmetry that related fermions to bosons to obtain a unified picture of nature.
It predicts superpartner particles for the known elementary particles: **Sparticles.**

Fermions ←→ Bosons







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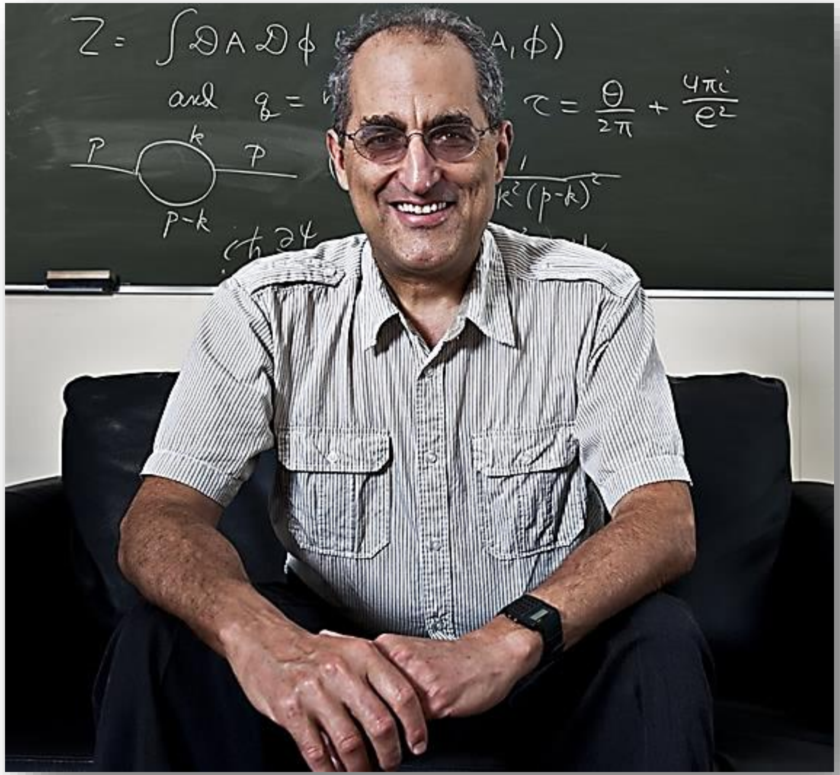
SUSY in QM

- *It turned out to be useful:*

1- Why some completely different potentials have almost the same energy spectra?

2- Why few potentials are exactly solvable and the others are not?

3- Why some potentials are reflectionless?



Edward Witten

SUSY in QM- The Basic Scenario

- For a particle of mass m in a box with zero potential:
- the Hamiltonian is: $\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$ and $H\,\psi(x) = E\,\psi(x)$
- Assume a normalized ground state $\psi(x)$ with zero ground state energy $E_0 = 0$, TISE and the potential V will be:

$$0 = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_0(x) + V(x)\psi_0(x) \Rightarrow V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)}$$

- Factorize that Hamiltonian:

$$\boxed{H_1 = A^\dagger A = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1(x)}$$

- Where:

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \qquad A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

SUSY in QM- The Basic Scenario

- The Hamiltonian will be: $H_1\psi(x) = (-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x))(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x))\psi(x)$
- Where W(x) is the superpotential. V can be: $V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x)$

The superpotential W(x):

$$A\psi_0(x) = 0 \Rightarrow H_1\psi_0(x) = A^\dagger A\psi_0(x) = 0 \Rightarrow \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\psi_0 + W(x)\psi_0 = 0 \Rightarrow W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'}{\psi_0}$$

$$H_2 = AA^\dagger = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_2(x) \Rightarrow H_2\psi(x) = [-\frac{\hbar^2}{\sqrt{2m}}\frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}}W'(x) + W^2(x)]\psi(x)$$

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x)$$

Where V1 and V2 are called supersymmetric partner potentials.

Energy Spectra

- **The energy spectra of both H_1 and H_2 are related, How?**
- For H_1, with eigenvalues E_n:

$$H_1\psi_n^{(1)}(x) = A^\dagger A\psi_n^{(1)}(x) = E_n^{(1)}\psi_n^{(1)}(x)$$

- $A\psi_n^{(1)}$ i

SUSY Breaking

$$H_2[A\psi_n^{(1)}(x)] = AA^\dagger A\psi_n^{(1)}(x) = E_n^{(1)}[A\psi_n^{(1)}(x)]$$

- Similarly for $\psi_n^{(2)}$:

$$H_2\psi_n^{(2)}(x) = A^\dagger A\psi_n^{(2)}(x) = E_n^{(2)}\psi_n^{(2)}(x)$$

$$H_1[A^\dagger\psi_n^{(2)}(x)] = A^\dagger AA^\dagger\psi_n^{(2)}(x) = E_n^{(2)}[A^\dagger\psi_n^{(2)}(x)]$$

$$E_2^{(1)} \underset{A^\dagger}{\overset{A}{\rightleftharpoons}} E_1^{(2)}$$

$$E_1^{(1)} \underset{A^\dagger}{\overset{A}{\rightleftharpoons}} E_0^{(2)}$$

$$E_0^{(1)} \underset{A^\dagger}{\overset{A}{\rightleftharpoons}} Nothing$$

$$\psi_n^{(2)}(x) = CA\psi_{n+1}^{(1)}(x)$$

SUSY for 1D Box

- For a particle with mass m in 1D infinite Square Well:

$$\psi_0^{(1)} = (\frac{2}{L})^{\frac{1}{2}} \sin(\frac{\pi x}{L}) \quad for \quad 0 \le x \le L$$

$$E_0 = \frac{\hbar^2 \pi^2}{2mL^2}$$

- Factorizing the Hamiltonian requires that:
- $H_1 = H - E_0$ and $E_n^{(1)} = \frac{n^2\hbar^2\pi^2}{2mL^2} - \frac{\hbar^2\pi^2}{2mL^2} = (n^2 - 1)\frac{\hbar^2\pi^2}{2mL^2}, \qquad n = 1, 2, 3, \ldots$
- Then,

$$\psi_n^{(1)} = (\frac{2}{L})^{\frac{1}{2}} \sin(\frac{(n+1)\pi x}{L}) \quad for \quad 0 \le x \le L$$

- Getting the 1st and the 2nd derivative will yield:

$$V_1(x) = \frac{\hbar^2}{2m} \frac{-(\frac{2}{L})^{\frac{1}{2}} \sin(\frac{\pi x}{L})\frac{\pi^2}{L^2}}{(\frac{2}{L})^{\frac{1}{2}} \sin(\frac{\pi x}{L})} = -\frac{\hbar^2\pi^2}{2mL^2}, \qquad 0 \le x \le L$$

Which is the same potential up to a constant.

SUSY for 1D Box

- The superpotential W(x):
- Then:

Recall that $V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x)$

$$V_2 = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}[2\csc^2(\frac{\pi x}{L}) - 1], \quad for \quad 0 \le x \le L.$$

- Note the Difference Between V_1 and V_2?

V_1 = Constant

- They have the same energy spectra.
- Thus:

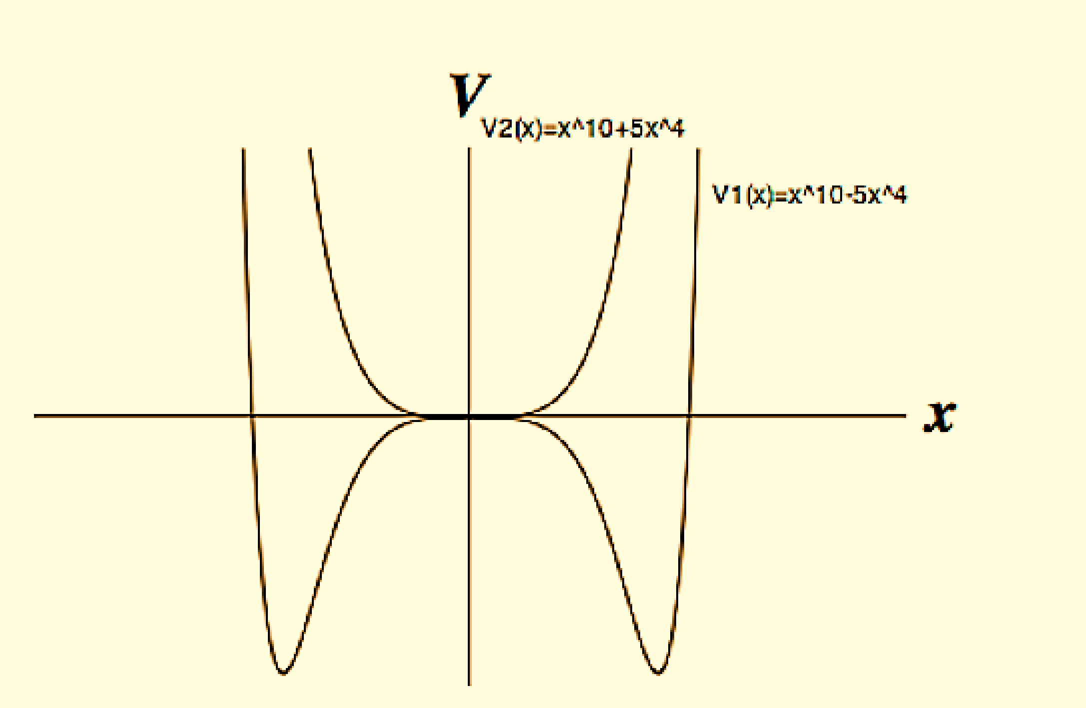
$$\psi_1^{(1)}(x) = (\frac{2}{L})^{\frac{1}{2}} \sin(\frac{2\pi x}{L}), \quad for \quad 0 \le x \le L \quad \Rightarrow \quad \psi_0^{(2)}(x) = -2\sqrt{\frac{2}{3L}} \sin^2(\frac{\pi x}{L}), \quad for \quad 0 \le x \le L$$

$$\longrightarrow \boxed{E_0^{(2)} = E_1^{(1)}}$$

The Shifted Oscillator

- Assume a superpotential: $W(x) = \beta x^5$
- Let, $\hbar/\sqrt{2m} = 1$
- Then,

$$V_1 = \beta^2 x^{10} - 5\beta x^4$$
$$V_2 = \beta^2 x^{10} + 5\beta x^4$$



- V_1 represents a double potential well.
- V_2 represents a single potential well.
- They have the same energy spectra.

The Varational Method

- TISE, generally, not easy to be solved.
- We estimate E_0.
- The Varational Principle gives the upper bound limit for E_0.
- It might be the exact value.

$$\boxed{E_{gs} \leq \langle\psi|H|\psi\rangle \equiv \langle H\rangle.}$$

- For SHO, $E_{gs} = \frac{1}{2}\ \hbar\omega.$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2.$$

- Pick up $\quad \psi(x) = Ae^{-bx^2}.$

$$\langle H\rangle = \langle T\rangle + \langle V\rangle.$$

$$\langle H\rangle = \frac{\hbar^2 b}{2m} + \frac{m\omega^2}{8b} \qquad \frac{d}{db}\langle H\rangle = \frac{\hbar^2}{2m} - \frac{m\omega^2}{8b^2} = 0 \Rightarrow b = \frac{m\omega}{2\hbar}$$

$$\langle H\rangle_{\min} = \frac{1}{2}\hbar\omega.$$

The Varational SUSY Method

- We can use SUSY to construct Superpartner Hamiltonians $H_1, H_2, \ldots etc$ to generate the full spectrum of the given anharmonic oscillator.
- Setting $\hbar = 2m = 1$, Assuming ground state energy $E_0^{(1)}$:

$$H_1 = A_1^\dagger A_1 + E_0^{(1)} = -\frac{d^2}{dx^2} + V_1(x)$$

where

$$A_1 = \frac{d}{dx} + W_1(x) \ , \ A_1^\dagger = -\frac{d}{dx} + W_1(x) \ , \ W_1(x) = -\frac{d \ln\psi_0^{(1)}}{dx}$$

- Then,

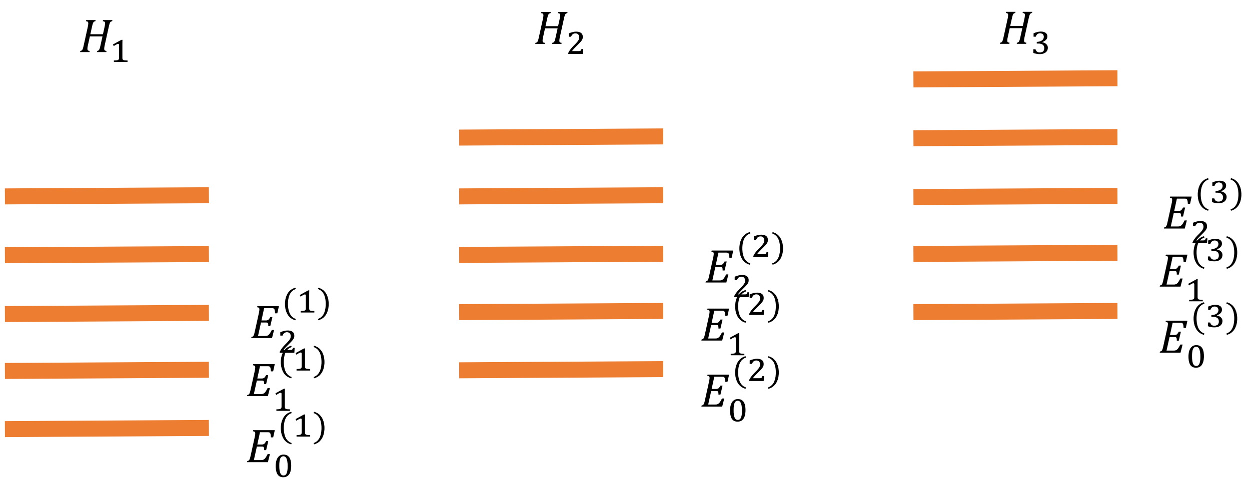
$$H_2 = A_1 A_1^\dagger + E_0^{(1)} = -\frac{d^2}{dx^2} + V_2(x)$$

The Varational SUSY Method

- Factorize H_2: $$H_2 = A_1 A_1^\dagger + E_0^{(1)} = A_2^\dagger A_2 + E_1^{(1)}$$
- By the same way, H_3: $$H_3 = A_2 A_2^\dagger + E_1^{(1)} = -\frac{d^2}{dx^2} + V_3(x)$$
- Furthermore,

$$\begin{aligned} E_n^{(3)} &= E_{n+1}^{(2)} = E_{n+2}^{(1)} , \\ \psi_n^{(3)} &= (E_{n+1}^{(2)} - E_0^{(2)})^{-1/2} A_2 \psi_{n+1}^{(2)} \\ &= (E_{n+2}^{(1)} - E_1^{(1)})^{-1/2} (E_{n+2}^{(1)} - E_0^{(1)})^{-1/2} A_2 A_1 \psi_{n+2}^{(1)} \end{aligned}$$

- Note that H_2 has a ground state that gives the same energy as the first excited state to H_2.



- Using the SUSY Variational principle,
- $E_0^{(1)} = < H_1 >$
- $E_0^{(2)} = E_1^{(1)} = < H_2 >$
- $E_0^{(3)} = E_1^{(2)} = E_2^{(1)} = < H_3 >$

Level	n	ρ	$\Delta(E)_{var}$	$\Delta(E)_{exact}$
0	1.183458	0.666721	0.669330	0.667986
1	0.995834	0.429829	1.727582	1.725658
2	1.000596	0.435604	2.316410	2.303151

Mathematical Techniques of SUSY have been used effectively in atomic, molecular, and nuclear physics.

References

- The main reference was **Supersymmetric Quantum Mechanics by Cooper** and his other papers. They can be found on the ARXIV.

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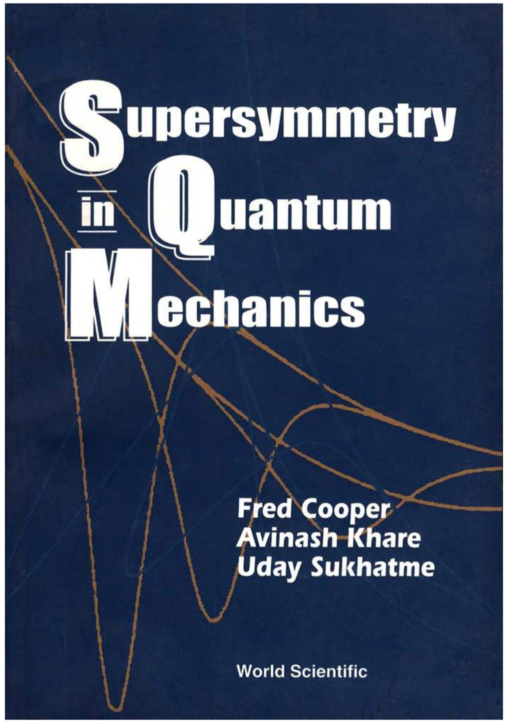

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Further Studies on Solving Eigenvalue Problems by Factorization
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On the factorization method in quantum mechanics

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