Relativistic causality and position space renormalization[1]

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1 Introduction

As Raymond had written[2] in his inimitable ironic style, he had *contributed to the "useful physics" (in his work with P. Moussa on angular distributions in 2-particle reactions) as well as to the "useless" quantum field theory (QFT), including the analysis of analytic properties of scattering amplitudes which follow from the causality principle – in joint work with Bros, Epstein, Glaser, Messiah* (see, e.g., [EGS]). Not surprisingly, our discussions at CERN were devoted to the useless part.

Perturbative ultraviolet renormalization in QFT was originally made known for momentum space integrals beginning with a high energy cutoff. But a causal position space approach has also been developed concurrently by Ernst Stueckelberg, a Swiss student of Sommerfeld, starting in the early forties (after a 1938 paper in German, anticipating the abelian Higgs-Kibble model, he switched to French - see [S45, S46, SR, SP]). This was taken up by a (French reading) mathematician, N. N. Bogolubov [B], who set himself to master QFT (while mobilized to work – with many others – on the Russian atomic project). The Russian work on renormalization (referred to in the book [BS] to whose list one should add [St]), perfected by Hepp [He], Zimmermann and Lowenstein [Z, LZ] (resulting in the /incomplete/ acronym BPHZ) is still substantially using the traditional momentum space picture. Even Epstein and Glaser [EG], who set the stage for the position space renormalization program based on locality, were proving Lorentz invariance of time-ordered products working in momentum space. It was only in [PS] – another famous unpublished preprint of Raymond's – that the problem was translated into a cohomological position space argument (see the historical survey in [G-BL]).

[1]Talk at the Raymond Stora Memorial meeting, CERN, December 4, 2015.
[2]I thank Paul Sorba for providing me with Stora's "bio0908" for the French Academy.

This led gradually to viewing renormalization as a problem of extending distributions defined originally for non-coinciding arguments, an approach that, in the words of Stora [S], "from a philosophical point of view, does not require the use – and the removal – of regularizations". The tortuous path from p- to x-space renormalization can be viewed, in modern parlance, as a duality transformation (the good old Fourier integral) mapping a large momentum onto a small distance problem. As relativistic causality does not require the existence of a Poincaré invariant vacuum state, the Stueckelberg-Bogolubov-Epstein-Glaser-Stora position space approach turned out to be the only one suited for the study of perturbative QFT on a curved background (which began flourishing during the last twenty years or so in the hands of Brunetti, Frerdenhagen, Dütsch, Hollands, Wald, among others).

Our collaboration started with Raymond reading Sect. 3.2 of the first volume of Hörmander's treatise [H] and pointing out that it is tailor-made for renormalization of a massless theory. It is based on the observation that the density

$$\mathbf{G}_m(x) := G_m(x)\frac{d^4x}{\pi^2} = \frac{1}{x^{2m}}\frac{d^4x}{\pi^2} \tag{1.1}$$

is a meromorphic distribution valued function of m with simple poles at $2m = 4, 5, 6, \ldots$. Subtracting the pole term, say at $m = 2$, we find a renormalized amplitude G_2^R defined up to a distribution with support at the origin. The ambiguity can be restricted by demanding that this distribution has the same degree of homogeneity as the function G_2 away from the origin (in our case -4). The resulting G_2^R is associate homogeneous of degree -4 and order one. More generally, a logarithmically divergent density $\mathbf{G}$ of an N-dimensional argument $\vec{x}$ defines an *associate homogeneous distribution G of degree $-N$ and order n* if

$$\lambda^N G(\lambda\vec{x}) = G(\vec{x}) + \sum_{j=1}^{n} R_j(G)(\vec{x})\frac{(\ln\lambda)^j}{j!},\ \lambda > 0, \tag{1.2}$$

where the distributions $R_j(G)$ can be viewed as generalized residues:

$$R_j(G) = Res[(\mathcal{E} + N)^{j-1}G(\vec{x})]\ , \quad \mathcal{E} = \sum_{\alpha=1}^{N} x^\alpha \partial_\alpha. \tag{1.3}$$

For a Feynman amplitude corresponding to a connected graph with V vertices $N = 4(V - 1)$. The order n of associate homogeneity corresponds to the number of (sub)divergences of the amplitude. One proves that only the coefficient to the highest power of the logarithm,

$$R_n(G) = res[(\mathcal{E} + N)^{n-1} G(\vec{x})]\delta(\vec{x}) \ , \tag{1.4}$$

is independent of the ambiguity of renormalization.

2 Causal factorization. Renormalization of associate homogeneous distributions

Let me start by sketching the recursive procedure based on the causal factorization principle in the (simpler) case of an euclidean perturbative QFT.

Denote the propagator between the points x_i and x_j of $\mathbb{R}^4$ by $G_{ij} = G_{ij}(x_{ij})$, $x_{ij} = x_i - x_j$. We assume it to be a smooth function away from the origin (i.e. off the diagonal $x_i = x_j$) of at most polynomial growth at infinity. In the case of a massless theory, treated in [NST], it is a rational homogeneous function of the type:

$$G_{ij}(x) = \frac{P_{ij}(x)}{(x^2)^{m_{ij}}} \ , \quad x^2 = \sum_{\alpha=1}^{4} (x^\alpha)^2, \ m_{ij} \in \mathbb{N}, \tag{2.1}$$

where $P_{ij}(x)$ is a homogeneous polynomial in the components x^α of x. (In a scalar QFT $P_{ij} = const$, $m_{ij} = 1$.) For the formulation of the principle of causal factorization one does not need the special form of the propagator.

We define ultraviolet renormalization by induction with respect to the number of vertices. Assume that all contributions of diagrams with less than n points are renormalized. If then Γ is an arbitrary connected n-point graph its renormalized contribution should satisfy the following inductive *causal factorization requirement.*

Let the index set $I = I(n) = \{1, \dots, n\}$ of Γ be split into any two non-empty non-intersecting subsets

$$I = I_1 \cup I_2 \ (I_1 \neq \emptyset \, , \ I_2 \neq \emptyset) \, , \ I_1 \cap I_2 = \emptyset \, .$$

Let $\mathcal{C}_{I_1,I_2} = \{(x_i) \in \mathbb{R}^{4n} \equiv (\mathbb{R}^4)^{\times n}; x_{j_1} \neq x_{j_2} \, \text{for} \, j_1 \in I_1, j_2 \in I_2\} = \mathcal{C}_{I_2,I_1}$. Let further G_1^R and G_2^R be the renormalized distributions associated with the subgraphs whose vertices belong to the subsets I_1 and I_2, respectively. We demand that for each such splitting our *euclidean* distribution G^R_Γ, defined on all partial diagonals, exhibits the *factorization property*:

$$G^R_\Gamma = G^R_1 \left(\prod_{\substack{i\in I_1 \\ j\in I_2}} G_{ij} \right) G^R_2 \quad \text{on} \quad \mathcal{C}_{I_1,I_2} \, , \tag{2.2}$$

where G_{ij} are factors (propagators) in the Feynman amplitude G_Γ which are smooth in $\mathcal{C}_{I_1,I_2}$ and can therefore be viewed as multiplicators.

Remark 1. In the Lorentzian signature case one demands that the points indexed by the set I_1 precede those of I_2 and uses Wightman functions instead of G_{ij} in the counterpart of (2.2) – justifying in this way the term *causal* (see Sect. 2.2 of [NST]).

In the case of a massless theory we add to this basic physical requirement two more *mathematical conventions* (MC) which restrict substantially the set of admissible renormalizations.

(MC1) *Renormalization maps rational homogeneous functions onto associate homogeneous distributions of the same degree of homogeneity; it extends associate homogeneous distributions defined off the small diagonal to associate homogeneous distributions of the same degree (but possibly of higher order) defined everywhere on* $\mathbb{R}^N$.

(MC2) *The renormalization map commutes with multiplication by polynomials.* If we extend the class of our distributions by allowing multiplication with smooth functions of no more than polynomial growth (in the domain of definition of the corresponding functionals), then this requirement will imply commutativity of the renormalization map with such multipliers.

The induction is based on the following *diagonal lemma*.

Proposition 1. *The complement* $C(\Delta_n)$ *of the small diagonal is the union of all* $\mathcal{C}_{I_1,I_2}$ *for all pairs of disjoint* I_1, I_2 *with* $I_1 \cup I_2 = \{1, \ldots, n\}$, *i.e.,*

$$C(\Delta_n) \;=\; \bigcup_{I_1 \dot\cup I_2 \,=\, \{1,\ldots,n\}} \mathcal{C}_{I_1,I_2} \, .$$

Proof. Let $(x_1, \ldots, x_n) \in C(\Delta_n)$. Then there are at least two different points $x_{i_1} \neq x_{j_1}$. We define I_1 as the set of all indices i of $I = I(n)$ for which $x_i \neq x_{j_1}$ and $I_2 := I \backslash I_1$. Hence, $C(\Delta_n)$ is included in the union of all such pairs. Each $\mathcal{C}_{I_1,I_2}$, on the other hand, is defined to belong to $C(\Delta_n)$. This completes the proof of our statement.

Remark 2. This simple proof did not satisfy Raymond. He felt that there should be a more natural combinatorial "diagonal lemma" that would serve both the euclidean and the Minkowski space framework allowing to complete each step of the renormalization by the extension of a distribution defined outside the full diagonal. Eventually, a beautiful general statement was indeed established (by Nikolov) - see Theorem A1 of [NST].

3 Renormalization of primitively divergent graphs

The above recursive procedure allows to reduce the elimination of divergences to the renormalization of primitively divergent graphs. We shall survey this initial step in the case of a euclidean massless QFT. A Feynman amplitude $G(\vec{x})$ is then a homogeneous function of $\vec{x} \in \mathbb{R}^N$. It is *superficially divergent* if G defines a homogeneous density in $\mathbb{R}^N$ of a non-positive degree of homogeneity:

$$G(\lambda\vec{x})\, d^N\lambda x = \lambda^{-\kappa} G(\vec{x})\, d^N x\ , \quad \kappa \geq 0 \quad (\lambda > 0)\,; \tag{3.1}$$

κ is called the (superficial) *degree of divergence.*

Proposition 2. *For any primitively divergent $G(\vec{x})$ and smooth (semi)norm $\rho(\vec{x})$ on $\mathbb{R}^N$ (allowed to vanish on a cone of lower dimension) one has*

$$[\rho(\vec{x})]^\epsilon G(\vec{x}) - \frac{1}{\epsilon}(Res\, G)(\vec{x}) = G^\rho(\vec{x}) + O(\epsilon). \tag{3.2}$$

Here $Res\, G$ is a distribution with support at the origin. Its calculation is reduced to the case $\kappa = 0$ of a logarithmically divergent graph by using the identity

$$(Res\, G)(\vec{x}) = \frac{(-1)^\kappa}{\kappa!}\, \partial_{i_1}...\partial_{i_\kappa} Res\, (x^{i_1}...x^{i_\kappa} G)(\vec{x}) \tag{3.3}$$

where summation is assumed (from 1 to N) over the repeated indices $i_1, ..., i_\kappa$. If G is homogeneous of degree $-N$ then

$$(Res\, G)(\vec{x}) = res\,(G)\,\delta(\vec{x}) \quad (\text{for } \partial_i(x^i G) = 0)\,. \tag{3.4}$$

Here the numerical residue $res\, G$ *is given by an integral over the hypersurface* $\Sigma_\rho = \{\vec{x} |\, \rho(\vec{x}) = 1\}$*:*

$$res\, G = \frac{1}{\pi^{N/2}} \int_{\Sigma_\rho} G(\vec{x}) \sum_{i=1}^{N} (-1)^{i-1} x^i dx^1 \wedge ... \hat{dx^i} ... \wedge dx^N, \qquad (3.5)$$

(a hat over an argument meaning, as usual, that this argument is omitted). The residue $res\, G$ *is independent of the (transverse to the dilation) surface* Σ_ρ *since the form in the integrant is closed in the projective space* $\mathbb{P}^{N-1}$*.*

We note that N is even, in fact divisible by 4, so that $\mathbb{P}^{N-1}$ is orientable.

Remark 3. The use of a homogeneous (semi)norm as a regulator is more flexible than dimensional regularization and should be also applicable in the presence of an axial (or chiral) anomaly.

The functional $res\, G$ is a period according to the definition of Kontsevich and Zagier [KZ]. The convention of accompanying the 4D volume d^4x by a π^{-2} factor ($2\pi^2$ being the volume of the unit sphere $\mathbb{S}^3$ in four dimensions) helps display the number theoretic character of residues. For one and two-loop graphs in a massless theory they are just rational numbers. For three, four and five loops in the φ^4 theory all residues are integer multiples of $\zeta(3), \zeta(5)$ and $\zeta(7)$, respectively. The first double zeta value, $\zeta(5, 3)$, appears at six loops (with a rational coefficient) (see the census of Schnetz who calls such residues *quantum periods* [Sch]). All *known* residues were (up to 2013) rational linear combinations of multiple zeta values of overall weight not exceeding $2\ell - 3$ [BK, Sch]. A seven loop graph was recently demonstrated [P, B14] to involve *multiple Deligne values* – i.e., values of *hyperlogarithms* at sixth roots of unity. An infinite series of ℓ-loop primitive φ^4 4-point *zig-zag graphs* were conjectured by Broadhurst and Kreimer [BK] and proven by Brown and Schnetz [BS12] to be proportional to $\zeta(2\ell - 3)$ with calculable rational coefficients (equal to $\binom{2\ell-2}{\ell-1}$ for $\ell = 3, 4$ – see [T] for an elementary derivation and further references).

The problem of integration over internal vertices in the φ^4 theory was tackled in [GGV, T15] elucidating an old result of [LZ].

There is a parallel between studying renormalzation of a *massless* QFT and neglecting friction by the founders of modern physics – starting with Galileo. Both idealizations allow to grasp the essence of the problem. Introducing friction in classical mechanics, and masses in the analysis of small distance behavior seems to be just adding technical details to the general picture. Raymond, however, *was worrying* about masses in QFT renormalization. (Nikolov is promising to throw soon some light on this problem.)

As we see, and the recent lectures of Claude Duhr [D] are confirming, good old (”useless”) QFT continues to serve both high energy physics and its healthy interaction with modern mathematics.

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