Non-Linear Realizations of Symmetries

(30 years with Raymond Stora)

Sergio FERRARA

R. Stora Memorial Meeting - CERN, December 4, 2015

My scientific intersection with Raymond Stora covers exactly thirty years, from our first joint paper *(with L. Girardello and O. Piguet)* in 1985 to our last recent one *(with M. Porrati, A. Sagnotti and A. Yeranyan)* in 2015. In between we co-authored two more papers *(with A. Masiero and M. Porrati in 1994, and with A. Marrani, E. Orazi and A. Yeranyan in 2011).*

All these papers have in common an algebraic approach to the problem at stake, which is combined with a wide use of geometrical and group-theoretical methods. These were the favorite tools of Raymond.

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THE NON POLYNOMIAL STRUCTURE OF SUPERSYMMETRIC CHIRAL ANOMALIES

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Supersymmetric chiral anomalies cannot be expressed as polynomials in the superconnection coefficients and suitable generalizations thereof which transform like the adjoint representation under global complex transformations.

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Bardeen anomaly and Wess–Zumino term in the supersymmetric standard model

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Abstract

We construct the Bardeen anomaly and its related Wess–Zumino term in the supersymmetric standard model. In particular we show that it can be written in terms of a composite linear superfield related to supersymmetrized Chern–Simons forms, in very much the same way as the Green–Schwarz term in four-dimensional string theory. Some physical applications, such as the contribution to the $g-2$ of gauginos when a heavy top is integrated out, are briefly discussed.

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Two-center black holes duality-invariants for *stu* model and its lower-rank descendants

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We classify 2-center extremal black hole charge configurations through duality-invariant homogeneous polynomials, which are the generalization of the unique invariant quartic polynomial for single-center black holes based on homogeneous symmetric cubic special Kä hler geometries. A crucial role is played by a horizontal $SL(p, \mathbb{R})$ symmetry group, which classifies invariants for p-center black holes. For $p = 2$, a (spin 2) quintet of quartic invariants emerge. We provide the minimal set of independent invariants for the rank-3 $\mathcal{N} = 2$, $d = 4$ *stu* model, and for its lower-rank descendants, namely, the rank-2 st^2 and rank-1 t^3 models; these models, respectively, exhibit seven, six, and five independent invariants. We also derive the polynomial relations among these and other duality invariants. In particular, the symplectic product of two charge vectors is not independent from the quartic quintet in the t^3 model, but rather it satisfies a degree-16 relation, corresponding to a quartic equation for the square of the symplectic product itself. © *2011 American Institute of Physics*. [doi:10.1063/1.3589319]

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Orignal Paper

Generalized Born–Infeld actions and projective cubic curves

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We investigate $U(1)^n$ supersymmetric Born–Infeld Lagrangians with a second non–linearly realized supersymmetry. The resulting non–linear structure is more complex than the square root present in the standard Born–Infeld action, and nonetheless the quadratic constraints determining these models can be solved exactly in all cases containing three vector multiplets. The corresponding models are classified by cubic holomorphic prepotentials. Their symmetry structures are associated to projective cubic varieties.

geometry and also affords a natural interpretation in terms of projective cubic $(n-2)$–varieties.

This article is organized as follows. In Section 2 we review the non–linear $N = 2$ constraints that define the $N = 2$ multi–field BI actions. In all $n = 2$ [5] and, as we shall see, also in the $n = 3$ cases, these constraints can be solved explicitly, in spite of the fact that they are coupled systems of n quadratic equations. In Section 3 we summarize the classification of projective cubic curves [13], whose singularity structure (see Table 1) underlies the possible independent sets of $N = 2$ constraints (see eqs. (8) and (9)). These are connected to the orbits of the three–fold symmetric projective representa-

It should not come as a surprise, then, that our common interest in Physics, over the years, was especially the supersymmetric extension of non-linear realizations of global and local symmetries. These symmetries afford a range of applications, which include the SM and the MSSM.

Non-linear realizations of supersymmetry play a central role in the supersymmetric Born-Infeld theory and in its multi-field generalizations in the N=2 case, where the tools of “special geometry” and the theory of projective invariant cubic polynomials come to rescue in some classification problems that they raise.

In the MSSM the role of non-linear realizations is related to the Kahlerian extension of the $SU(2)xU(1)_Y/U(1)_{e.m.}$ coset, whose three complex coordinates correspond to three Goldstone bosons and their supersymmetric partners (completing massive vector multiplets for the $W^{\pm}$,Z^0 vector bosons, (1,2(1/2),0)).

The complexified coset is then $SL(2,C)xGL(1,C)_Y/GL(1,C)_{e.m.}$, and we know that the linearly realized versions of the SM and the MSSM require the addition of an extra scalar boson in the former, and of a chiral superfield S in the latter.

Indeed the (linearly realized) MSSM rests on the introduction of a GL(2,C) matrix Φ (rather than SL(2,C)), so that

$$\Phi = S\,U \quad (\mathrm{det}U = 1) \;\longrightarrow\; \mathrm{det}\Phi = S^2$$

In this fashion Φ carries a linear realization of SU(2) x U(1), with two chiral doublets $H_{(1)}$ and $H_{(2)}$,

$$\Phi = \begin{pmatrix} H_{(1)1} & H_{(2)1} \\ H_{(1)2} & H_{(2)2} \end{pmatrix} \qquad \mathrm{det}\Phi = H_{(1)} \times H_{(2)}$$

Therefore the non-linear σ-model is obtained from the linear one integrating out the S multiplet.

In the collaboration with *Masiero* and *Porrati* we constructed the Bardeen anomaly and the Wess-Zumino term of the MSSM in its non-linear phase. This anomaly and the associated Wess-Zumino term give effects that replace fermion loops when chiral fermions (contributing to the anomaly) are integrated out (for instance, the top quark multiplet).

Physical effects of such Wess-Zumino terms is to make an anomalous (gauge and supersymmetric) theory gauge invariant, as occurs for the Green-Schwarz mechanism in String Theory.

This confirmed a previous result of *Bilchak, Gastmans and Van Proeyen*, who showed that the MSSM, with an anomalous fermion content (but classically supersymmetric) gives loop corrections generating magnetic moment transitions which violate supersymmetric sum rules *(SF, Porrati)*.

We showed that “supersymmetric” Wess-Zumino terms give additional (classical) contributions to these (loop) magnetic moment transitions which restore supersymmetry (and the supersymmetry is not anomalous if the gauge symmetry is not)

Magnetic moment sum rules (connect the magnetic moments of partners in the same massive multiplet with top spin J+1/2: J+1/2, J, J, J-1/2). Defining the gyromagnetic ratio g_J of a given particle of mass M and spin J:

$$\mathbf{u} = \frac{e}{2M} g_J \, \mathbf{J}$$

Then for $J_{max}>1$ we have the sum rule ($J_{max}=J+1/2$)

$$g_{J+1/2} - 2 = 2J\, h_J \,, \; g_J - 2 = (2J+1)\, h_J \,, \; g_{J-1/2} - 2 = (2J+2)\, h_J$$

$$E.g. \text{ for } J_{\max} = 1 : g_{1/2} - 2 = 2\, h \,, \; g_1 - 2 = h \,; \text{ for } J_{\max} = 1/2 : g_{1/2} - 2 = 0$$

(SF, Remiddi)

Here h_J is the transition magnetic moment for (J+1/2,J-1/2) states. In an anomaly free theory h≠0 from loop effects, while in an anomalous theory loop effects violate the sum rules but extra contributions from Wess-Zumino terms restore supersymmetry.

The link between this early work with Raymond and the later collaborations in 2011 and in 2014 is to be found in the applications of algebraic and geometrical methods to the problems we were confronting.

An interesting problem was the classification of Extremal (asymptotically flat) Black-Hole Orbits in D=4, which in the (continuous) classical limit of their dyonic charges were classified by invariants of the electric-magnetic duality groups, such as $E_{7(7)}$ in the N=8 supergravity.

In fact, due to a pioneering work by *Gaillard and Zumino* all duality groups must exhibit a “symplectic representation” for their electric and magnetic charges, and so it happens in all N-extended supergravities (coupled to matter or not).

This representation is dyonic and the non-compact nature of these groups rest on the presence of scalars which are not inert under their action, but rather transform in a non-linear realization. The 70 scalars of N=8 supergravity, which span the $E_{7(7)}/SU(8)$ coset, are just an example.

We classified (with *Marrani, Orazi, Yeranyan*) the orbits of N=2 duality group Sl(2)xSl(2)xSl(2) and its descendants, corresponding to the so-called stu, st^2, t^3 models of N=2 supergravity, where the dyonic black-hole octet $Q_{a_1 a_2 a_3}$ is in the (1/2,1/2,1/2) of Sl(2)xSl(2)xSl(2) and the classic Bekenstein-Hawking Entropy-area relation takes the form

$$A_{stu} = \frac{1}{4\pi} \sqrt{I_4}$$

where I_4 is the Cayley Hyperdeterminant

$$I_4 \sim Q_{a_1 a_2 a_3} Q_{b_1 b_2 b_3} Q_{c_1 c_2 c_3} Q_{d_1 d_2 d_3} \epsilon^{a_1 b_1} \epsilon^{a_1 b_1} \epsilon^{a_2 c_1} \epsilon^{a_3 d_1} \epsilon^{b_2 c_2} \epsilon^{b_3 d_1} \epsilon^{c_3 d_3}$$

The main result was the extension from one-center (Q_1) to two-center (Q_1,Q_2) charge configurations, making use of a horizontal extra symmetry $Sl_H(2)$ exchanging Q_1 and Q_2.

It was shown that these models, with charge vectors of the types (1/2,1/2,1/2) of Sl(2)xSl(2)xSl(2), (1/2,1) of Sl(2)xSl(2), and (3/2) of Sl(2) respectively have one center specified by a single invariant while for two centers they have, respectively, 7 (a quintet and two singlets), 6 and 5 independent invariants. These results are consistent with an earlier analysis by *V. Kac* on the theory of nilpotent orbits of semi-simple Lie algebras.

In N=8 under $E_{7(7)} \rightarrow Sl(2)^3 x SO(4,4)$ [Sl(2) triality connected to SO(4,4) triality].

The last work with Raymond (with *Porrati*, *Sagnotti* and *Yeranyan*) is also the one that he found most fascinating. It is connected with the problem of classifying multifield (N=2) supersymmetric Born-Infeld models with *n* vector multiplets resorting to the theory of projective cubic invariant polynomials related to the singular structure of cubic *n*-2 varieties.

This connection emerges from the special geometry of the Lagrangians for N=2 vector multiplets in special limits.

The chiral multiplet part of the N=2 vector multiplets (2,1/2) [m=0] and (1/2,0,0) [m=μ] is integrated out, giving rise to an effective self-interacting theory for leftover N=1 vector multiplets.

The end result describes an *n*-field generalization of the *n*=1 Born-Infeld action, which incorporates a hidden N=2 supersymmetry, spontaneously broken to a manifest N=1 theory.

We now describe these generalizations in some detail.

Let us begin by observing that the standard bosonic (single field) Born-Infeld theory can be obtained solving the constraint *(Rocek; Ivanov, Kaupstnikov; Rocek, Tseytlin; Bagger, Galperin; Kuzenko, Theisen)*

$$G_+^2 \; + \; F\left(m \; - \; \overline{F}\right) \; = \; 0$$

where *F* is a complex auxiliary field. The Lagrangian is then

$$\begin{aligned}\mathcal{L}_{BI} \;&=\; Im\left[(e_1 + ie_2)(F_1 + iF_2)\right] \;=\; e_1\, ImF \;+\; e_2\, ReF \\ &=\; -\,\frac{e_1}{m}\, G\widetilde{G} \;+\; \frac{e_2 m}{2}\left[1 \;-\; \sqrt{1 \;+\; \frac{4}{m^2}\, G^2 \;-\; \frac{4}{m^4}\,\left(G\widetilde{G}\right)^2}\,\right]\end{aligned}$$

This rests on a (microscopic) action with prepotential dominated, in the infrared, by $\quad \mathcal{F} \;=\; \frac{i}{2}\, X^2 \;+\; \frac{1}{M}\, X^3 \;+\ldots$

For *n*>1 the proposed multi-field generalization reads

$$d_{ABC}\left[G_+^B\, G_+^C \; + \; F^B\left(m^c \; - \; \overline{F}^C\right)\right] \; = \; 0$$

This is part of the superfield constraint for N=2 vector multiplets

$$d_{ABC}\left[W^B\, W^C \; + \; Y^B\left(m^c \; - \; \overline{D}^2\, \overline{Y}^C\right)\right] \; = \; 0$$

with $V^A_{N=2}\left(Y^A, W^A_\alpha\right)$

N=2 rigid supersymmetry also implies the constraints

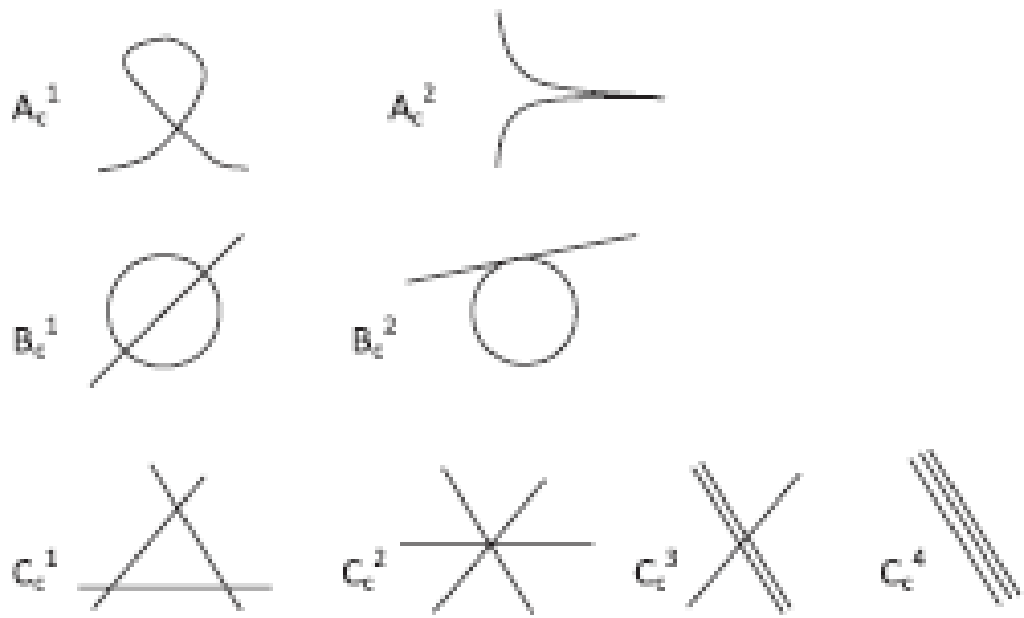
$$d_{ABC}\, W^B\, X^C \; = \; 0 \; , \quad d_{ABC}\, X^B\, X^C \; = \; 0$$

and note that with $W \; = \; \frac{1}{3!}\, d_{ABC}\, X^A\, X^B\, X^C$

$$\frac{\partial W}{\partial X^A} \; \equiv \; d_{ABC}\, X^B\, X^C \; = \; 0$$

Cubic polynomials can be classified in terms of tri-fold symmetric projective representations of Sl(n,R) and their invariants. For n=2 one has points, for n=3 curves, for n=4 surfaces, and so on. The degenerations of the invariants correspond to cubic varieties with different singularity structures. There is actually one invariant quartic polynomial (I_4) for n=2, but there are two invariant polynomials (P_4,Q_6), quartic and sextic, for n=3. One combination of them is the discriminant of the cubic, of degree 12 ($I_{12} = P_4^3 - 6\, Q_6^2$) *(B. Sturmfels, "Algorithms in Invariant Theory").* The number of inequivalent Born-Infeld's depends on the number of orbits of $P(Sym^3(R^n))$ of Sl(n,R). For n=3, (P_4,Q_6), the classification gives 9 cases over the complex ($1+8_{deg}$) and 15 cases over the reals ($2+13_{deg}$).

For a generic non–singular cubic $I_{12} \neq 0$, while for singular ones $I_{12}=0$. The singular cases can be further classified in terms of three types of degenerations, depending on whether the curve U = 0 has singular points (A), is the product of a conic and a line, U = L × Q (B), or is a product of three lines, U = L × L × L. As in other contexts, it is convenient to begin by considering the classification over the complex numbers before turning to its finer counterpart over the reals.



Over the complex there are eight degenerate cases *(J. Harris, "Algebraic Geometry, a First Course")*. Two of them are of type (A), that is singular points which are either a node or a cusp. Two are of type (B), and one can distinguish them further according to whether the line intersects the conic or is tangent to it. Finally, four are of type (C), and one can distinguish them further according to whether the lines intersect pairwise (triangle), or all intersect at the same point, or two are concurrent and intersect the third, or finally all three are concurrent.

All these cases fall in two groups, distinguished by the pair of values of P_4 and Q_6 . When these do not vanish, ∂I_{12} and $\partial^2 I_{12}$ may or may not vanish. Moreover, when P_4 and Q_6 both vanish, different derivatives of these invariant polynomials may or may not vanish.

The classifications allow to solve in a closed form the quadratic constraints

$$d_{ABC}\left[G_+^B G_+^C + F^B\left(m^c - \overline{F}^C\right)\right] = 0$$

The explicit solutions are give in the Appendix of the last paper with *Raymond Stora*.

Let us observe that our multi-field generalization of the Born-Infeld theory differs from the previous proposal of *Aschieri, Brace, Morariu, Zumino*, since it preserves a hidden N=2 supersymmetry and thus can be regarded aa a generalization of the Goldstino action for N=2 → N=1 breaking.

The local versions of these models should correspond to a general treatment of the partial super-Higgs effect when one gravitino stays massless, while the other gets a mass but still completes a massive multiplet of the unbroken N=1:

$$\left(2, \frac{3}{2}\right)_{m=0} + \left(\frac{3}{2}, 1, 1, \frac{1}{2}\right)_{m \neq 0} + \ldots$$

With his deep knowledge of modern mathematical tools, *Raymond* gave an essential contribution also to this field.