An elementary introduction to BRST symmetry

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Landau's gauge quantization of electrodynamics

- Landau's gauge quantization of QED was suggested by Landau and refined by Nakanishi (Nakanishi N, 1974).
- It produces a quantum structure whose extension to non-abelian gauge theories gives a natural framework for the BRST construction.

QED

- The starting point is the Lagrangian density:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + b\, \partial_\mu A^\mu + A_\mu j_\mu$$

in which $j_\mu(x)$ is the electromagnetic current density which is conserved due to the equations of motion of the charged matter fields.

Gupta-Bleuler

- The corresponding Euler-Lagrange equations are

$$\partial_\mu A_\mu(x) = 0$$
$$\partial^2 A_\mu(x) = j_\mu(x) + \partial_\mu b(x)$$
$$\partial^2 b(x) = 0$$

- The Lorentz's gauge fixing condition is strongly implemented by the Euler-Lagrange equation of the b field (first equation).
- This field gives an additive contribution to the current in the Euler-Lagrange equation of the vector potential (second equation) and satisfies the d'Alembert's wave equation (third equation) due to current conservation.

Normal modes

- The standard field quantization procedure suited for Feynman's perturbation theory is based on the normal mode decomposition of the solutions of the free field equations, which are obtained by taking $j_\mu(x) = 0$.
- In relativistic quantum field theory free quantum fields play the physical role of asymptotic fields. Asymptotic fields are built with creation and annihilation operators of the Fock space of scattering states. Since there are two kinds of scattering states, the in-going and the out-going ones, there also are two different kinds of asymptotic free fields which are known as the in and out fields.

Normal modes in the Landau gauge

- $A_\mu(x) = A_\mu^{(ph)}(x) + A_\mu^{(u)}(x)$
- $A_\mu^{(ph)}(x) =$
  $\int \frac{d^4k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{-i\,k\,x}\, \theta(k_0) \left[\delta(k^2) \sum_{h=\pm 1} \epsilon_\mu(\vec{k},h)\, a(\vec{k},h)\right] + c.\text{-}c.$
- $A_\mu^{(u)}(x) = i \int \frac{d^4k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{-i\,k\,x}\, \theta(k_0) \left[\delta(k^2) \left(k_\mu\, \alpha(\vec{k}) + \right.\right.$
  $\left.\left. -\bar{k}_\mu\, \frac{\beta(\vec{k})}{k\,\bar{k}}\right) - k_\mu\, \delta'(k^2)\, \beta(\vec{k})\right] + c.\text{-}c.$
- $b(x) = \int \frac{d^4k}{(2\,\pi)^{3/2}}\, \mathrm{e}^{-i\,k\,x}\, \theta(k_0)\, \delta(k^2)\, \beta(\vec{k}) + c.\text{-}c.$

Normal modes in the Landau gauge

where:

- $\epsilon^\mu(\vec{k}, h)$ for $h = \pm 1$ are space-like circular polarization vectors such that: $\epsilon\, k = \epsilon\, \bar{k} = 0$, $\epsilon^*_\mu(\vec{k}, h) = \epsilon_\mu(-\vec{k}, h)$;
- $\bar{k}$ is the parity reflected image of k.
- The polarization vectors define the unpolarized photon density matrix

$$\sum_{h=\pm 1} \epsilon_\mu(\vec{k}, h)\epsilon^*_\nu(\vec{k}, h) = -g_{\mu\nu} + \frac{k_\mu\, \bar{k}_\nu + \bar{k}_\mu\, k_\nu}{k\, \bar{k}}$$

Normal modes in the Landau gauge

- The Fourier analysis of the general solution $A_\mu(x)$ involves therefore four complex coefficients.
- Two of them, $a(\vec{k}, h)$ with $h = \pm 1$, are associated with the physical states with helicity h;
- $\alpha(\vec{k})$ corresponds to the longitudinally polarized states;
- $\beta(\vec{k})$ is the Fourier transform of $b(x)$.

Quantization

- In the quantized system the vector potential components correspond to operators which satisfy the equal-time canonical commutation relations

$$[A_i(\vec{r}), \partial_0 A_j(\vec{r'}) - \partial_j A_0(\vec{r'})] = i\,\delta_{i,j}\,\delta(\vec{r} - \vec{r'})$$
$$[A_0(\vec{r}), b(\vec{r'})] = i\,\delta(\vec{r} - \vec{r'})$$

Hilbert spaces with indefinite metric

- Expressed in terms of the Fourier coefficients of the general solution,

$$[a(\vec{k},h), a^+(\vec{q},h')] = 2\,|\vec{k}|\,\delta_{h,h'}\,\delta(\vec{k}-\vec{q})$$
$$[\alpha(\vec{k}), \beta^+(\vec{q})] = [\beta(\vec{k}), \alpha^+(\vec{q})] = 2\,|\vec{k}|\,\delta(\vec{k}-\vec{q})$$

- $a^+(\vec{k},h)$ and $(\alpha^+(\vec{k})+\beta^+(\vec{k}))/\sqrt{2}$ satisfy standard commutation rules and can be identified with creation operators.
- $(\alpha(\vec{k})-\beta(\vec{k}))/\sqrt{2}$ obeys a commutator with the wrong sign.
- This means that $(\alpha^+(\vec{k})-\beta^+(\vec{k}))/\sqrt{2}$ creates negative norm states: the space underlying the quantization is not a Fock-Hilbert space.

Gupta-Bleuler's solution

- The appearance of negative norm states contrasts with the usual probabilistic interpretation of the inner product in Quantum Mechanics.
- The Gupta-Bleuler way out of this paradox consists in identifying the physical state vector space with the subset of the Fock space annihilated by $\beta(\vec{k})$.
- This choice is possible since $b(x)$ is a free field and hence the prescription selects a subspace of the Fock space which is physically invariant, i.e. invariant under the action of the observables.

Gupta-Bleuler's solution

- The physical invariant subspace is spanned by states generated by polynomials of $a^{+}(\vec{k}, h)$ which are positive norm states and mixed polynomials of $a^{+}(\vec{k}, h)$ and $\beta^{+}(\vec{k})$ which are orthogonal to the rest of the physical space and have zero norm.
- These states can be freely added to the positive norm physical states without changing their inner products and correspond to generic gauge variations of the physical states.

The Yang-Mills model

- The situation is completely different for non-abelian models such as YM theories: In this case the field analogous to $b(x)$ is not a free field.
- Disregarding for simplicity the possible couplings to matter, one starts from the Lagrangian density

$$L = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu\right) \times$$
$$\times\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu\right) + b^a\, \partial^\mu A^a_\mu$$

The Yang-Mills field equations

- The field equations are

$$\partial^2 A^a_\mu + g\, f^{abc} A^b_\nu (\partial_\nu A^c_\mu - \partial_\mu A^c_\nu) - g^2 f^{abc} f^{cde} A^d_\mu A^b_\nu A^e_\nu = \\ = \partial_\mu b^a$$

$$\partial^\mu A^a_\mu = 0$$

- These two equations do not imply anymore that $\partial^2 b^a(x) = 0$ except when $g = 0$.

The problem with GB

- For $g = 0$ the above Lagrangian reduces to the sum of N pure QED Lagrangians.
- As mentioned, in perturbation theory one starts from the free theory describing the quantum mechanics of the asymptotic scattering states. Thus the space of asymptotic in-going states of YM theory consists of the tensor product of N indefinite-metric Fock spaces analogous to that of QED.

The non-abelian difficulty

The difficulty arises when one tries to identify a physical subspace with semi-definite norm. The extension of the Gupta-Bleuler choice valid for quantum electrodynamics does not identify a physically invariant subspace of the non-abelian theory since $b^a(x)$ is not a free field.

Fadeev-Popov's solution

- The solution to this difficulty was found by Faddeev and Popov, using the Feynman functional quantization method.
- The point they made is that the correct identification of the physical subspace requires an extension of the indefinite-metric space of states.
- This is obtained in a perfectly local approach by introducing ghost fields which compensate the unphysical degrees of freedom.

A quantum mechanical model for the Kugo-Ojima quartet mechanism

- The idea of compensating fields, the Kugo-Ojima quartet mechanism, is based on the extension to field theory of the following example with a finite number of degrees of freedom.

The Bosonic 2-oscillator

- Consider a two-dimensional bosonic harmonic oscillator quantized through canonical commutation relations:

$$H_B = \bar{P}\,P + \omega^2 \bar{z}\,z = \omega\,(A^\dagger\,A + \bar{A}^\dagger \bar{A} + 1)$$
$$z = (x + iy)/\sqrt{2} \quad P = (p_x - i\,p_y)/\sqrt{2}$$
$$E_{n,m} = \omega\,(n + m + 1),\, n, m = 0, 1, 2, \ldots$$

- The ground state $|0\rangle$ is identified with the vacuum state of a Fock space.

The Fermionic oscillator

- Consider a system with the same Hamiltonian quantized by replacing canonical commutators with anti-commutators:

$$H_F = \bar{\Pi}\,\Pi + \omega^2\,\bar{\zeta}\,\zeta = \omega\,[\bar{a}^\dagger\,\bar{a} + a^\dagger\,a - 1]$$
$$E_{n,m} = \omega\,(n + m - 1) \qquad n, m = 0, 1$$

- The canonical coordinates, being nilpotent, cannot correspond to Hermitian operators.

The super-oscillator

- The bosonic and the fermionic oscillators can be combined into a supersymmetric oscillator:

$$H_{B-F} = H_B + H_F = \omega\,(A^\dagger A + \bar{A}^\dagger \bar{A} + \bar{a}^\dagger\, \bar{a} + a^\dagger\, a)$$

- The resulting theory is invariant under transformations interchanging bosonic and fermionic degrees of freedom generated by the BRST operator

$$Q = i(\bar{A}^\dagger\, \bar{a} - a^\dagger\, A)$$

- This charge is nilpotent

$$Q^2 = 0$$

The supersymmetry algebra

- Q is not Hermitian in the Fock-Hilbert space:

$$Q^{\dagger} = i\,(A^{\dagger}\, a - \bar{a}^{\dagger}\, \bar{A})$$

- $Q^{\dagger}$ is also nilpotent and conserved.
- Q and $Q^{\dagger}$ generate an extended supersymmetry algebra characterized by the anti-commutation relation

$$\{Q, Q^{\dagger}\} = \frac{H}{\omega}$$

Hodge decomposition

- Since Q is nilpotent, $\ker Q$ contains $\operatorname{im} Q$:

$$\ker Q = \ker H \oplus \operatorname{im} Q$$

- The supersymmetry algebra implies that the full Hilbert space of states admits the orthogonal decomposition

$$\ker H \oplus \operatorname{im} Q \oplus \operatorname{im} Q^\dagger$$

which is the Hodge decomposition associated with Q.

Eliminating excited states

- The Fock vacuum $|0\rangle$ is the only vector of the Fock space of the B-F model invariant under the action of both Q and $Q^\dagger$.
- $\ker H$ for this model is therefore one-dimensional and it is generated by $|0\rangle$.
- An obvious way of disregarding the unwanted excited states of the B-F oscillator is to select supersymmetric invariant vectors, that is, vector states annihilated by both Q and $Q^\dagger$. However, in the field theory context, this choice — which turns out to be equivalent to the Dirac-Bergmann construction — is an exceedingly rigid constraint which contrasts with explicit Lorentz covariance.

Selecting physical states

- If, on the contrary, we weaken our condition, by selecting the set of the vector states annihilated by only one of the two charges, say Q, too many unphysical vector states — the elements of $\mathrm{im}\, Q$ — survive the selection.
- The way to overcome this difficulty is to void $\mathrm{im}\, Q$ of physical content.

The indefinite inner product

- This is accomplished by introducing an indefinite inner product in the Fock space, relative to which Q is required to be pseudo-Hermitian:

$$\bar{A}^{+} = A^{\dagger} \qquad A^{+} = \bar{A}^{\dagger} \qquad \bar{a}^{+} = a^{\dagger} \qquad a^{+} = \bar{a}^{\dagger}$$

- One verifies immediately that both H_{B-F} and Q are pseudo-Hermitian operators.

BRST cohomology

- The Fock vacuum $|0\rangle$ has positive pseudo-norm, $\langle 0|0\rangle > 0$. Furthermore $Q|0\rangle = 0$.
- The states of $\text{im } Q$ are pseudo-orthogonal to those of $\ker Q$ since Q is pseudo-Hermitian.
- Since $\text{im } Q$ is pseudo-orthogonal to $\ker Q$, $\ker Q$ is not a Hilbert space.
- However the quotient space

$$H_{phys} = \ker Q / \text{im } Q$$

known as the BRST cohomology space, is the space which can be naturally identified with the space of physical states.

BRST cohomology

- The Hodge decomposition establishes an isomorphism between H_{phys} and $\ker H$.
- Adding arbitrary states in $\operatorname{im} Q$ to states in $\ker Q$ does not change their pseudo-inner products. Therefore, from the point of view of the physical interpretation based on the probabilistic interpretation of the pseudo-inner product, two states in $\ker Q$ whose difference belongs to $\operatorname{im} Q$ must be considered equivalent.
- If the inner product induced on $\ker Q / \operatorname{im} Q$ by the pseudo-inner product on the original space is definite positive, H_{phys} is a Hilbert space. This has to be investigated on a case by case basis.

Interactions

- In the case of the B-F model, $\ker Q$ coincides with the equivalence class of $|0\rangle$ and H_{phys} is a one-dimensional Hilbert space: complete compensation of degrees of freedom occurs in this particular model.
- A less trivial system can be obtained by coupling the B-F oscillator to some further mechanical system, e.g. to a bosonic one-dimensional, physical, harmonic oscillator

$$h = \frac{1}{2}\,(p_{phy}^2 + \Omega^2\, x_{phy}^2) = \Omega\,(A_{phy}^{\dagger}\, A_{phy} + \frac{1}{2})$$

Interactions

- The space of states of this model is the tensor product of the B-F Fock space with that of the physical oscillator. This is an indefinite-metric vector space.
- The physical space, $\ker Q / \operatorname{im} Q$, coincides with the equivalence classes of the vector space spanned by $(A^{\dagger}_{phy})^n|0\rangle$ for any $n \geq 0$.
- Thus H_{phys} is equivalent to the Hilbert-Fock space of the physical oscillator.

Interactions

- One can couple the physical oscillator to the B-F oscillator, e.g. through the interaction

$$V = i\,\lambda\{Q, x\,\bar{z}\,\zeta\} = \lambda\,x\,(z\,\bar{z} + \bar{\zeta}\,\zeta)$$

  which is pseudo-Hermitian and, being an anticommutator $\{Q, X\}$, commutes with Q.

- The dynamics induced by the complete Hamiltonian

$$H_C = H_{B-F} + h + V$$

  on the full Fock space is affected by V.

Gauge-independence

- However the dynamics in H_{phys} does not depend on the coupling V since all the matrix elements $\langle p|V|p'\rangle$ with $p, p' \in H_{phys}$ vanish.
- Therefore the dynamics of the coupled physical oscillator in H_{phys} is perfectly equivalent to that of the uncoupled physical oscillator and one concludes that bosonic and fermionic degrees of freedom of the F-B oscillator compensate each other.

Quantum field theory

- This compensation mechanism is naturally extended to quantum field theory.
- As an example, consider a neutral scalar field Φ interacting with a couple of unphysical complex, scalar fields ϕ and ψ quantized with opposite statistics.

$$L = L(\Phi) + \partial_\mu \bar{\phi}\, \partial_\mu \phi - m^2 \bar{\phi}\phi + \\ + \partial_\mu \bar{\psi}\, \partial_\mu \psi - m^2 \bar{\psi}\psi + g\,\Phi(\bar{\phi}\phi + \bar{\psi}\psi)$$

The quartet mechanism in quantum field theory

- In view of a perturbation expansion in powers of g, one begins quantizing the free ($g = 0$) theory.
- For each value $\vec{k}$ of the spatial momentum, one finds four oscillators corresponding to the fields $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$ plus one further oscillator associated with the field Φ.
- For any $\vec{k}$ the system composed of the first four oscillators is equivalent to the B-F oscillator described above, while the oscillator associated with Φ plays the role of the physical oscillator.

The BRST operator in quantum field theory

- The nilpotent BRST operator is

$$Q = i \int_{k_0=\sqrt{|\vec{k}|^2+m^2}} \frac{d\vec{k}}{2\,k_0} (\bar{A}^\dagger(\vec{k})\,\bar{a}(\vec{k}) - A(\vec{k})\,a^\dagger(\vec{k}))$$

  where $A, \bar{A}$ are the annihilation operators of $\phi\,, \bar{\phi}$ while $a, \bar{a}$ are those of $\psi\,, \bar{\psi}$.
- The BRST formalism requires the conservation of Q.

The BRST invariant action

- The operator Q is the generator of following field transformations

$$\delta\,\bar{\phi}\,(x) = \epsilon\,Q\,\bar{\phi}(x) = \epsilon\,\bar{\psi}(x)$$
$$\delta\,\psi\,(x) = \epsilon\,Q\,\psi(x) = \epsilon\,\phi(x)$$

  the other fields remaining unchanged. The parameter ϵ should be considered anticommuting.

- The invariance of the the classical action follows from the equation

$$L = L(\Phi) + iQ[\partial_\mu\,\bar{\phi}\,\partial_\mu\,\psi - m^2\bar{\phi}\,\psi + g\,\Phi\,\bar{\phi}\,\psi]$$

  and from the nilpotent character of Q.

- This shows that Q, defined as an operator on the asymptotic Fock space, is Pseudo-Hermitian, conserved and nilpotent.
- Therefore the physical content of this theory can be analysed in much the same way as done above for the physical oscillator coupled to the B-F oscillator.

Physical states in QFT

- $\ker Q$ is physically invariant since Q is conserved.
- The physical state space is identified with the linear set of equivalence classes $\ker Q / \operatorname{im} Q$.
- It is a Hilbert space since it coincides with the set of equivalence classes of the states of the asymptotic Fock space associated to the field Φ.

Perturbation theory

- Notice that in the perturbation theory based on Feynman diagrams, the Φ self-interactions induced by the unphysical fields correspond to one loop diagrams built with either ϕ or ψ internal lines.
- In particular the Φ^n coupling is given by the sum of two n-vertex polygonal diagrams with sides corresponding either to ϕ , or to ψ, lines.
- This sum vanishes for any n since the contributions from the fermionic fields ψ exactly cancel those from the bosonic ones ϕ. In this sense ϕ and ψ compensate each other.

Renormalization

- Considering this point of view with more care one sees that the Feynman diagram expression for Φ^n couplings with $n \leq 2$ are ill-defined since the corresponding one-loop diagrams are divergent. In renormalization theory they are defined up to additive terms depending on three free coefficients.
- Therefore the exact cancellation of bosonic and fermionic loop contributions requires a renormalization prescription for which the above mentioned free coefficients compensate each other. This is a clear, however simple, example of the role of renormalization in the BRST formalism.

The ghost number

- One last basic ingredient of the BRST construction is the ghost number operator. For the B-F oscillator this is defined as follows

$$N_g = \frac{1}{2} (a^\dagger a - \bar{a}^\dagger \bar{a})$$

- N_g commutes with H_C and satisfies $[N_g, Q] = Q$.
- Thus it is conserved and it leaves the subspace $\ker Q$ invariant.

The ghost number grading

- This implies that N_g induces a grading on $\ker Q$ which decomposes into the direct sum of eigenspaces of N_g.
- For this model only the ghost number zero subspace is of physical interest, the rest of $\ker Q$ belonging to $\operatorname{im} Q$.

Yang-Mills theory in BRST formalism

- One starts from the asymptotic free $g = 0$ theory which corresponds to the sum of the free Landau Lagrangian densities depending on the fields A^a_μ and b^a for $a = 1, \ldots, N$.
- Following the compensating field strategy one introduces two more sets of anticommuting scalar fields: the ghost field $c^a(x)$ and the anti-ghost field $\bar{c}^b(x)$.
- Proceeding in strict analogy with the B-F oscillator one adds to the Lagrangian density the compensating term

$$-\bar{c}^a\,\partial^2\,c^a$$

Normal modes of YM

- The normal modes are

$$c^a(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{k^0=|\vec{k}|} \frac{d\vec{k}}{2\,k^0} \left(\gamma^a(\vec{k})\, \mathrm{e}^{-ik\,x} + (\gamma^a)^+(\vec{k})\, \mathrm{e}^{i\,k\,x}\right)$$

$$\bar{c}^a(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{k^0=|\vec{k}|} \frac{d\vec{k}}{2\,k^0} \left(\bar{\gamma}^a(\vec{k})\, \mathrm{e}^{-ik\,x} - (\bar{\gamma}^a)^+(\vec{k})\, \mathrm{e}^{i\,k\,x}\right)$$

- The (anti)commutation rules

$$\{\gamma^a(\vec{k}), (\bar{\gamma}^b)^+(\vec{q})\} = \{\bar{\gamma}^a(\vec{k}), (\gamma^b)^+(\vec{q})\} = 2\,|\vec{k}\,|\delta^{ab}\delta(\vec{k} - \vec{q})$$
$$[\alpha^a(\vec{k}), (\beta^b)^+(\vec{q})] = [\beta^a(\vec{k}), (\alpha^b)^+(\vec{q})] = 2\,|\vec{k}|\,\delta(\vec{k} - \vec{q})$$

and those for $a^a(\vec{k}, h)$ complete the canonical quantization prescriptions of the asymptotic theory.

The asymptotic BRST operator

- The BRST operator Q

$$Q = -i \int_{k^0=|\vec{k}|} \frac{d\vec{k}}{2\,k^0} \left[(\beta^a)^+(\vec{k})\,\gamma^a(\vec{k}) - (\gamma^a)^+(\vec{k})\,\beta^a(\vec{k}) \right] =$$

$$= -i \int d^4x \left[\partial_\mu\, c^a(x) \frac{\delta}{\delta\, A^a_\mu(x)} - b^a(x) \frac{\delta}{\delta\, \bar{c}^a(x)} \right]$$

annihilates the free Lagrangian density

$$L_0 = -\frac{1}{2} \partial_\mu\, A^a_\nu (\partial_\mu\, A^a_\nu - \partial_\nu\, A^a_\mu) - i\, Q\, (\bar{c}^a \partial_\mu\, A^a_\mu)$$

- Indeed the first term in $i\,Q$ is the generator of abelian gauge transformation $A^a_\mu \to A^a_\mu + \partial_\mu\, c^a$ which leaves invariant the first term in L_0, while the second term in L_0 is annihilated by the nilpotent Q.

The interacting theory

- Therefore, for what concerns the asymptotic theory, $\ker Q / \operatorname{im} Q$ is a Hilbert physical state corresponding to the Fock space of gluons with helicity ± 1.
- One is left with the problem of extending this result to the fully interacting theory. The Lagrangian density

$$L = -\frac{1}{4}(F^a)^{\mu\nu} F^a_{\mu\nu} - i\, Q\,(\bar{c}^a \partial_\mu A^a_\mu)$$

is not annihilated by Q since $-\frac{1}{4}(F^a)^{\mu\nu} F^a_{\mu\nu}$ is not invariant under the abelian gauge transformations.

The non-abelian gauge symmetry

- The solution to this problem consists in replacing, at the interacting level, the abelian generator with the non-abelian one:

$$X(c) = \int d^4x\, D_\mu c^c(x)\, \frac{\delta}{\delta A_\mu^c(x)}$$

and deforming the BRST operator is accordingly

$$Q \to -i\left(X(c) - \int d^4x\, b^a(x)\, \frac{\delta}{\delta \bar{c}^a(x)}\right)$$

The interacting BRST operator

- $-\frac{1}{4}(F^a)^{\mu\nu} F^a_{\mu\nu}$ is now annihilated by $X(c)$ since it is gauge-invariant.
- One is not finished yet, since $X(c)$, and thus Q too, is not nilpotent, due to the non-abelian character of the gauge transformations:

$$[X^a(x), X^b(y)] = \delta(x-y)\, g\, f^{abc}\, X^c(y)$$
$$X^2(c) = X(c \wedge c/2)$$

where $(c \wedge c)^a(x) = g\, f^{abc}\, c^b(x)\, c^c(x)$.

The interacting BRST operator

- This requires adding an extra term to Q

$$D(c) = X(c) - \frac{g}{2} \int d^4x\, f^{abc}\, c^a(x)\, c^b(x)\, \frac{\delta}{\delta c^c(x)}$$

$$Q = -i \left(D(c) - \int d^4y\, b^a(y)\, \frac{\delta}{\delta \bar{c}^a(y)} \right)$$

- $D^2(c) = Q^2 = 0$.
- $D(c)$ annihilates $-\frac{1}{4}(F^a)^{\mu\nu} F^a_{\mu\nu}$ since $X(c)$ does it.
- $D(c)$ is the coboundary operator of Chevalley gauge Lie algebra cohomology.

The Faddeev-Popov Lagrangian density

- One arrives at

$$L = -\frac{1}{4}(F^a)^{\mu\nu} F^a_{\mu\nu} - i\, Q\, (\bar{c}^a \partial_\mu A^a_\mu)$$

- The first term is the original gauge invariant Yang-Mills term; the second Q-trivial term is the gauge-fixing term implementing the compensation mechanism which relies on the nilpotency of Q .

Gauge independence

- Replacing the Q-trivial term in the Lagrangian density with a different term with the same structure and ghost number, e.g. $-i\,Q(\bar{c}^a(\partial_\mu A^a_\mu + \xi\, b^a))$, corresponds to a change in the unphysical part of the asymptotic Fock space: it does not change the physical results. For example, the parameter ξ does not affect the physics.
- This construction extends to the full quantum theory through renormalization. This is a non-trivial result which will not be reviewed here.

Generalizations

- The BRST formalism applies directly to arbitrary gauge theories and to Lagrangian systems whose constraints are first class and form a Lie algebra.
- Further generalizations are possible, e.g. to cases in which the algebra of constraints is not a Lie algebra since its structure constants are field dependent.

Summary

The relation between the BRST construction and gauge-invariance

- Covariant quantization of gauge fields requires the compensation of unphysical degrees of freedom.
- This is ensured by the Kugo-Ojima quartet compensation mechanism.
- This is not related to any gauge symmetry, but is based on the existence of a physically invariant subspace identified with the kernel of the BRST operator Q.
- In turn, the existence of such a BRST operator in the interacting theory relies on the gauge invariance of the underlying classical theory.